Stability analysis of a three-dimensional chemotaxis system

Symphony Chakraborty & T. W. -H. Sheu

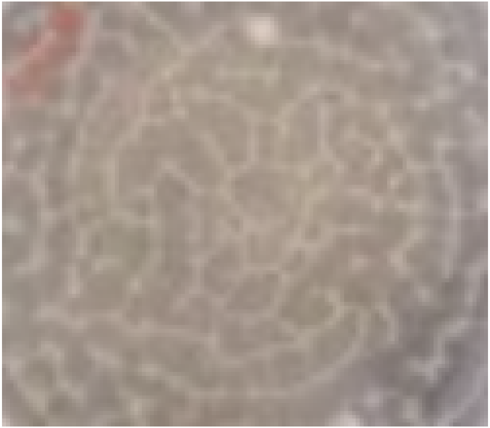
CASTS, National Taiwan University, Taipei, Taiwan.

March 17^{th}, 2019

2019 Spring Progress in Mathematical and Computational Studies on Science and Engineering Problems

Taiwan-India Joint Conference - Recent Progress on Flow Simulation and Stability Analysis

Motivation



https://tetrahymenaasset.vet.cornell.edu/science-modules/by-name/pattern-formation/

Introduction

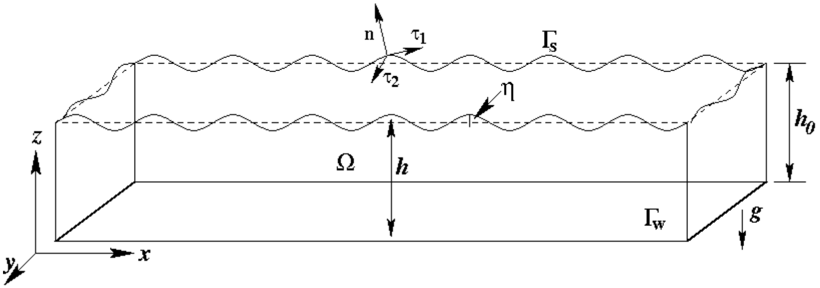


Figure 1: Schematic diagram of a three-dimensional chemotaxis system with liquid-air interface Γ_s, where the oxygen concentration is equal to that of air, not crossed by bacteria. No-slip boundary condition is imposed at the container walls Γ_w.

Governing Equations

The dimensional NS along with the KS equations are:

$$\nabla\cdot\mathbf{u} = 0, \quad (1a)$$

$$\rho\left(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}\right)+\nabla p-\mu\nabla^2\mathbf{u}+\rho g = -nV_b g(\rho_b-\rho)\mathbf{k}, \quad (1b)$$

$$n_t+\nabla\cdot[\mathbf{u}n-D_b\nabla n+S_{dim}r(c)n\nabla c] = 0, \quad (1c)$$

$$c_t+\nabla\cdot(\mathbf{u}c-D_O\nabla c) = -n\kappa r(c). \quad (1d)$$

Boundary conditions prescribed on the deformed free-surface $z=h(x,y,t)$,

$$\mathbf{n}\cdot\bar{\tau}\cdot\mathbf{n} = -(\nabla\cdot\mathbf{n})\sigma, \quad (1e)$$

$$\mathbf{t}_1\cdot\bar{\tau}\cdot\mathbf{n} = 0, \quad (1f)$$

$$\mathbf{t}_2\cdot\bar{\tau}\cdot\mathbf{n} = 0, \quad (1g)$$

$$S_{dim}nr(c)\nabla c\cdot\mathbf{n} = D_b\nabla n\cdot\mathbf{n}, \quad (1h)$$

$$c = c_{air}. \quad (1i)$$

Governing Equations

$\mathbf{n} = \frac{(-h_x,\, -h_y,\, 1)}{N}$, the unit outward normal vector,
$\mathbf{t}_1 = \frac{(1,\, 0,\, h_x)}{\sqrt{1+{h_x}^2}}$, $\mathbf{t}_2 = \frac{(0,\, 1,\, h_y)}{\sqrt{1+{h_y}^2}}$, the unit tangential vectors on the interface and $N = \sqrt{1 + {h_x}^2 + {h_y}^2}$.
The kinematic boundary condition is

$$h_t = w - uh_x - vh_y\,, \tag{2a}$$

Other boundary conditions at the wall,

$$\mathbf{u} = 0, \qquad \nabla n \cdot \mathbf{n} = 0, \qquad \nabla c \cdot \mathbf{n} = 0, \quad \text{at} \quad z = 0\,. \tag{2b}$$

