Wave-seafloor interactions

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September 6, 2017

References

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- Liu, Park & Cowen, JFM, vol. 574, pp. 449
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Modeling water waves

- Liner approximation
  - Airy wave theory

- Nonlinear models
  - Stokes waves
  - Depth-integrated equations

    Shallow water equations

    Boussinesq equations

Water wave theories

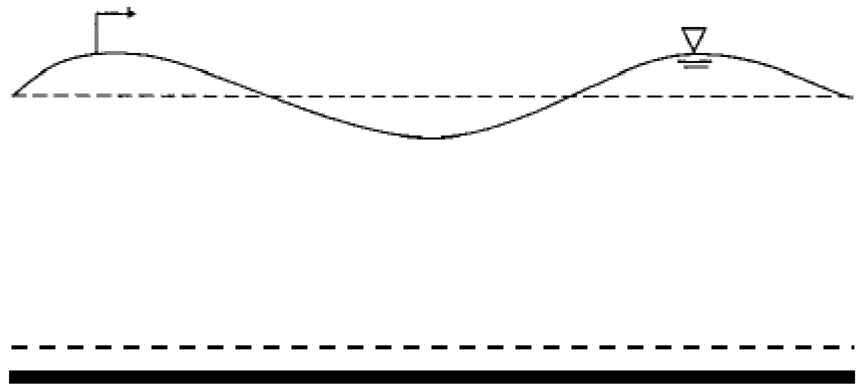
- Incompressible fluid
  - Constant density
- Zero viscosity
- Irrotational flow

- Rigid, impermeable bed

Effects of a seabed

- Rigid, impermeable bed
  - Bottom boundary layer
- Rigid, porous bottom
  - Sandy seabed
- Muddy seafloor
  - Fluidized soil bottom

Viscous damping



$$\eta = a\cos(kx - \omega t)$$

$$\Rightarrow a = a_0 e^{-\alpha t}, \quad \alpha = \frac{\nu k \sqrt{\omega / 2\nu}}{\sinh 2kh}$$

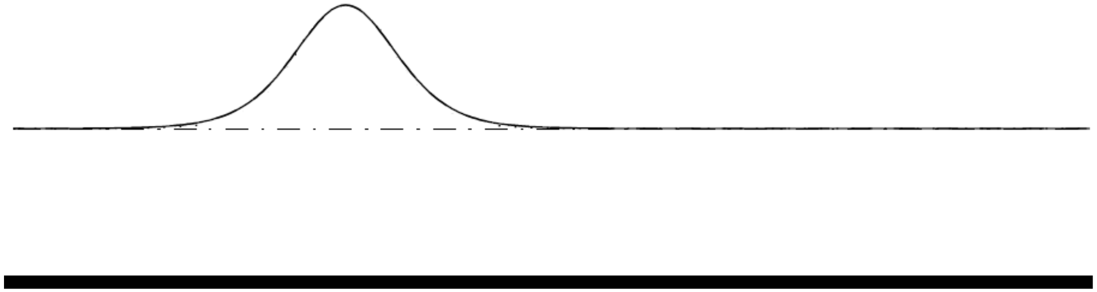
$$\boxed{a = a_0 e^{-\alpha t}, \quad \alpha = \frac{\nu k \sqrt{\omega/2\nu}}{\sinh 2kh}}$$

