ψ-v computation of flow past sharp edges in uniform and accelerated flow

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$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{3}$$

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$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \qquad (3)$$

Where the non dimensional quantities t, p, u, v are respectively time, pressure, and velocities along the x- & y-directions. $Re = UL/\nu$ is the Reynolds number, ν being the kinematic viscosity.

The Streamfunction-Vorticity and Streamfunction-Velocity Formulations

Introducing streamfunction ψ and vorticity ω as $u = \dfrac{\partial \psi}{\partial y}$, $v = -\dfrac{\partial \psi}{\partial x}$, $\omega = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$,

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the above N-S can be rewritten as:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega \tag{4}$$

$$\frac{\partial \omega}{\partial t} = \frac{1}{Re}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) - \left(u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y}\right) \tag{5}$$

This is known as Streamfunction-Vorticity (ψ-ω) formulation.

The pure Streamfunction Formulation

The vorticity can be further eliminated to yield the so called Streamfunction Velocity (ψ-v) formulation

$$\frac{\partial}{\partial t}(\nabla^2\psi) = \frac{1}{Re}\nabla^4\psi + u\nabla^2 v - v\nabla^2 u. \tag{6}$$

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where the non dimensional quantities t, ψ, u, v are respectively time, streamfunction, and velocities along the x- & y-directions. $Re = UL/\nu$ is the Reynolds number, ν being the kinematic viscosity. The ψ-v formulation can easily be written in pure Streamfunction form as

$$\frac{\partial}{\partial t}(\nabla^2\psi) = \frac{1}{Re}\nabla^4\psi - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi \qquad (7)$$

Advantages

The main advantages of this formulation are:

- Avoids difficulties associated with primitive variables u, v, p.
- Avoids difficulties associated with vorticity boundary conditions.
- Iterations involve only the single variable ψ.
- Computationally easier to implement.

The steady-state equation

$$\nabla^4\psi = Re\left[\frac{\partial\psi}{\partial y}\left(\frac{\partial^3\psi}{\partial x^3}+\frac{\partial^3\psi}{\partial x\partial y^2}\right)-\frac{\partial\psi}{\partial x}\left(\frac{\partial^3\psi}{\partial x^2\partial y}+\frac{\partial^3\psi}{\partial y^3}\right)\right]. \tag{8}$$

Consider a rectangular domain $[a_1, a_2] \times [b_1, b_2]$ in the xy-plane. We divide the interval $[a_1, a_2]$ into m sub-intervals, not necessarily of equal lengths, by the points $a_1 = x_0, x_1, x_2, x_3, \ ...., x_{m-1}, x_m = a_2$ and similarly $[b_1, b_2]$ into n subintervals by the points $b_1 = y_0, y_1, y_2, y_3, \ ...., y_{n-1}, y_n = b_2$.

The compact steady-state stencil

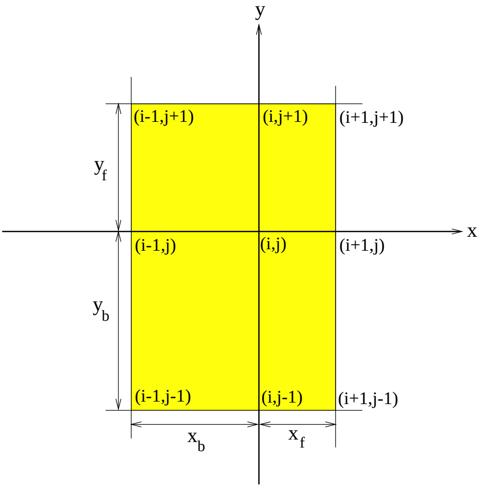

Figure: The stencil used for the steady ψ-v formulation.

Discretization of the steady-state equation [6]

The finite difference approximation of (8) is given by

$$
\begin{aligned}
p\delta_x^4\psi + 2\delta_x^2\delta_y^2 + q\delta_y^4\psi = Re&\left[v\left(\delta_x^2 u + \delta_y^2 u\right) - u\left(\delta_x^2 v + \delta_y^2 v\right)\right] \\
&- \frac{6p(x_f - x_b)}{x_f x_b}\delta_x^2 v + \frac{12p}{x_f x_b}\delta_x v \\
&- \frac{24p(x_f - x_b)}{x_f^2 x_b^2} v + \frac{6q(y_f - y_b)}{y_f y_b}\delta_y^2 u \\
&- \frac{12q}{y_f y_b}\delta_y u + \frac{24q(y_f - y_b)}{y_f^2 y_b^2} u \\
&+ O(x_f - x_b, y_f - y_b)
\end{aligned} \tag{9}
$$

where $p = 1 + \frac{Re}{3}\left(x_f - x_b\right) u$ and $q = 1 + \frac{Re}{3}\left(y_f - y_b\right) v$.

and the operators δ_x, δ_y, δ_x^2, δ_y^2, δ_x^4, δ_y^4, and $\delta_x^2\delta_y^2$ are given by

$$\delta_x\psi = \frac{\psi_{i+1,j} - \psi_{i-1,j}}{x_f + x_b},$$

$$\delta_y\psi_{ij} = \frac{\psi_{i,j+1} - \psi_{i,j-1}}{y_f + y_b},$$

$$\delta_x^2\psi = \frac{2}{(x_f + x_b)}\left[\frac{\psi_{i+1,j}}{x_f} - \left(\frac{1}{x_f} + \frac{1}{x_b}\right)\psi_{i,j} + \frac{\psi_{i-1,j}}{x_b}\right],$$

$$\delta_y^2\psi_{ij} = \frac{2}{(y_f + y_b)}\left[\frac{\psi_{i,j+1}}{y_f} - \left(\frac{1}{y_f} + \frac{1}{y_b}\right)\psi_{i,j} + \frac{\psi_{i,j-1}}{y_b}\right],$$