Governing Equations

Scaling:

$$(x,y) = L(x',y'), \quad z = h_0 z', \quad h = h_0 h', \quad n = \bar{n}_0 n', \quad c = c_{air} c',$$
$$t = \frac{{h_0}^2}{\varepsilon D_b} t', \quad p = \frac{\mu D_b}{\varepsilon {h_0}^2} p', \quad (u,v) = \frac{D_b}{h_0}(u',v') \quad w = \frac{\varepsilon D_b}{h_0} w', \tag{3}$$

Dimensionless Navier-Stokes equation reads as

$$\begin{aligned}
u_x + v_y + w_z &= 0, && \text{(4a)}\\
\varepsilon\left(u_t + uu_x + vu_y + wu_z\right) + Pr_\tau p_x &= Pr_\tau\left[\varepsilon^2\left(u_{xx} + u_{yy}\right) + u_{zz}\right], && \text{(4b)}\\
\varepsilon\left(v_t + uv_x + vv_y + wv_z\right) + Pr_\tau p_y &= Pr_\tau\left[\varepsilon^2\left(v_{xx} + v_{yy}\right) + v_{zz}\right], && \text{(4c)}\\
\varepsilon^3\left(w_t + uw_x + vw_y + ww_z\right) + Pr_\tau p_z &= \varepsilon^2 Pr_\tau\left[\varepsilon^2\left(w_{xx} + w_{yy}\right) + w_{zz}\right]\\
&\quad -\varepsilon {Fr_\tau}^{-2} - \varepsilon Pr_\tau Ra_\tau n, && \text{(4d)}
\end{aligned}$$

Governing Equations

Dimensionless Keller-Segel equation reads as

$$
\begin{aligned}
\varepsilon\left(n_t + un_x + vn_y + wn_z\right) &= \left[\varepsilon^2\left(n_{xx} + n_{yy}\right) + n_{zz}\right] \\
&\quad - S_\tau r(c)\Big(n\left[\varepsilon^2\left(c_{xx} + c_{yy}\right) + c_{zz}\right] \\
&\quad + \left[\varepsilon^2\left(c_x n_x + c_y n_y\right) + c_z n_z\right]\Big), \qquad (5a) \\
\varepsilon\left(c_t + uc_x + vc_y + wc_z\right) &= -H_\tau r(c) n \\
&\quad + Le_\tau\left[\varepsilon^2\left(c_{xx} + c_{yy}\right) + c_{zz}\right]. \qquad (5b)
\end{aligned}
$$

Boundary conditions

Dimensionless boundary conditions at $z = h(x, y, t)$,

$$p + \frac{\varepsilon^3\left[\left(1+\varepsilon^2 h_x^2\right)h_{yy} - 2\varepsilon^2 h_x h_y h_{xy} + \left(1+\varepsilon^2 h_y^2\right)h_{xx}\right]}{Ca_\tau\left[1+\varepsilon^2\left(h_x^2+h_y^2\right)\right]^{\frac{3}{2}}}$$

$$= \frac{2\varepsilon^2\left[\varepsilon^2\left(u_x h_x^2 + \left(u_y+v_x\right)h_x h_y + v_y h_y^2\right) - \left(u_z+\varepsilon^2 w_x\right)h_x - \left(v_z+\varepsilon^2 w_y\right)h_y + w_z\right]}{\left[1+\varepsilon^2\left(h_x^2+h_y^2\right)\right]}, \quad \text{(6a)}$$

$$\varepsilon^2\left[\left(u_y+v_x\right)h_y - 2\left(w_z-u_x\right)h_x + \left(v_z+\varepsilon^2 w_y\right)h_x h_y\right]$$
$$= \left(1-\varepsilon^2 {h_x}^2\right)\left(u_z+\varepsilon^2 w_x\right), \quad \text{(6b)}$$

$$\varepsilon^2\left[\left(u_y+v_x\right)h_x - 2\left(w_z-v_y\right)h_y + \left(u_z+\varepsilon^2 w_x\right)h_x h_y\right]$$
$$= \left(1-\varepsilon^2 {h_y}^2\right)\left(v_z+\varepsilon^2 w_y\right), \quad \text{(6c)}$$