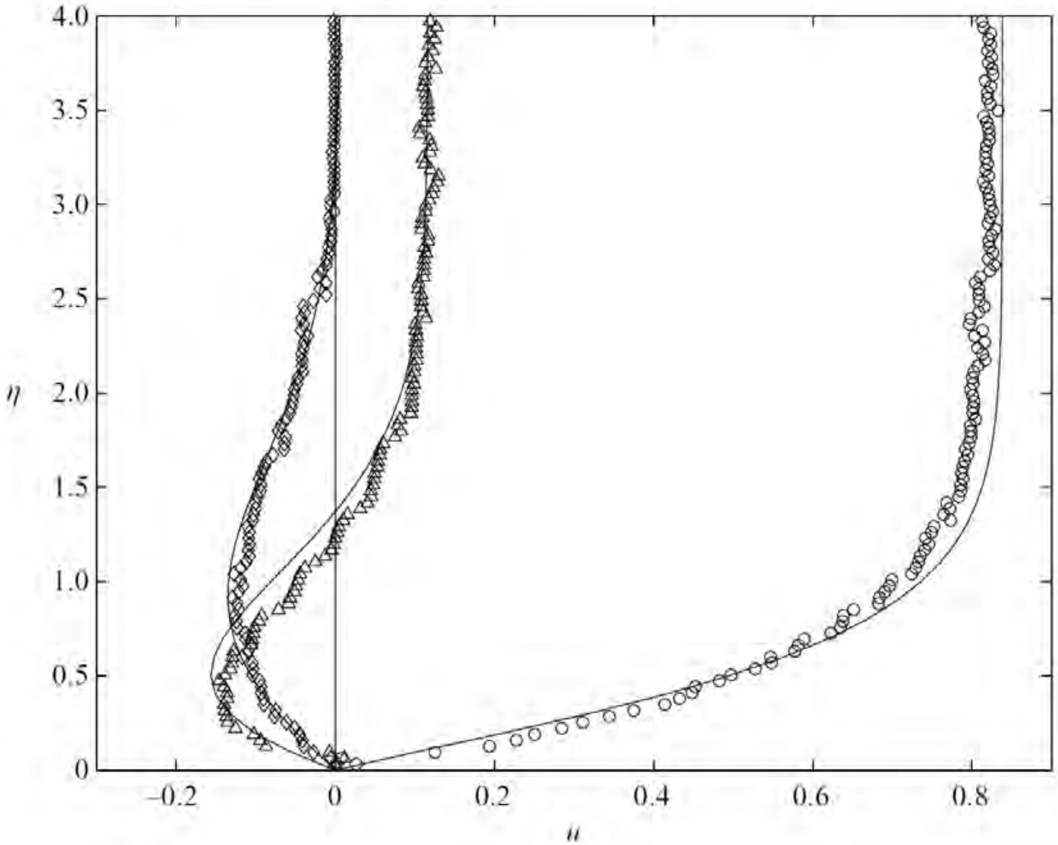
$$\nu = 10^{-5}\ \mathrm{m^2/s},\ \ h = 5\ \mathrm{m},\ \ T = 5\ \mathrm{s}$$

$$\Rightarrow \delta = \sqrt{\nu/2\omega} = 1.3\ \mathrm{mm}$$

$$\Rightarrow \alpha = 4.8 \times 10^{-5}\ \mathrm{s^{-1}}$$

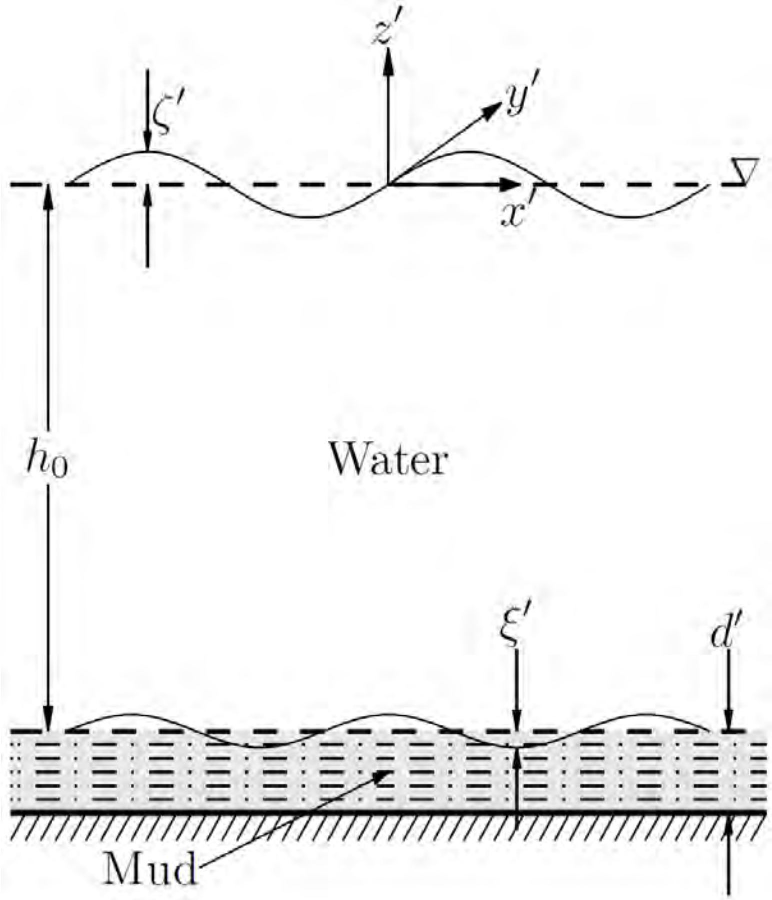
$$\Rightarrow \begin{cases} 10\%:\ 2195\ \mathrm{s} = 439T \\ 5\%:\ 1069\ \mathrm{s} = 214T \\ 1\%:\ 209\ \mathrm{s} = 42T \\ 0.005\%:\ 5\ \mathrm{s} = 1T \end{cases}$$