Discretization

$$\delta_x^4\psi_{ij} = \frac{12}{h}\left[-\frac{1}{x_f^3}\psi_{i+1,j} - \frac{1}{x_b^3}\psi_{i-1,j} + \left(\frac{1}{x_f^3}+\frac{1}{x_b^3}\right)\psi_{i,j}\right],$$

$$\delta_y^4\psi_{ij} = \frac{12}{k}\left[-\frac{1}{y_f^3}\psi_{i,j+1} - \frac{1}{y_b^3}\psi_{i,j-1} + \left(\frac{1}{y_f^3}+\frac{1}{y_b^3}\right)\psi_{i,j}\right],$$

and

$$\begin{aligned}\delta_x^2\delta_y^2\psi_{ij} = \frac{1}{hk}\Bigg[&\frac{\psi_{i+1,j+1}}{x_f y_f} + \frac{\psi_{i-1,j+1}}{x_b y_f} - \left(\frac{1}{x_f y_f}+\frac{1}{x_b y_f}\right)\psi_{i,j+1} \\ &- \left(\frac{1}{x_f y_f}+\frac{1}{x_f y_b}\right)\psi_{i+1,j} + \left(\frac{1}{x_f y_f}+\frac{1}{x_f y_b}+\frac{1}{x_b y_f}+\frac{1}{x_b y_b}\right)\psi_{i,j} \\ &- \left(\frac{1}{x_f y_b}+\frac{1}{x_b y_b}\right)\psi_{i,j-1} - \left(\frac{1}{x_b y_f}+\frac{1}{x_b y_b}\right)\psi_{i-1,j} + \frac{\psi_{i+1,j-1}}{x_f y_b} \\ &+ \frac{\psi_{i-1,j-1}}{\cdots}\Bigg].\end{aligned} \quad (10)$$

Discretization

Making use of the approximations for the operators δ_x, δ_y, δ_x^2, δ_y^2, δ_x^4, δ_y^4, and $\delta_x^2\delta_y^2$ as provided above, (13) can be discretized compactly on the nonuniform nine-point stencil as

$$
\begin{aligned}
A\psi_{i+1,j+1} + B\psi_{i,j+1} + C\psi_{i-1,j+1} + D\psi_{i+1,j} + E\psi_{i,j} \\
+ F\psi_{i-1,j} + G\psi_{i+1,j-1} + H\psi_{i,j-1} + I\psi_{i-1,j-1} = f_{i,j}
\end{aligned} \qquad (11)
$$

Discretization

where

$$A = \frac{2}{hkx_f y_f}$$

$$B = -\frac{12q}{ky_f^3} - \frac{2}{hk}\left(\frac{1}{x_f y_f} + \frac{1}{x_b y_f}\right)$$

$$C = \frac{2}{hkx_b y_f}$$

$$D = -\frac{12p}{hx_f^3} - \frac{2}{hk}\left(\frac{1}{x_f y_f} + \frac{1}{x_f y_b}\right)$$

$$E = \frac{12p}{h}\left(\frac{1}{x_f^3} + \frac{1}{x_b^3}\right) + \frac{12q}{k}\left(\frac{1}{y_f^3} + \frac{1}{y_b^3}\right) + \frac{2}{hk}\left(\frac{1}{x_f y_f} + \frac{1}{x_f y_b} + \frac{1}{x_b y_f} + \frac{1}{x_b y_b}\right)$$

Discretization

$$F = -\frac{12p}{hx_b^3} - \frac{2}{hk}\left(\frac{1}{x_b y_f} + \frac{1}{x_b y_b}\right)$$

$$G = \frac{2}{hkx_f y_b}$$

$$H = -\frac{12q}{ky_b^3} - \frac{2}{hk}\left(\frac{1}{x_f y_b} + \frac{1}{x_b y_b}\right)$$

$$I = \frac{2}{hkx_b y_b}$$

Discretization

$$
\begin{aligned}
f_{i,j} = {} & \frac{6p}{h}\left(\frac{v_{i+1,j}-v_{i-1,j}}{x_f x_b}\right) - \frac{Re.u_{i,j}}{k}\left[\frac{v_{i,j+1}}{y_f}+\frac{v_{i,j-1}}{y_b}-\left(\frac{1}{y_f}+\frac{1}{y_b}\right)v_{i,j}\right] \\
& - \frac{1}{h}\left(\frac{6p(x_f-x_b)}{x_f x_b}+Re.u_{i,j}\right)\left[\frac{v_{i+1,j}}{x_f}+\frac{v_{i-1,j}}{x_b}-\left(\frac{1}{x_f}+\frac{1}{x_b}\right)v_{i,j}\right] \\
& - \frac{6q}{k}\left(\frac{u_{i,j+1}-u_{i,j-1}}{y_f y_b}\right) + \frac{Re.v_{i,j}}{h}\left[\frac{u_{i+1,j}}{x_f}+\frac{u_{i-1,j}}{x_b}-\left(\frac{1}{x_f}+\frac{1}{x_b}\right)u_{i,j}\right] \\
& + \frac{1}{k}\left(\frac{6q(y_f-y_b)}{y_f y_b}+Re.v_{i,j}\right)\left[\frac{u_{i,j+1}}{y_f}+\frac{u_{i,j-1}}{y_b}-\left(\frac{1}{y_f}+\frac{1}{y_b}\right)u_{i,j}\right] \\
& - \frac{24p(x_f-x_b)}{x_f^2 x_b^2}v_{i,j} + \frac{24q(y_f-y_b)}{y_f^2 y_b^2}u_{i,j}
\end{aligned}
$$

Approximation of velocities u and v

$$\frac{y_b}{6k}u_{i,j+1} - \left(1 - \frac{y_f + y_b}{6k}\right)u_{i,j} + \frac{y_f}{6k}u_{i,j-1}$$

$$= \frac{\left(\psi_{i,j+1} - \psi_{i.j-1}\right)}{2k}$$

$$- \frac{(y_f - y_b)}{2k}\left[\frac{\psi_{i,j+1}}{y_f} + \frac{\psi_{i,j-1}}{y_b} - (\frac{1}{y_f} + \frac{1}{y_b})\psi_{i,j}\right],$$

$$\frac{x_b}{6h}v_{i+1,j} - \left(1 - \frac{x_f + x_b}{6h}\right)v_{i,j} + \frac{x_f}{6h}v_{i-1,j}$$

$$= \frac{\left(\psi_{i-1,j} - \psi_{i+1,j}\right)}{2h}$$

$$+ \frac{(x_f - x_b)}{2h}\left[\frac{\psi_{i+1,j}}{x_f} + \frac{\psi_{i-1,j}}{x_b} - (\frac{1}{x_f} + \frac{1}{x_b})\psi_{i,j}\right].$$

The unsteady stencil

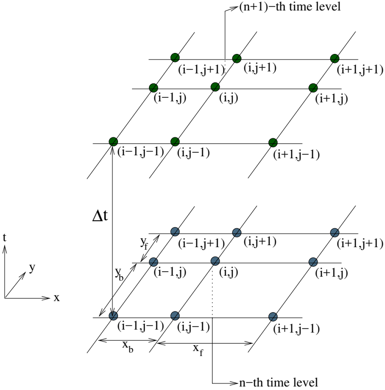

Figure: The stencil used for the unsteady ψ-v formulation.