Boundary conditions

Dimensionless boundary conditions at $z = h(x, y, t)$,

$$S_\tau r(c) n\left[c_z - \varepsilon^2\left(c_x h_x + c_y h_y\right)\right] = \left[n_z - \varepsilon^2\left(n_x h_x + n_y h_y\right)\right], \quad (7a)$$

$$c = 1. \quad (7b)$$

The kinematic boundary condition is

$$h_t + u h_x + v h_y = w. \quad (7c)$$

Other boundary conditions at $z = 0$,

$$u = 0, \quad v = 0, \quad w = 0, \quad (7d)$$

$$n_z = \varepsilon^2\left(n_x h_x + n_y h_y\right), \quad (7e)$$

$$c_z = \varepsilon^2\left(c_x h_x + c_y h_y\right). \quad (7f)$$

Dimensionless Parameters

Prandtl number, $Pr_\tau = \frac{\nu}{D_b}$,

Froude number, $Fr_\tau = \frac{D_b}{\sqrt{g{h_0}^3}}$,

Rayleigh number, $Ra_\tau = \frac{gV_b\bar{n}_0(\rho_b - \rho){h_0}^3}{D_b\mu}$,

Chemotaxis sensitivity, $S_\tau = \frac{S_{dim}c_{air}}{D_b}$,

Chemotaxis head, $H_\tau = \frac{\kappa\bar{n}_0{h_0}^2}{c_{air}D_b}$,

Lewis number, $Le_\tau = \frac{D_O}{D_b}$,

Capillary number, $Ca_\tau = \frac{\mu D_b}{\sigma h_0}$.

Steady state solution

$$p(z) = \varepsilon\left(\frac{1}{Pr_\tau {Fr_\tau}^2} + Ra_\tau\right)(1-z), \quad (8a)$$

$$c(z) = 1 - \frac{2}{S_\tau}\ln\left(\frac{\cos\left(\frac{S_\tau}{2}A_1 z\right)}{\cos\left(\frac{S_\tau}{2}A_1\right)}\right), \quad (8b)$$

$$n(z) = \frac{S_\tau}{2}\frac{Le_\tau {A_1}^2}{H_\tau}\sec^2\left(\frac{S_\tau}{2}A_1 z\right), \quad (8c)$$

where the unknown constant A_1 is determined from the transcendental equation: $\tan\left(\frac{S_\tau}{2}A_1\right) = \frac{H_\tau}{Le_\tau}\frac{1}{A_1}$.

$$n(z) = 1+\frac{S_\tau}{6}\left(3z^2-1\right)\frac{H_\tau}{Le_\tau}+\frac{{S_\tau}^2}{18}\left(1-3z^2+3z^4\right)\frac{{H_\tau}^2}{{Le_\tau}^2}+O\left(\frac{H_\tau}{Le_\tau}\right)^3$$

$$c(z) = 1+\frac{1}{2}\left(z^2-1\right)\frac{H_\tau}{Le_\tau}+\frac{S_\tau}{24}\left(z^2-1\right)^2\frac{{H_\tau}^2}{{Le_\tau}^2}+O\left(\frac{H_\tau}{Le_\tau}\right)^3$$

Linear stability analysis

Perturb the field variables at the basic state:

$$\begin{aligned} &\mathbf{u}(\mathbf{x},t) = \varepsilon\tilde{\mathbf{u}}(\mathbf{x},t), \quad p(\mathbf{x},t) = p(z) + \varepsilon\tilde{p}(\mathbf{x},t), \\ &n(\mathbf{x},t) = n(z) + \varepsilon\tilde{n}(\mathbf{x},t), \quad c(\mathbf{x},t) = c(z) + \varepsilon\tilde{c}(\mathbf{x},t), \\ &h(\mathbf{x},t) = 1 + \varepsilon\tilde{\eta}(\mathbf{x},t). \end{aligned} \tag{9}$$