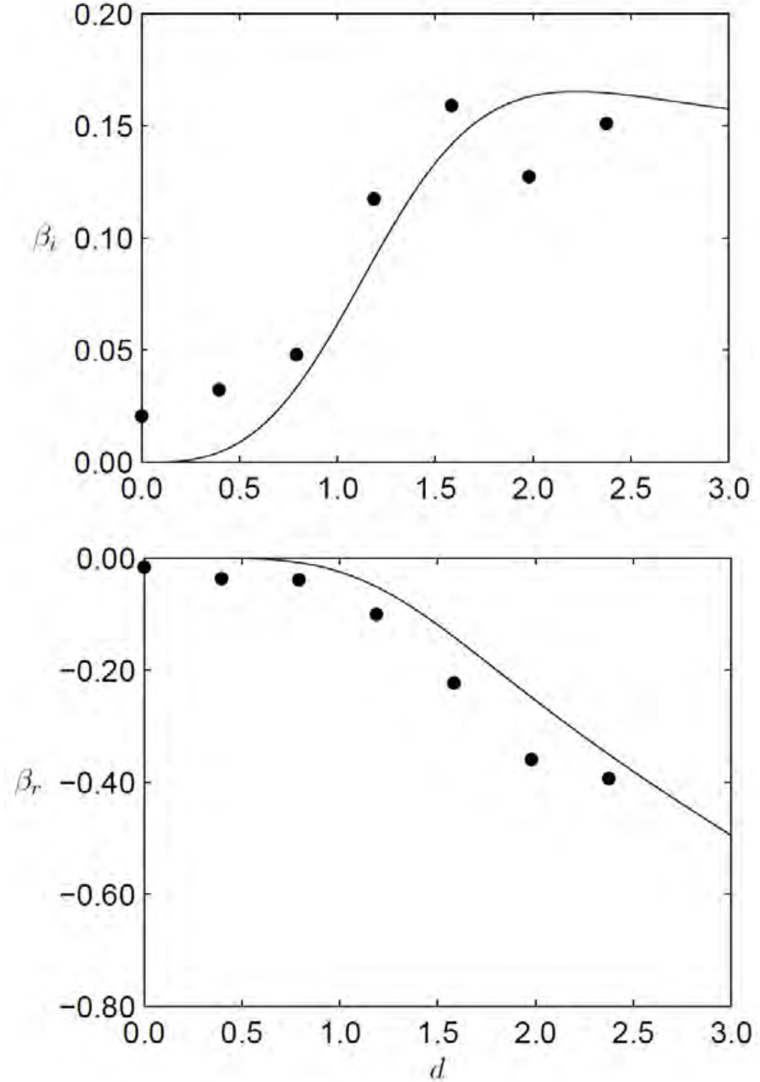


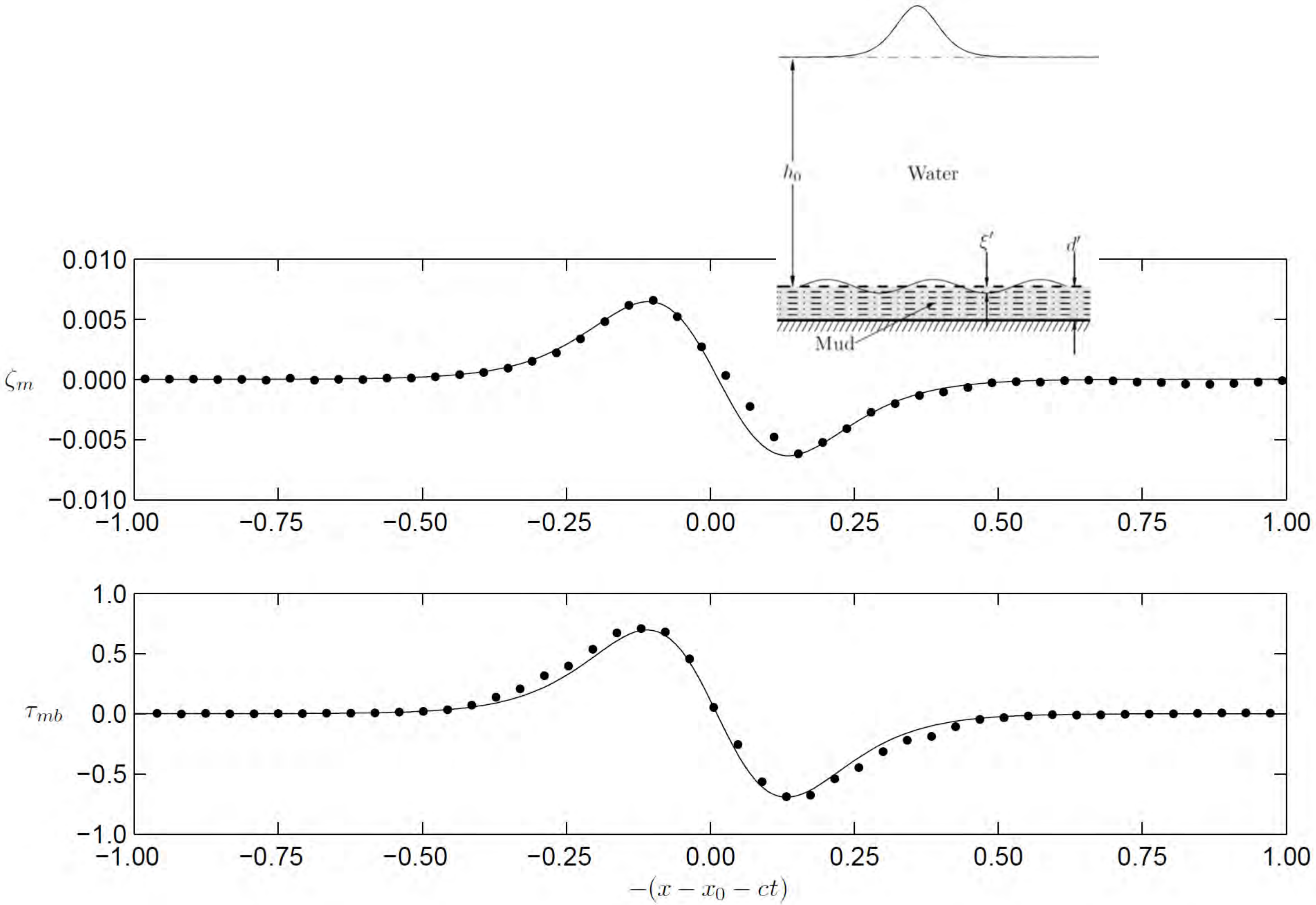

4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
η
−0.2
0
0.2
0.4
0.6
0.8
u