The unsteady biharmonic equation

the transient biharmonic form of N-S equation may be written as

$$\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}+\frac{\partial^4\psi}{\partial y^4}-Re\left[\frac{\partial\psi}{\partial y}\left(\frac{\partial^3\psi}{\partial x^3}+\frac{\partial^3\psi}{\partial x\partial y^2}\right)-\frac{\partial\psi}{\partial x}\left(\frac{\partial^3\psi}{\partial x^2\partial y}+\frac{\partial^3\psi}{\partial y^3}\right)\right]=Re\left[\frac{\partial}{\partial t}(\nabla^2\psi)\right]. \tag{12}$$

Discretization of the unsteady equation [7]

Making use of (13), equation (12) can be discretized as

$$\left(p\delta_x^4\psi + 2\delta_x^2\delta_y^2 + q\delta_y^4\psi - Re\left[v\nabla^2 u - u\nabla^2 v\right]\right.$$
$$+ \frac{6p(x_f - x_b)}{x_f x_b}\delta_x^2 v - \frac{12p}{x_f x_b}\delta_x v$$
$$+ \frac{24p(x_f - x_b)}{x_f^2 x_b^2} v - \frac{6q(y_f - y_b)}{y_f y_b}\delta_y^2 u + \frac{12q}{y_f y_b}\delta_y u \tag{13}$$
$$\left. - \frac{24q(y_f - y_b)}{y_f^2 y_b^2} u\right)^n = Re\delta_t\left(\nabla^2\psi\right)$$

Discretization

Using forward differences for the time derivative and making use of the approximations for the operators δ_x, δ_y, δ_x^2, δ_y^2, δ_x^4, δ_y^4, and $\delta_x^2\delta_y^2$ as described for the steady-state scheme, (13) can be discretized compactly on the nonuniform nine-point stencil as

$$\frac{1}{h}\left[\frac{\psi_{i+1,j}^{n+1}}{x_f}+\frac{\psi_{i-1,j}^{n+1}}{x_b}-\left(\frac{1}{x_f}+\frac{1}{x_b}\right)\psi_{i,j}^{n+1}\right]$$
$$+\frac{1}{k}\left[\frac{\psi_{i,j+1}^{n+1}}{y_f}+\frac{\psi_{i,j-1}^{n+1}}{y_b}-\left(\frac{1}{y_f}+\frac{1}{y_b}\right)\psi_{i,j}^{n+1}\right]=\Big(A\psi_{i+1,j+1}^{n}$$
$$+B\psi_{i,j+1}^{n}+C\psi_{i-1,j+1}^{n}+D\psi_{i+1,j}^{n}+E\psi_{i,j}^{n}+F\psi_{i-1,j}^{n}$$
$$+G\psi_{i+1,j-1}^{n}+H\psi_{i,j-1}^{n}+I\psi_{i-1,j-1}^{n}\Big)-\phi_{i,j}^{n} \quad (14)$$

Discretization

where

$$A = \frac{2}{hkx_f y_f} \frac{\Delta t}{Re}$$

$$B = \left[-\frac{12}{ky_f^3} - \frac{2}{hk} \left(\frac{1}{x_f y_f} + \frac{1}{x_b y_f} \right) \right] \frac{\Delta t}{Re}$$

$$C = \frac{2}{hkx_b y_f} \frac{\Delta t}{Re}$$

$$D = \left[-\frac{12}{hx_f^3} - \frac{2}{hk} \left(\frac{1}{x_f y_f} + \frac{1}{x_f y_b} \right) \right] \frac{\Delta t}{Re}$$

Discretization

$$E = \left[\frac{12}{h} \left(\frac{1}{x_f^3} + \frac{1}{x_b^3} \right) + \frac{12}{k} \left(\frac{1}{y_f^3} + \frac{1}{y_b^3} \right) + \frac{2}{hk} \left(\frac{1}{x_f y_f} + \frac{1}{x_f y_b} + \frac{1}{x_b y_f} + \frac{1}{x_b y_b} \right) \right] \frac{\Delta t}{Re}$$

Discretization

$$F = \left[-\frac{12}{hx_b^3} - \frac{2}{hk}\left(\frac{1}{x_b y_f} + \frac{1}{x_b y_b}\right)\right]\frac{\Delta t}{Re}$$

$$G = \frac{2}{hkx_f y_b}\frac{\Delta t}{Re}$$

$$H = \left[-\frac{12}{ky_b^3} - \frac{2}{hk}\left(\frac{1}{x_f y_b} + \frac{1}{x_b y_b}\right)\right]\frac{\Delta t}{Re}$$

$$I = \frac{2}{hkx_b y_b}\frac{\Delta t}{Re}$$

Discretization

and

$$\phi_{i,j}^n = \frac{\Delta t}{Re}\left[\frac{6}{h}\left(\frac{v_{i+1,j}^n - v_{i-1,j}^n}{x_f x_b}\right) - \frac{Re.u_{i,j}^n}{k}\left\{\frac{v_{i,j+1}^n}{y_f} + \frac{v_{i,j-1}^n}{y_b} - \left(\frac{1}{y_f}+\frac{1}{y_b}\right)v_{i,j}^n\right\}\right.$$
$$-\frac{1}{h}\left(\frac{6(x_f - x_b)}{x_f x_b} + Re.u_{i,j}^n\right)\left\{\frac{v_{i+1,j}^n}{x_f} + \frac{v_{i-1,j}^n}{x_b} - \left(\frac{1}{x_f}+\frac{1}{x_b}\right)v_{i,j}^n\right\} - \frac{24(x_f - x_b)}{x_f^2 x_b^2}v_{i,j}^n$$
$$-\frac{6}{k}\left(\frac{u_{i,j+1}^n - u_{i,j-1}^n}{y_f y_b}\right) + \frac{1}{k}\left(\frac{6(y_f - y_b)}{y_f y_b} + Re.v_{i,j}^n\right)\left\{\frac{u_{i,j+1}^n}{y_f} + \frac{u_{i,j-1}^n}{y_b} - \left(\frac{1}{y_f}+\frac{1}{y_b}\right)u_{i,j}^n\right\}$$
$$\left. + \frac{Re.v_{i,j}^n}{h}\left\{\frac{u_{i+1,j}^n}{x_f} + \frac{u_{i-1,j}^n}{x_b} - \left(\frac{1}{x_f}+\frac{1}{x_b}\right)u_{i,j}^n\right\} + \frac{24(y_f - y_b)}{y_f^2 y_b^2}u_{i,j}^n\right]$$