Eliminate of $\tilde{p}$, $\tilde{u}$, and $\tilde{v}$, and then the governing equations can be expressed in terms of $\tilde{w}$, $\tilde{n}$, and $\tilde{c}$ only. Take the divergence of the linearized NS equations, *i.e.*, $[\partial_x(x\text{-mom}) + \partial_y(y\text{-mom}) + \partial_z(z\text{-mom})]$ and use of the continuity equation will give

$$\nabla^2\tilde{p} = \left(1-\varepsilon^2\right)\left[\frac{\varepsilon}{Pr_\tau}\tilde{w}_t - \varepsilon^2\left(\tilde{w}_{xx}+\tilde{w}_{yy}\right) - \tilde{w}_{zz}\right]_z - \varepsilon Ra_\tau \tilde{n}_z\,. \tag{10}$$

Linear stability analysis

Apply Laplacian operator on the z-component of the linearized NS equation and use (10) to eliminate the pressure.

$$\varepsilon\left[\varepsilon^2\left(\tilde{w}_{xx}+\tilde{w}_{yy}\right)+\tilde{w}_{zz}\right]_t - Pr_\tau\left[\varepsilon^4\left(\tilde{w}_{xxxx}+\tilde{w}_{yyyy}\right)+\tilde{w}_{zzzz}\right]$$
$$= -\varepsilon Pr_\tau Ra_\tau\left(\tilde{n}_{xx}+\tilde{n}_{yy}\right)+2\varepsilon^2 Pr_\tau\left(\tilde{w}_{xxzz}+\tilde{w}_{yyzz}+\varepsilon^2\tilde{w}_{xxyy}\right). \quad (11)$$

Adding of $\partial_z(z\text{-mom})$ with the two-dimensional Laplacian operator of the linearized normal stress boundary condition and using (10) will give

$$Pr_\tau\left[3\varepsilon^2\left(\tilde{w}_{xx}+\tilde{w}_{yy}\right)+\tilde{w}_{zz}\right]_z - \varepsilon\tilde{w}_{zt}$$
$$= \varepsilon^3\frac{Pr_\tau}{Ca_\tau}\left(\tilde{\eta}_{xxxx}+2\tilde{\eta}_{xxyy}+\tilde{\eta}_{yyyy}\right) \quad (12)$$

Linear stability analysis

Decomposed into normal modes of the form,

$$\left[\tilde{w},\tilde{n},\tilde{c},\tilde{\eta}\right] = \left[W(z),N(z),C(z),\eta\right]e^{\mathrm{i}(k_x x+k_y y)+\omega t}. \quad (13)$$

The system of equations become

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}-\left(\frac{\varepsilon\omega}{Pr_\tau}+\varepsilon^2k^2\right)\right]\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-\varepsilon^2k^2\right)W = -\varepsilon k^2 Ra_\tau N, \quad (14a)$$

$$\frac{\mathrm{d}^2N}{\mathrm{d}z^2}-S_\tau\frac{\mathrm{d}c}{\mathrm{d}z}\frac{\mathrm{d}N}{\mathrm{d}z}-\left(\varepsilon^2k^2+\varepsilon\omega+S_\tau\frac{\mathrm{d}^2c}{\mathrm{d}z^2}\right)N = \varepsilon\frac{\mathrm{d}n}{\mathrm{d}z}W$$

$$+S_\tau\left(n(z)\frac{\mathrm{d}^2C}{\mathrm{d}z^2}+\frac{\mathrm{d}n}{\mathrm{d}z}\frac{\mathrm{d}C}{\mathrm{d}z}-\varepsilon^2k^2n(z)C\right), \quad (14b)$$

$$\frac{\mathrm{d}^2C}{\mathrm{d}z^2}-\left(\frac{\varepsilon\omega}{Le_\tau}+\varepsilon^2k^2\right)C-\frac{H_\tau}{Le_\tau}N = \frac{\varepsilon}{Le_\tau}\frac{\mathrm{d}c}{\mathrm{d}z}W, \quad (14c)$$

