Nonlinear ODE in Complex Form

The Painlevé Equations - Nonlinear Special Functions (IV)

2014 - 09 - 01 (Mon.)
10:00 - 11:30
R440, Astronomy and Mathematics Building

The six Painlevé equations (PI-PVI) were first discovered around the beginning of the twentieth century by Painlevé, Gambier and their colleagues in an investigation of nonlinear second-order ordinary differential equations. Recently there has been considerable interest in the Painlevé equations primarily due to the fact that they arise as reductions of the soliton equations which solvable by inverse scattering. Consequently the Painlevé equations can be regarded as completely integrable equations and possess solutions which can be expressed in terms of solutions of linear integral equations, despite being nonlinear equations. Although first discovered from strictly mathematical considerations, the Painlevé equations have arisen in a variety of important physical applications including statistical mechanics, random matrices, plasma physics, nonlinear waves, quantum gravity, quantum field theory, general relativity, nonlinear optics and fibre optics.

The Painlevé equations may be thought of a nonlinear analogues of the classical special functions [3, 6] and are included in the “Digital Library of Mathematical Functions”, which was published online in 2010 and as the book “NIST Handbook of Mathematical Functions” [9]. This is the update of Abramowitz and Stegun's classic book “Handbook of Mathematical Functions”.

The Painlevé equations possess a plethora of interesting properties including a Hamiltonian structure and associated isomonodromy problems, which express the Painlevé equations as the compatibility condition of two linear systems. Solutions of the Painlevé equations have some interesting asymptotics which are useful in applications. They possess hierarchies of rational solutions and one-parameter families of solutions expressible in terms of the classical special functions, for special values of the parameters. Further the Painlevé equations admit symmetries under affine Weyl groups which are related to the associated Bäcklund transformations.

In these lectures I shall: (i), discuss the origin and history of the Painlevé equations; (ii), review many of the remarkable properties which the Painlevé equations possess; (iii), describe the numerical solution of the Painlevé equations; and (iv), discuss some applications of the Painlevé equations, including to nonlinear waves, orthogonal polynomials, random matrices and vortex dynamics.

References

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