Discretization

This formulation is $O(x_f^2, x_b^2, \Delta t)$. Here , the superscript n and $n+1$ represent the $n-$ and $(n+1)^{th}$ time levels, respectively. The second order time discretization is obtained by using Crank-Nicolson scheme.

$$\begin{aligned}\Big(a\psi_{i+1,j+1}^{n+1} + b\psi_{i,j+1}^{n+1} + c\psi_{i-1,j+1}^{n+1} + d\psi_{i+1,j}^{n+1} + e\psi_{i,j}^{n+1} + f\psi_{i-1,j}^{n+1} \\ + g\psi_{i+1,j-1}^{n+1} + h\psi_{i,j-1}^{n+1} + i\psi_{i-1,j-1}^{n+1}\Big) = \Big(A\psi_{i+1,j+1}^{n} + B\psi_{i,j+1}^{n} \\ + C\psi_{i-1,j+1}^{n} + D\psi_{i+1,j}^{n} + E\psi_{i,j}^{n} + F\psi_{i-1,j}^{n} + G\psi_{i+1,j-1}^{n} \\ + H\psi_{i,j-1}^{n} + I\psi_{i-1,j-1}^{n}\Big) - \phi_{i,j}^{n}\end{aligned} \tag{15}$$

Discretization

where

$$a = -\frac{\Delta t}{2Re}\frac{2}{hkx_f y_f}$$

$$b = \frac{1}{ky_f} + \frac{\Delta t}{2Re}\left[\frac{12}{ky_f^3} + \frac{2}{hk}\left(\frac{1}{x_f y_f} + \frac{1}{x_b y_f}\right)\right]$$

$$c = -\frac{\Delta t}{2Re}\frac{2}{hkx_b y_f}$$

$$d = \frac{1}{hx_f} + \frac{\Delta t}{2Re}\left[\frac{12}{hx_f^3} + \frac{2}{hk}\left(\frac{1}{x_f y_f} + \frac{1}{x_f y_b}\right)\right]$$

Discretization

$$e = -\frac{1}{h}\left(\frac{1}{x_f}+\frac{1}{x_b}\right) - \frac{1}{k}\left(\frac{1}{y_f}+\frac{1}{y_b}\right) - \frac{\Delta t}{2Re}\left[\frac{12}{h}\left(\frac{1}{x_f^3}+\frac{1}{x_b^3}\right) + \frac{12}{k}\left(\frac{1}{y_f^3}+\frac{1}{y_b^3}\right) + \frac{2}{hk}\left(\frac{1}{x_f y_f}+\frac{1}{x_f y_b}+\frac{1}{x_b y_f}+\frac{1}{x_b y_b}\right)\right]$$

Discretization

$$f = \frac{1}{hx_b} + \frac{\Delta t}{2Re}\left[\frac{12}{hx_b^3} + \frac{2}{hk}\left(\frac{1}{x_b y_f} + \frac{1}{x_b y_b}\right)\right]$$

$$g = -\frac{\Delta t}{2Re}\frac{2}{hkx_f y_b}$$

$$h = \frac{1}{ky_b} + \frac{\Delta t}{2Re}\left[\frac{12}{ky_b^3} + \frac{2}{hk}\left(\frac{1}{x_f y_b} + \frac{1}{x_b y_b}\right)\right]$$

$$i = -\frac{\Delta t}{2Re}\frac{2}{hkx_b y_b}$$

Discretization

$$A = \frac{\Delta t}{2Re} \frac{2}{hkx_f y_f}$$

$$B = \frac{1}{ky_f} - \frac{\Delta t}{2Re} \left[\frac{12}{ky_f^3} + \frac{2}{hk} \left(\frac{1}{x_f y_f} + \frac{1}{x_b y_f} \right) \right]$$

$$C = \frac{\Delta t}{2Re} \frac{2}{hkx_b y_f}$$

$$D = \frac{1}{hx_f} - \frac{\Delta t}{2Re} \left[\frac{12}{hx_f^3} + \frac{2}{hk} \left(\frac{1}{x_f y_f} + \frac{1}{x_f y_b} \right) \right]$$

Discretization

$$E = -\frac{1}{h}\left(\frac{1}{x_f}+\frac{1}{x_b}\right)-\frac{1}{k}\left(\frac{1}{y_f}+\frac{1}{y_b}\right)+\frac{\Delta t}{2Re}\left[\frac{12}{h}\left(\frac{1}{x_f^3}+\frac{1}{x_b^3}\right)\right.$$
$$\left.+\frac{12}{k}\left(\frac{1}{y_f^3}+\frac{1}{y_b^3}\right)+\frac{2}{hk}\left(\frac{1}{x_fy_f}+\frac{1}{x_fy_b}+\frac{1}{x_by_f}+\frac{1}{x_by_b}\right)\right]$$

Discretization

$$F = \frac{1}{hx_b} - \frac{\Delta t}{2Re}\left[\frac{12}{hx_b^3} + \frac{2}{hk}\left(\frac{1}{x_b y_f} + \frac{1}{x_b y_b}\right)\right]$$

$$G = \frac{2}{hkx_f y_b}\frac{\Delta t}{2Re}$$

$$H = \frac{1}{ky_b} - \frac{\Delta t}{2Re}\left[\frac{12}{ky_b^3} + \frac{2}{hk}\left(\frac{1}{x_f y_b} + \frac{1}{x_b y_b}\right)\right]$$