Linear stability analysis

The boundary conditions prescribed at the interface $z = 1$

$$\frac{\mathrm{d}^4 W(1)}{\mathrm{d}z^4} - \left(\frac{\varepsilon\omega}{Pr_\tau} + 3\varepsilon^2 k^2\right)\frac{\mathrm{d}^2 W(1)}{\mathrm{d}z^2} = 0, \quad (15a)$$

$$\frac{\mathrm{d}^3 W(1)}{\mathrm{d}z^3} + \varepsilon^2 k^2 \frac{\mathrm{d}W(1)}{\mathrm{d}z} = 0, \quad (15b)$$

$$\frac{\mathrm{d}N(1)}{\mathrm{d}z} - S_\tau \frac{\mathrm{d}c(1)}{\mathrm{d}z} N(1) - S_\tau n \frac{\mathrm{d}C(1)}{\mathrm{d}z} = \frac{S_\tau}{\omega}\frac{\mathrm{d}c(1)}{\mathrm{d}z}\frac{\mathrm{d}n(1)}{\mathrm{d}z} W(1) + \frac{S_\tau}{\omega}\left(n\frac{\mathrm{d}^2 c(1)}{\mathrm{d}z^2} - \frac{1}{S_\tau}\frac{\mathrm{d}^2 n(1)}{\mathrm{d}z^2}\right) W(1), \quad (15c)$$

$$\omega C(1) + \frac{\mathrm{d}c(1)}{\mathrm{d}z} W(1) = 0, \quad (15d)$$

Linear stability analysis

And, at the other boundaries $z = 0$, we prescribed

$$W(0) = 0, \quad \frac{\mathrm{d}W(0)}{\mathrm{d}z} = 0, \quad \frac{\mathrm{d}N(0)}{\mathrm{d}z} = 0, \quad \frac{\mathrm{d}C(0)}{\mathrm{d}z} = 0. \qquad (16)$$

where $k = \sqrt{{k_x}^2 + {k_y}^2}$, k_x and k_y are the wave numbers of the disturbance in the x- and y-directions, and ω is complex.

Linear stability analysis

Consider possible assumptions with the leading terms which gives non-trivial solutions.
Case I: cell conservation equation is purely diffusive, and oxygen conservation balances between diffusion and consumption of oxygen.

$$Ra_\tau \sim O(1), \quad \omega \sim O\left(\frac{S_\tau H_\tau}{Le_\tau}\right), \qquad D^2 C = 0, \quad D^2 N = 0\,,$$

and, Case II: cell conservation equation is the same, but in oxygen conservation, advection is important as well as the other terms.

$$Ra_\tau \sim O\left(\frac{Le_\tau}{S_\tau H_\tau}\right), \quad \omega \sim O(\frac{S_\tau H_\tau}{Le_\tau}), \qquad D^2 C = N, \quad D^2 N = 0\,.$$

Linear stability analysis

Expansions of the form

$$W(z)=\sum_{i=0}^{\infty}W_i(z)\left(\frac{S_\tau H_\tau}{Le_\tau}\right)^i,\quad N(z)=\sum_{i=0}^{\infty}N_i(z)\left(\frac{S_\tau H_\tau}{Le_\tau}\right)^i,$$

$$C(z)=\sum_{i=0}^{\infty}C_i(z)\left(\frac{S_\tau H_\tau}{Le_\tau}\right)^i,\quad \omega(k)=\sum_{i=1}^{\infty}\omega_i(k)\left(\frac{S_\tau H_\tau}{Le_\tau}\right)^i,$$

$$\text{and,}\quad Ra_\tau(k)=\sum_{i=-1}^{\infty}Ra_{\tau i}(k)\left(\frac{S_\tau H_\tau}{Le_\tau}\right)^i. \tag{17}$$

So, the governing equations become

$$\frac{\mathrm{d}^4 W_0}{\mathrm{d}z^4} = -\varepsilon k^2 N_0 Ra_{\tau-1},\quad \frac{\mathrm{d}^2 C_0}{\mathrm{d}z^2}=N_0,\quad \frac{\mathrm{d}^2 N_0}{\mathrm{d}z^2}=0, \tag{18}$$

Linear stability analysis

With the boundary conditions

$$\frac{\mathrm{d}^3 W_0(1)}{\mathrm{d}z^3}=0, \quad \frac{\mathrm{d}^2 W_0(1)}{\mathrm{d}z^2}=0, \quad C_0(1)=0, \quad \frac{\mathrm{d}N_0(1)}{\mathrm{d}z}=0,$$
$$\frac{\mathrm{d}W_0(0)}{\mathrm{d}z}=0, \quad W_0(0)=0, \quad \frac{\mathrm{d}N_0(0)}{\mathrm{d}z}=0, \quad \frac{\mathrm{d}C_0(0)}{\mathrm{d}z}=0. \quad (19)$$