Viscous muddy bed

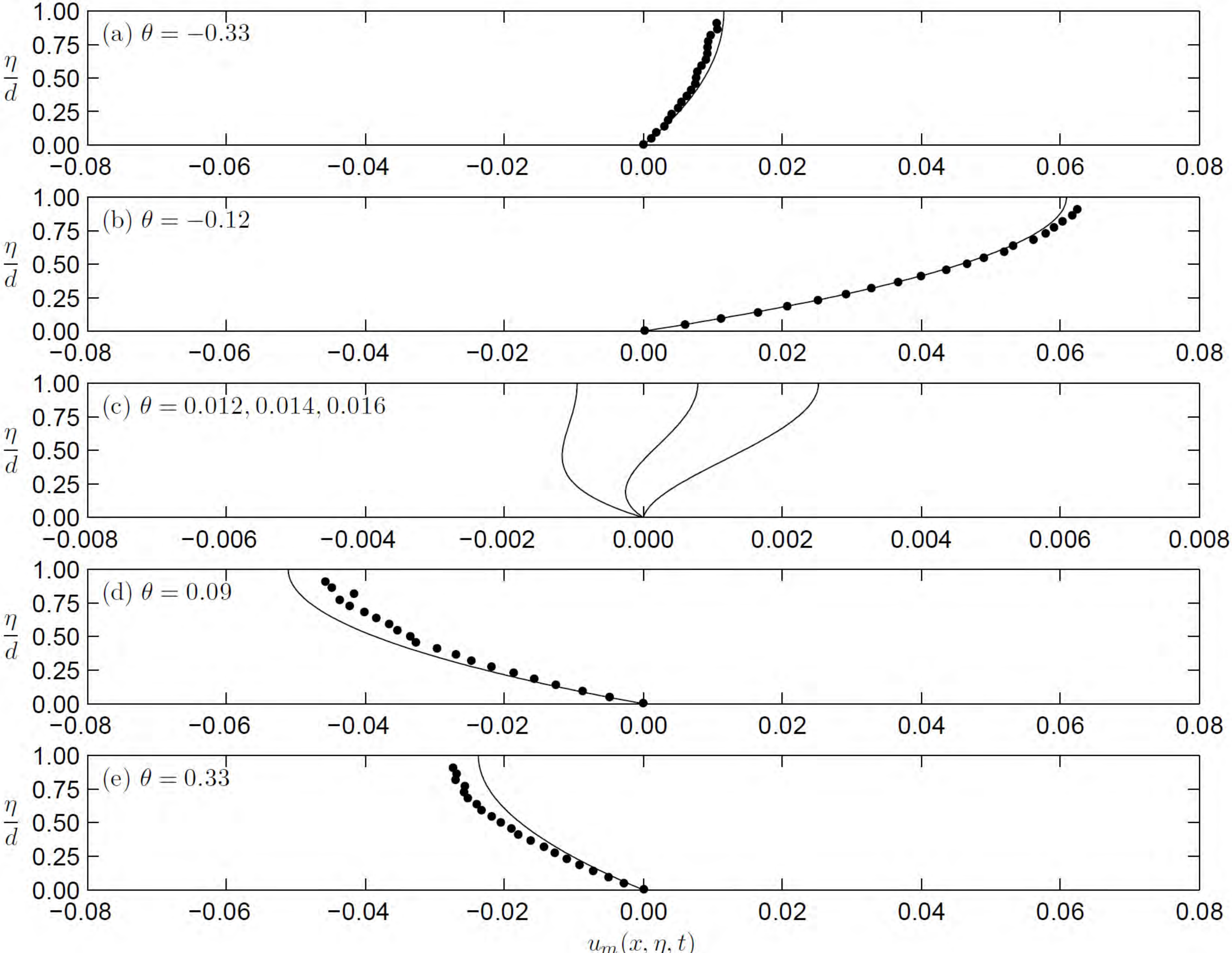
$$\zeta = a_0 e^{-\beta_i \xi} e^{i(\sigma + \beta_r \xi)}$$