$$I = \frac{2}{hkx_b y_b}\frac{\Delta t}{2Re}$$

Discretization

$$
\begin{aligned}
\phi_{i,j} = \frac{\Delta t}{2Re} \Bigg[& \frac{6}{h}\left(\frac{v^n_{i+1,j} - v^n_{i-1,j}}{x_f x_b}\right) - \frac{Re.u^n_{i,j}}{k}\left\{\frac{v^n_{i,j+1}}{y_f} + \frac{v^n_{i,j-1}}{y_b} - \left(\frac{1}{y_f} + \frac{1}{y_b}\right) v^n_{i,j}\right\} \\
& - \frac{1}{h}\left(\frac{6(x_f - x_b)}{x_f x_b} + Re.u^n_{i,j}\right)\left\{\frac{v^n_{i+1,j}}{x_f} + \frac{v^n_{i-1,j}}{x_b} - \left(\frac{1}{x_f} + \frac{1}{x_b}\right) v^n_{i,j}\right\} - \frac{24(x_f - x_b)}{x_f^2 x_b^2} v^n_{i,j} \\
& - \frac{6}{k}\left(\frac{u^n_{i,j+1} - u^n_{i,j-1}}{y_f y_b}\right) + \frac{1}{k}\left(\frac{6(y_f - y_b)}{y_f y_b} + Re.v^n_{i,j}\right)\left\{\frac{u^n_{i,j+1}}{y_f} + \frac{u^n_{i,j-1}}{y_b} - \left(\frac{1}{y_f} + \frac{1}{y_b}\right) u^n_{i,j}\right\} \\
& + \frac{24(y_f - y_b)}{y_f^2 y_b^2} u^n_{i,j} \Bigg] + \frac{\Delta t}{2Re}\Bigg[\frac{6}{h}\left(\frac{v^{n+1}_{i+1,j} - v^{n+1}_{i-1,j}}{x_f x_b}\right) - \frac{6}{k}\left(\frac{u^{n+1}_{i,j+1} - u^{n+1}_{i,j-1}}{y_f y_b}\right) \\
& - \frac{1}{h}\left(\frac{6(x_f - x_b)}{x_f x_b} + Re.u^{n+1}_{i,j}\right)\left\{\frac{v^{n+1}_{i+1,j}}{x_f} + \frac{v^{n+1}_{i-1,j}}{x_b} - \left(\frac{1}{x_f} + \frac{1}{x_b}\right) v^{n+1}_{i,j}\right\} \\
& - \frac{Re.u^{n+1}_{i,j}}{k}\left\{\frac{v^{n+1}_{i,j+1}}{y_f} + \frac{v^{n+1}_{i,j-1}}{y_b} - \left(\frac{1}{y_f} + \frac{1}{y_b}\right) v^{n+1}_{i,j}\right\} - \frac{24(x_f - x_b)}{x_f^2 x_b^2} v^{n+1}_{i,j} \\
& + \frac{1}{k}\left(\frac{6(y_f - y_b)}{y_f y_b} + Re.v^{n+1}_{i,j}\right)\left\{\frac{u^{n+1}_{i,j+1}}{y_f} + \frac{u^{n+1}_{i,j-1}}{y_b} - \left(\frac{1}{y_f} + \frac{1}{y_b}\right) u^{n+1}_{i,j}\right\} \\
& + \frac{Re.v^{n+1}_{i,j}}{h}\left\{\frac{u^{n+1}_{i+1,j}}{x_f} + \frac{u^{n+1}_{i-1,j}}{x_b} - \left(\frac{1}{x_f} + \frac{1}{x_b}\right) u^{n+1}_{i,j}\right\} + \frac{24(y_f - y_b)}{y_f^2 y_b^2} u^{n+1}_{i,j} \Bigg]
\end{aligned}
$$

Laminar flow past a flat plate in uniform flow

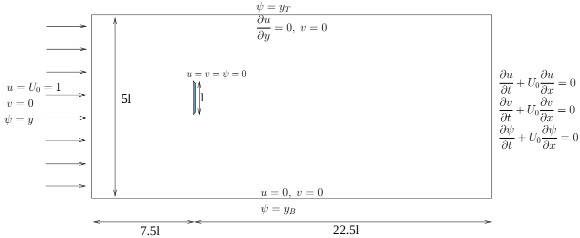

Figure: Configuration of the flow past a flat plate in uniform flow problem ([10, 11, 12]).

The Boundary Conditions

Inlet BC

At the inlet $u = U_0, \ v = 0$.

The Boundary Conditions

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At the inlet $u = U_0,\ v = 0$.

Outlet BC

At the outlet, $\dfrac{\partial \phi}{\partial t} + U_0 \dfrac{\partial \phi}{\partial x} = 0$ with ϕ standing for u, v or ψ.

The Boundary Conditions

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At the inlet $u = U_0, \; v = 0$.

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At the outlet, $\dfrac{\partial \phi}{\partial t} + U_0 \dfrac{\partial \phi}{\partial x} = 0$ with ϕ standing for u, v or ψ.

Surface and other BC

On the surface of the plate, $u = v = 0$. At the other boundaries of the computational domain $\dfrac{\partial u}{\partial y} = 0$ and $v = 0$.

The Grid used for the flat plate flow

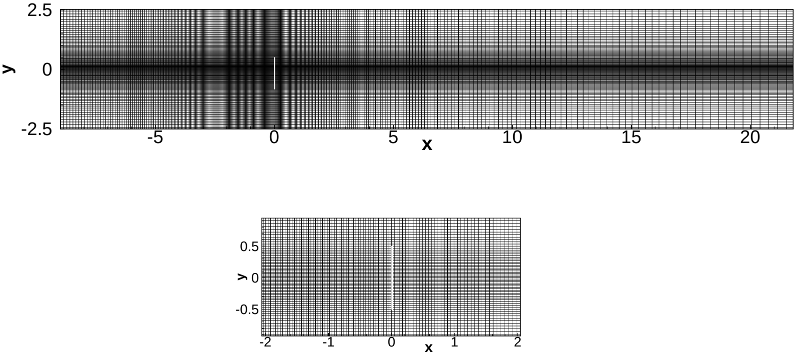


Figure: A typical 301×101 grid (top) and Close up view of the grid near the surface of the plate (bottom).

Steady state streamlines

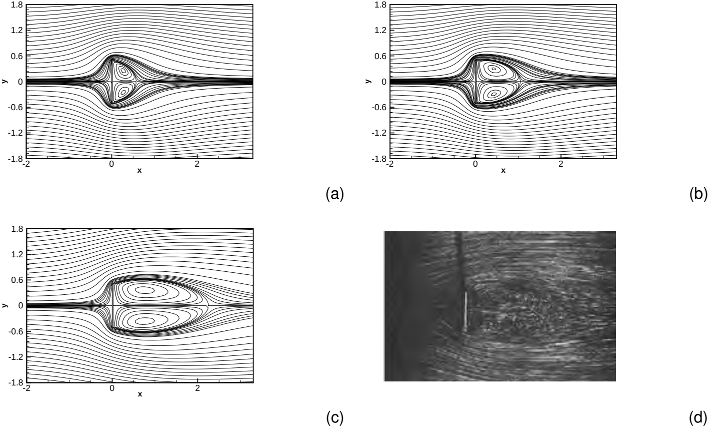


Figure: Steady-state streamlines for the flow past flat plate in a uniform flow for (a) $Re = 5$, (b) $Re = 10$, (c) $Re = 20$ and (d) $Re = 20$ (exp., [12]).