Arbitrarily set $N_0=1$ at $z=1$ and the corresponding solutions are

$$W_0(z) = -\frac{k^2\varepsilon Ra_{\tau-1}}{24}z^2(z^2-4z+6), \quad (20a)$$
$$C_0(z) = \frac{1}{2}(z^2-1), \quad (20b)$$
$$N_0(z) = 1. \quad (20c)$$

Linear stability analysis

At the next order, the governing equations are

$$\begin{aligned}
\frac{\mathrm{d}^4 W_1}{\mathrm{d}z^4} &= \left(\frac{\varepsilon\omega_1}{Pr_\tau}+2k^2\varepsilon^2\right)\frac{\mathrm{d}^2 W_0}{\mathrm{d}z^2}-k^2\varepsilon\left(Ra_{\tau 0}N_0+Ra_{\tau -1}N_1\right),\\
\frac{\mathrm{d}^2 C_1}{\mathrm{d}z^2} &= N_1+k^2\varepsilon^2 C_0\,,\\
\frac{\mathrm{d}^2 N_1}{\mathrm{d}z^2} &= \left(2+\varepsilon\omega_1+k^2\varepsilon^2\right)N_0+z\frac{\mathrm{d}N_0}{\mathrm{d}z}+z\varepsilon W_0\,,
\end{aligned} \tag{21}$$

with the boundary conditions

$$\begin{aligned}
&\frac{\mathrm{d}^3 W_1(1)}{\mathrm{d}z^3}+k^2\varepsilon^2\frac{\mathrm{d}W_0(1)}{\mathrm{d}z}=0,\quad \frac{\mathrm{d}^2 W_1(1)}{\mathrm{d}z^2}-\left(3k^2\varepsilon^2+\varepsilon\omega_1\right)W_0(1)=0,\\
&\omega_1\frac{\mathrm{d}N_1(1)}{\mathrm{d}z}-\omega_1\frac{\mathrm{d}C_0(1)}{\mathrm{d}z}+\omega_2\frac{\mathrm{d}N_0(1)}{\mathrm{d}z}-\omega_1 N_0(1)=0,\\
&C_1(1)=0,\quad \frac{\mathrm{d}W_1(0)}{\mathrm{d}z}=0,\quad W_1(0)=0,\quad \frac{\mathrm{d}N_1(0)}{\mathrm{d}z}=0,\quad \frac{\mathrm{d}C_1(0)}{\mathrm{d}z}=0.
\end{aligned} \tag{22}$$

Linear stability analysis

From the solution of the problem at the first order gives the functions W_1, C_1 and N_1, and we can obtain ω_1 as

$$\omega_1 = \varepsilon k^2 \left(\frac{13}{360} Ra_{\tau_{-1}} - 1 \right). \tag{23}$$

At the second order, we have found ω_2 as follows:

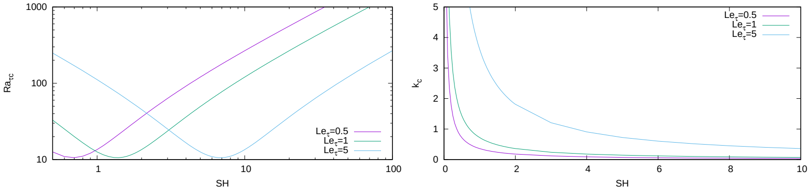
$$\begin{aligned} \omega_2 = & \frac{k^2}{13!360 Pr_\tau \omega_1} \Big[6! Ra_{\tau}{}^2_{-1} \left(1325467 \varepsilon^2 k^2 Pr_\tau + \varepsilon^4 k^4 \left(1879648 Pr_\tau - 474045 \right) \right) \\ & + 13 \varepsilon^4 k^4 Ra_{\tau}{}^3_{-1} \left(624226 Pr_\tau + 474045 \right) - 13! \varepsilon^2 k^2 Pr_\tau \left(13 Ra_{\tau 0} - 5! \right) \\ & - 6! 1716 Ra_{\tau_{-1}} \Big(15 \Big(Pr_\tau \left(2358 \varepsilon^4 k^4 + 2882 \varepsilon^2 k^2 + 7560 \right) - 255 \varepsilon^4 k^4 \Big) \\ & - 2366 \varepsilon^2 k^2 Pr_\tau Ra_{\tau 0} \Big) \Big]. \end{aligned} \tag{24}$$