Flow evolution for $Re = 40$.

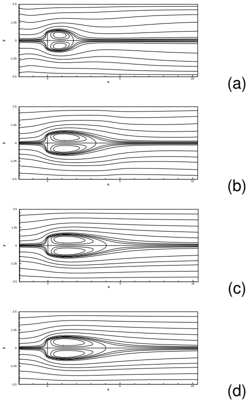


Figure: Streamlines for $Re = 40$ at (a) $t = 5$, (b) $t = 10$, (c) $t = 50$ and (d) $t = 100$.

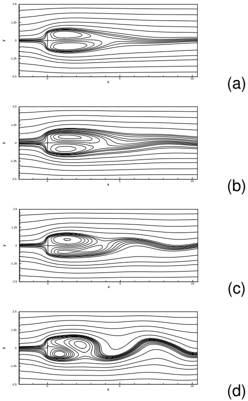


Figure: Streamlines for $Re = 40$ at (a) $t = 200$, (b) $t = 210$, (c) $t = 220$ and (d) $t = 240$.

Periodic vortex shedding for $Re = 40$ and 100

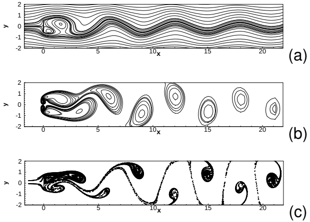


Figure: For $Re = 40$: (a) the streamlines, (b) the vorticity contours and (c) the streaklines behind the flat plate depicting vortex shedding.

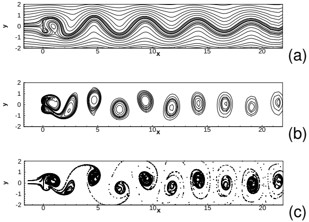


Figure: For $Re = 100$: (a) the streamlines, (b) the vorticity contours and (c) the streaklines behind the flat plate depicting vortex shedding.

Click to see vortex shedding for $Re = 100$

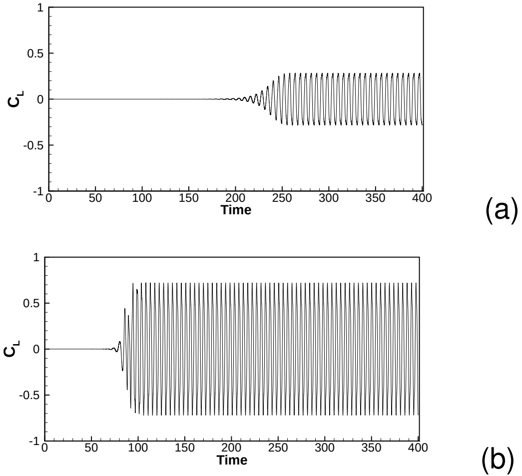


Figure: *History of lift coefficients for (a) $Re = 40$ and (b) $Re = 100$.*

Periodic flow study

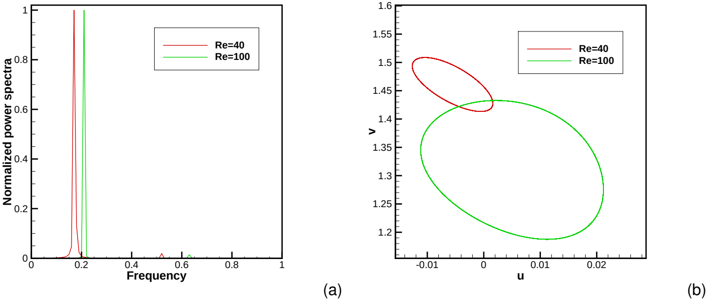


Figure: (a) Spectral density of the lift coefficient and (b) Phase plane of u- and v-velocities at the point $(1.0,\ 0.0)$ behind the flat plate for $Re = 40$ and $Re = 200$ in the time interval $[300,\ 400]$.

A look at the critical Reynolds number

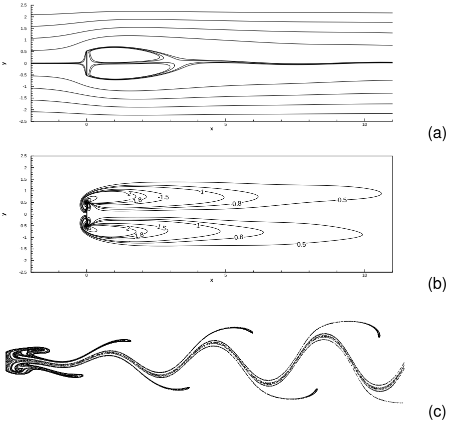


Figure: For $Re = 32$: (a) the streamlines and (b) the vorticity contours at the onset of symmetry breaking at time $t = 650$, and (c) the streaklines behind the flat plate depicting vortex shedding.

Periodic flow study

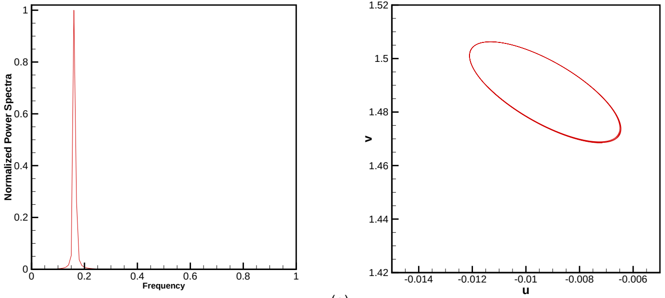


Figure: (a) Spectral density of the lift coefficient and (b) Phase plane of u- and v-velocities at the point $(1.0,\ 0.0)$ behind the flat plate for $Re = 32$ in the time interval $[1100,\ 1200]$.

The schematic diagram: Laminar flow past a flat plate in uniformly accelerated flow ([8, 10, 11])

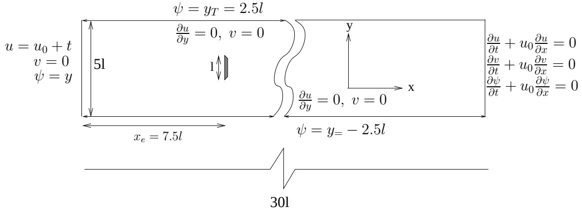


Figure: Configuration of the flow past a flat plate in uniformly accelerated flow problem.

A note on Non-dimensionalization: $u = u_0 + at$, characteristic time-scale $T = \left(\frac{L}{a}\right)^{\frac{1}{2}}$, $Re = \frac{L^2}{\nu T} = \frac{a^{\frac{1}{2}} L^{\frac{3}{2}}}{\nu}$.

$t^* = \frac{t}{T}$, $x^* = \frac{x}{L}$ and $u^* = \frac{uT}{L}$ will lead to $u^* = u_0^* + t^*$.

Flow evolution for accelerated flat plate for $Re = 400$.

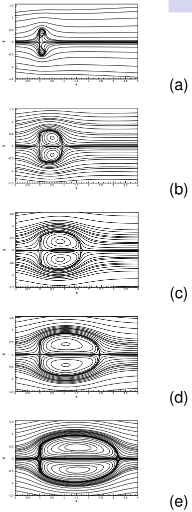


Figure: Streamlines for $Re = 400$ at (a) $t = 1$, (b) $t = 2$, (c) $t = 3$ (d) $t = 4$ and (e) $t = 5$.

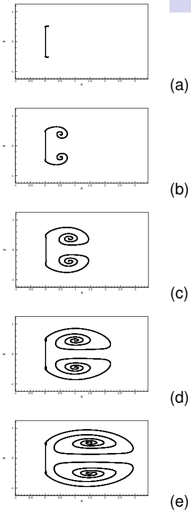


Figure: Streaklines for $Re = 400$ at (a) $t = 1$, (b) $t = 2$, (c) $t = 3$ (d) $t = 4$ and (e) $t = 5$.

Streakline evolution for accelerated flat plate for $Re = 400$ and 500.

Click to see vortex shedding for $Re = 400$
Click to see vortex shedding for $Re = 500$

Flow past a wedge-like sharp edge in accelerated flow

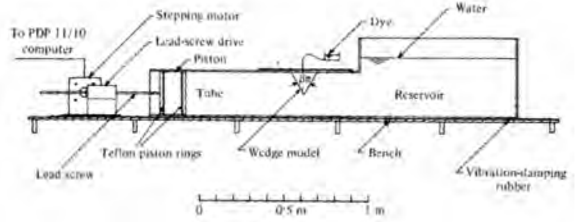


Figure: Experimental set-up for starting vortex flow visualization by Pullin and Perry ([13]).

Flow past a wedge-like sharp edge in accelerated flow

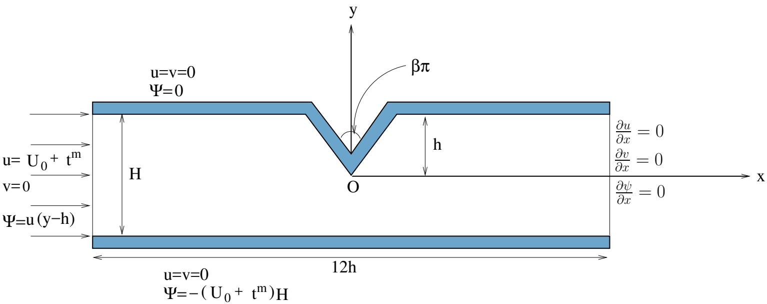
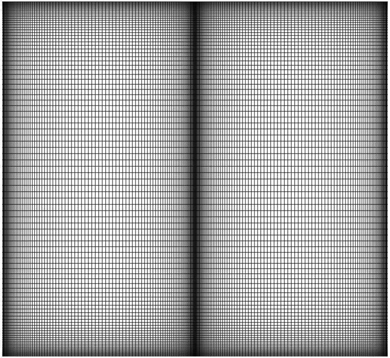
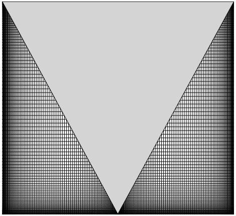


Figure: Configuration of the flow past a wedge in accelerated flow problem ([8, 9, 13, 14]).

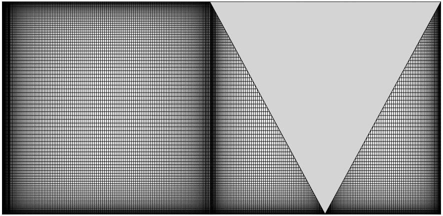
Wedge Grid



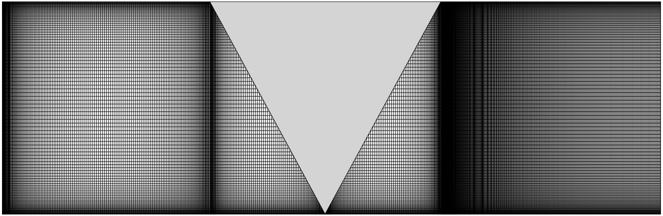
Wedge Grid (801×201)



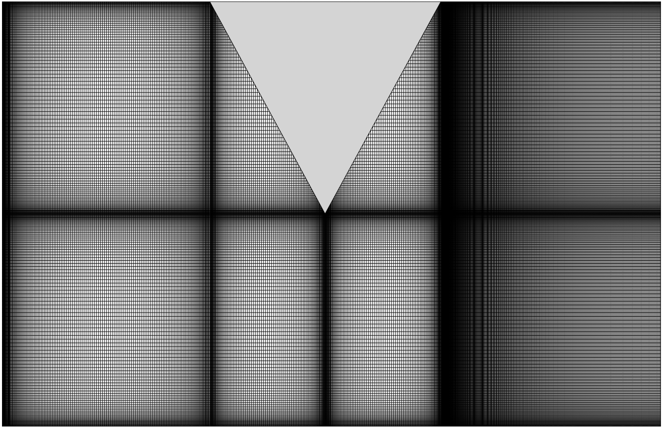
Wedge Grid (801×201)



Wedge Grid (801×201)



Wedge Grid (801×201)



Effect of the parameter m

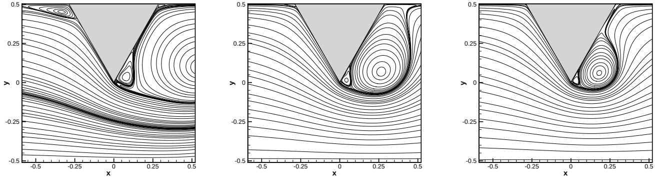
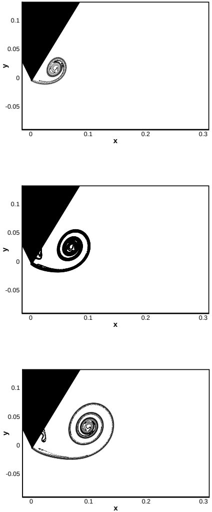


Figure: Streamlines for accelerated flow past a 60^o wedge at time $t = 0.8$ for $Re = 1560$ (left, m=0), $Re = 6621$ (middle, m=0.45) and $Re = 6873$ (right, m=0.88).