Linear stability analysis

The system is unstable to small wavenumber disturbances if $Ra_{\tau_{-1}} > (360/13)$. The resultant instability indicates that it is non-oscillatory. It would be interesting to study the marginal stability in which $\omega_r = 0$ and can calculate ω_i. Marginal stability $(\omega = 0)$ occurs when

$$Ra_\tau(k) = \frac{360}{13(S_\tau H_\tau / Le_\tau)} + O(1). \tag{25}$$

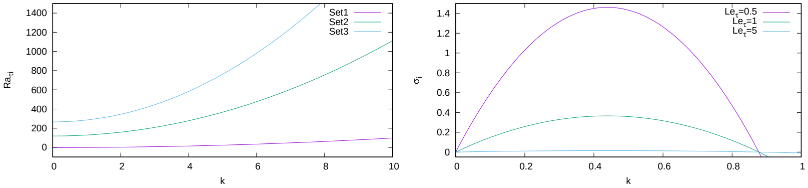
Results: Linear stability



(a) (b)

Figure 2: (a) $Ra_{\tau c}$ and (b) corresponding values of k_c are computed for the varying values of Le_τ at $\varepsilon = 0.1$. Minimum value of the curve is $\frac{S_\tau H_\tau}{Le_\tau} \approx 1.036$.

Results: Linear stability



(a) (b)

Figure 3: (a) $Ra_{\tau i}(k)$ curves at the varying sets of parameters, Set 1: $S_\tau = 1$, $H_\tau = 1$, $Le_\tau = 0.5$, $Pr_\tau = 500$; Set 2: $S_\tau = 5$, $H_\tau = 4$, $Le_\tau = 1$, $Pr_\tau = 7700$; and Set 3: $S_\tau = 10$, $H_\tau = 20$, $Le_\tau = 5$, $Pr_\tau = 500$; and (b) $\omega_i(k)$ curves for varying Le_τ with fixed value of $S_\tau = 5$, $H_\tau = 4$, $\varepsilon = 0.1$.

Results: Linear stability

Le_τ	$S_\tau H_\tau$	$Ra_{\tau c}$-HP	k_c-HP	$Ra_{\tau c}$-Present	k_c-Present	ε
1	0.05	1.02×10^4	1.37	524.738	3.85395	0.1
1	0.05			524.738	1.92697	0.2
1	0.05			524.738	0.77079	0.5
1	1	625	1.58	12.613	0.72176	0.1
1	10	200	1.90	120.596	0.07254	0.1
1	50	328	1.94	709.072	0.01451	0.1
1	100	522	1.91	1447.160	0.00725	0.1
5	10	221	1.24	13.535	0.36268	0.1
5	50	354	1.35	120.596	0.00725	0.1

Table 1: Values of $Ra_{\tau c}$, k_c evaluated for the shallow chamber case. Here, -HP represents the values from Hillesdon and Pedley (1996) for flat free surface and -Present represents the values of the present study for deformed free surface.

Weakly nonlinear analysis

$$\begin{aligned} \mathbf{u} = \mathbf{u}_0 + \tilde{\mathbf{u}}, \quad p &= p_0 + \tilde{p}, \quad n = n_0 + \tilde{n}, \\ \text{and} \quad c &= c_0 + \tilde{\mathbf{c}}, \quad h = h_0 + \tilde{h}. \end{aligned} \tag{26}$$

The scales of these perturbations are

$$\frac{\mathbf{u}'}{\mathbf{u}_0} \sim \frac{p'}{p_0} \sim \frac{n'}{n_0} \sim \frac{c'}{c_0} \sim \varepsilon \,.$$

Express $\mathbf{u}$, p, n, c, and h in terms of ε as

$$\left[\mathbf{u}, n, c, p, h\right] = \sum_{n=0}^{3} \varepsilon^n \left[\mathbf{u}_n, n_n, c_n, p_n, h_n\right] \left(x, y, z, t, T\right) + O(\varepsilon^4), \tag{27a}$$

$$Ra_\tau = Ra_{\tau c} + \varepsilon Ra_{\tau 1} + \ldots, \tag{27b}$$

A slow timescale T is chosen to be $T = \varepsilon t$.