Starting vortex evolution for flow past a wedge for $Re = 6873$.



(a)

(b)

(c)

Figure: Streaklines for $Re = 6873$ at (a) $t = 2.8$, (b) $t = 4$ and (d) $t = 5$ (Experimental [13]).

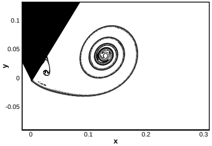
(a)

(b)

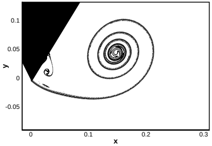
(c)

Figure: Streaklines for $Re = 6873$ at (a) $t = 2.8$, (b) $t = 4$ and (d) $t = 5$ (Present).

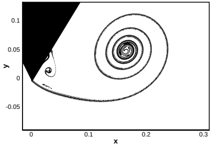
Starting vortex evolution for flow past a wedge for $Re = 6873$.



(a)

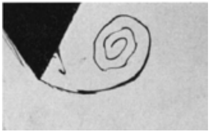


(b)

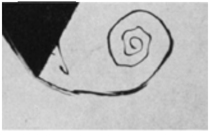


(c)

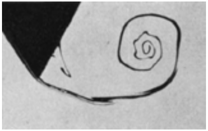
Figure: Streaklines for $Re = 6873$ at (a) $t = 6$, (b) $t = 6.6$ and (d) $t = 7$ (Experimental [13]).



(a)



(b)



(c)

Figure: Streaklines for $Re = 6873$ at (a) $t = 6$, (b) $t = 6.6$ and (d) $t = 7$ (Present).

The structure of the starting vortex

- The initial stage: The vortex sheet shedded from the edge rolls up into a spiral shape.

The structure of the starting vortex

- The initial stage: The vortex sheet shedded from the edge rolls up into a spiral shape.
- The second stage: Generation of small vortices.

The structure of the starting vortex

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- The second stage: Generation of small vortices.
- The final stage: The three fold structure.

The structure of the starting vortex

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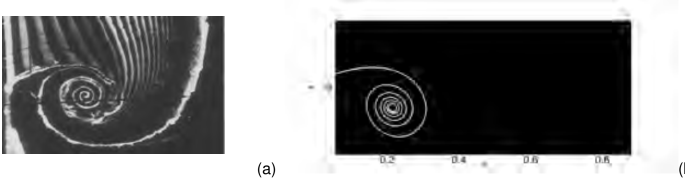

(a) (b)

Figure: (a) The initial stage: Comparison of experimental result of Lian and Huan [14] at time $t = 1.44$ and our numerical simulation for $Re = 6873$.

The second stage: Generation of small vortices

- The outermost part of the starting vortex becomes wavy due to the instability of the shear layer.

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- The outermost part of the starting vortex becomes wavy due to the instability of the shear layer.
- Afterwards a part of the outermost vortex sheet breaks and rolls up into small vortices.
- The small vortices are spaced almost uniformly, and their centers are located along the spiral curve of the large starting vortex, thus they are a part of this large starting vortex. Each small vortex has rolled up into a spiral shape, with an apparently very thin shear layer.

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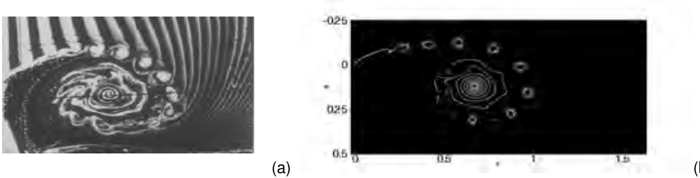


Figure: (a) The second stage: Comparison of experimental result of Lian and Huan [14] at time $t = 2.09$ and our numerical simulation for $Re = 6873$.

Click here to visualize shear layer instability

The third stage: The thee-fold structure

- The outermost vortex layer.
- The core.
- The turbulent annular region.


(a)

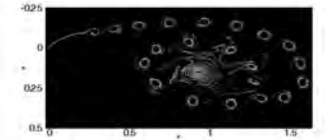
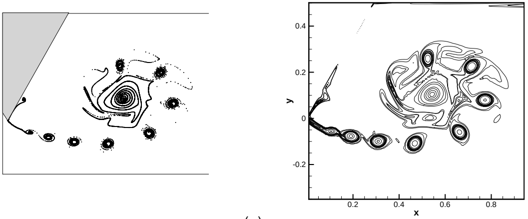

(b)

Figure: (a) Comparison of experimental result of Lian and Huan [14] at time $t = 2.885$ and our numerical simulation for $Re = 6873$.

Existence of coherent structure



(a) (b)

Figure: (a) Streaklines and (b) vorticity contours for $Re = 6873$ at $t = 14.08$.

Existence of coherent structure

The Q criterion [15]: The eigenvalue analysis of the perturbed velocity field gives $\lambda = \pm\sqrt{Q}$, where

$$Q = \left(\frac{\partial^2 \psi}{\partial x \partial y}\right)^2 - \frac{\partial^2 \psi}{\partial x^2}\frac{\partial^2 \psi}{\partial y^2}.$$

It follows that in the regions of the fluid where $Q < 0$, the distance between two particles embedded in the original velocity field will not diverge as a function of time.

Existence of coherent structure

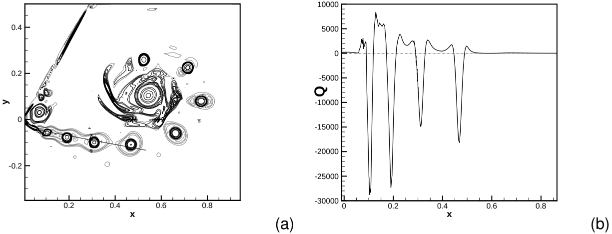


Figure: (a) Contour maps of Q at $t = 14.08$, negative values of Q (solid curves) corresponds to stable eigenvalues while positive values (dotted curves) corresponds to unstable ones; (b) Cross section of Q along the line shown in (a). Note that large instabilities occur only at the edge of the vortices.

Summarizing the whole phenomenon

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Thank You