Weakly nonlinear analysis

Obtain the steady-state equations for $n(z)$ and $c(z)$ at $O(1)$. At $O(\varepsilon)$, obtain the linear system of equations where the variables w_1, n_1, and c_1 are expressed as

$$
\begin{aligned}
w_1 &= W(z)\Re\left(A_1(T)e^{iky} + A_2(T)e^{-\frac{\sqrt{3}}{2}ikx-\frac{1}{2}iky} + A_3(T)e^{\frac{\sqrt{3}}{2}ikx-\frac{1}{2}iky}\right), \\
n_1 &= N(z)\Re\left(A_1(T)e^{iky} + A_2(T)e^{-\frac{\sqrt{3}}{2}ikx-\frac{1}{2}iky} + A_3(T)e^{\frac{\sqrt{3}}{2}ikx-\frac{1}{2}iky}\right), \\
c_1 &= C(z)\Re\left(A_1(T)e^{iky} + A_2(T)e^{-\frac{\sqrt{3}}{2}ikx-\frac{1}{2}iky} + A_3(T)e^{\frac{\sqrt{3}}{2}ikx-\frac{1}{2}iky}\right),
\end{aligned}
$$

where the amplitudes $A_i (i = 1, 2, 3)$ for rolls become $A_1 = A$, $A_2 = A_3 = 0$ and for hexagons $A_1 = A_2 = A_3 = A$.

Weakly nonlinear analysis

At $O(\varepsilon^2)$, the equations give rise to the solvability condition as follows:

$$\int \mathbf{v} \cdot RHSdx + \int \int \left(v_2 n_1 \frac{\partial c_1}{\partial y} \right) \bigg|_{z=0} dxdy \quad = \quad 0 \,, \qquad (29)$$

where the first-order adjoint $\mathbf{v} = (v_1, v_2, v_3)$ denotes the variables (w_1, n_1, c_1).

Three solvability conditions can be obtained by setting $\mathbf{v} = V(z)\bar{A}_1 e^{-\mathrm{i}ky}$, $\mathbf{v} = V(z)\bar{A}_2(T)e^{\frac{\sqrt{3}}{2}\mathrm{i}kx + \frac{1}{2}\mathrm{i}ky}$, and $\mathbf{v} = V(z)\bar{A}_3 e^{-\frac{\sqrt{3}}{2}\mathrm{i}kx + \frac{1}{2}\mathrm{i}ky}$ and substituting the nine expressions into (29) and then performing integration to obtain

$$\xi_1 \frac{dA_1}{dT} + Ra_{\tau 1}\xi_3 + \xi_2 \bar{A}_2 \bar{A}_3 \quad = \quad 0 \,. \qquad (30)$$

Weakly nonlinear analysis

For steady rolls where $A_1 = A$, $A_2 = A_3 = 0$ and $dA/dT = 0$, the existing solvability conditions imply that $Ra_{\tau 1} = 0$. For steady hexagons where $A_1 = A_2 = A_3 = A$ and $dA/dT = 0$,

$$Ra_{\tau 1} = -\frac{\xi_2}{\xi_3} A. \tag{31}$$

A stability analysis of (30) exhibits that the branches of roll bifurcation are unstable.
Redefine $Ra_\tau = Ra_{\tau c} + \varepsilon^2 Ra_{\tau 2}$ and consider $T = \varepsilon^2 t$.

Conclusions

- Parametric study: S_τ, H_τ and Le_τ are independently varied to determine Ra_τ and k.
- The instability generates as the values of S_τ, H_τ and Le_τ increase, although this effect is reversed as the length of the chamber increases.
- The wave number increases with the length of the chamber as well as the peak of temporal growth.

Weakly nonlinear analysis and bifurcation study is in progress.

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Symphony Chakraborty & T. W. -H. Sheu

CASTS, National Taiwan University, Taipei, Taiwan.

March 17^{th}, 2019

2019 Spring Progress in Mathematical and Computational Studies on Science and Engineering Problems

Taiwan-India Joint Conference - Recent Progress on Flow Simulation and Stability Analysis

Thank you