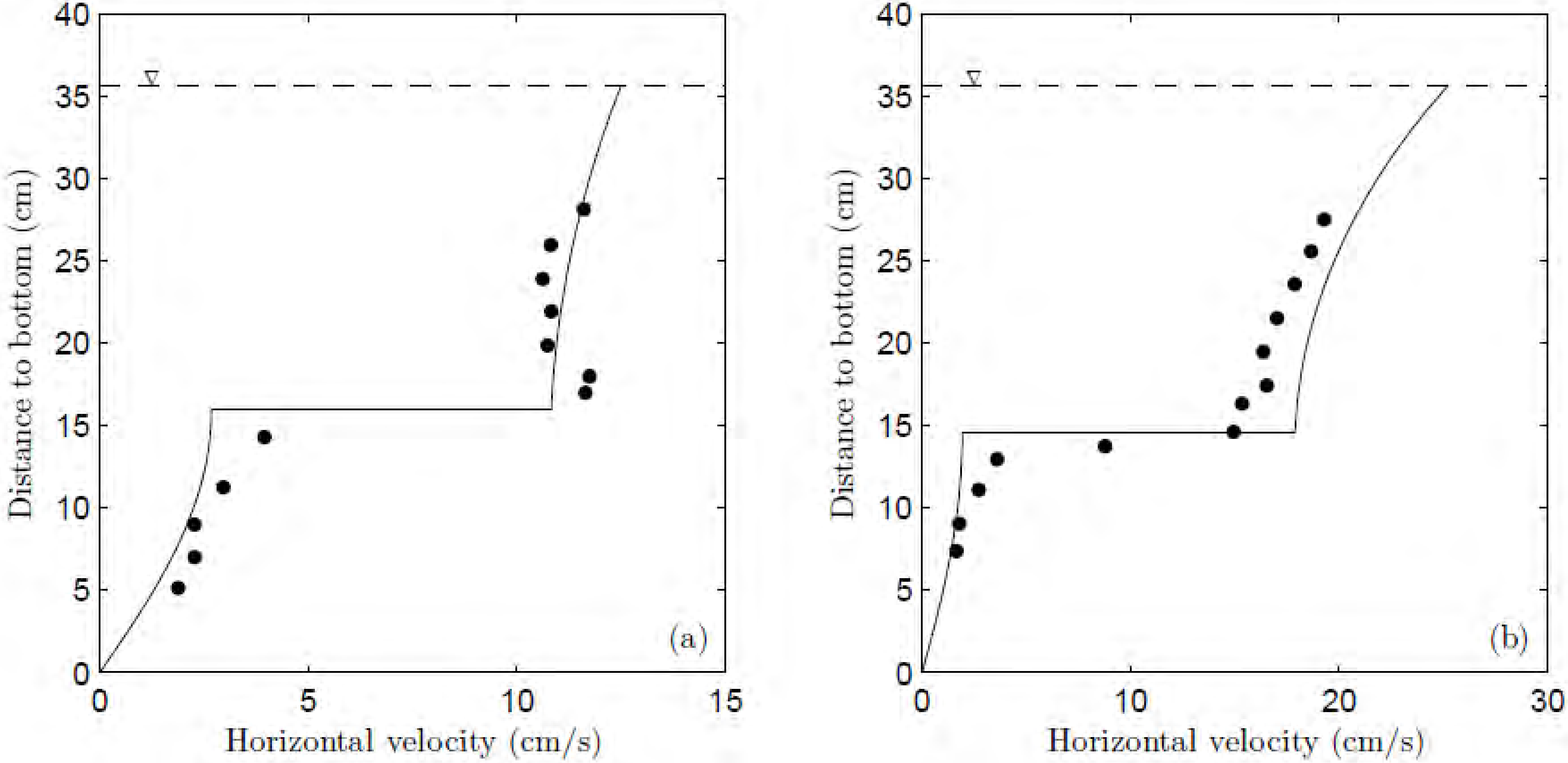


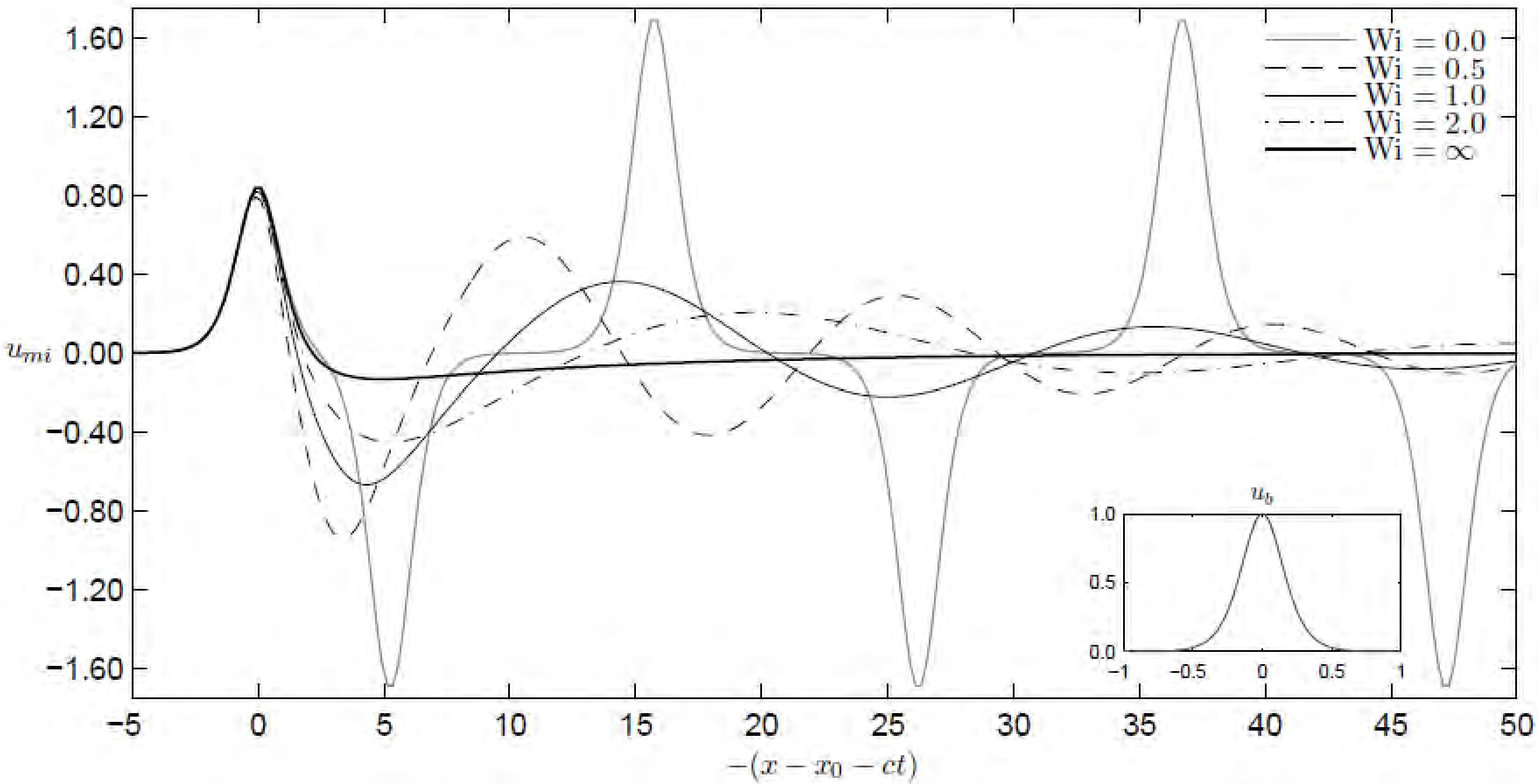


Water
h_0
ξ'
d'
Mud
ζ_m
0.010
0.005
0.000
−0.005
−0.010
τ_{mb}
1.0
0.5
0.0
−0.5
−1.0
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
$-(x - x_0 - ct)$


(a) $\theta = -0.33$
(b) $\theta = -0.12$
(c) $\theta = 0.012, 0.014, 0.016$
(d) $\theta = 0.09$
(e) $\theta = 0.33$
$\frac{\eta}{d}$
$u_m(x, \eta, t)$

Liner waves over a viscoelastic bed



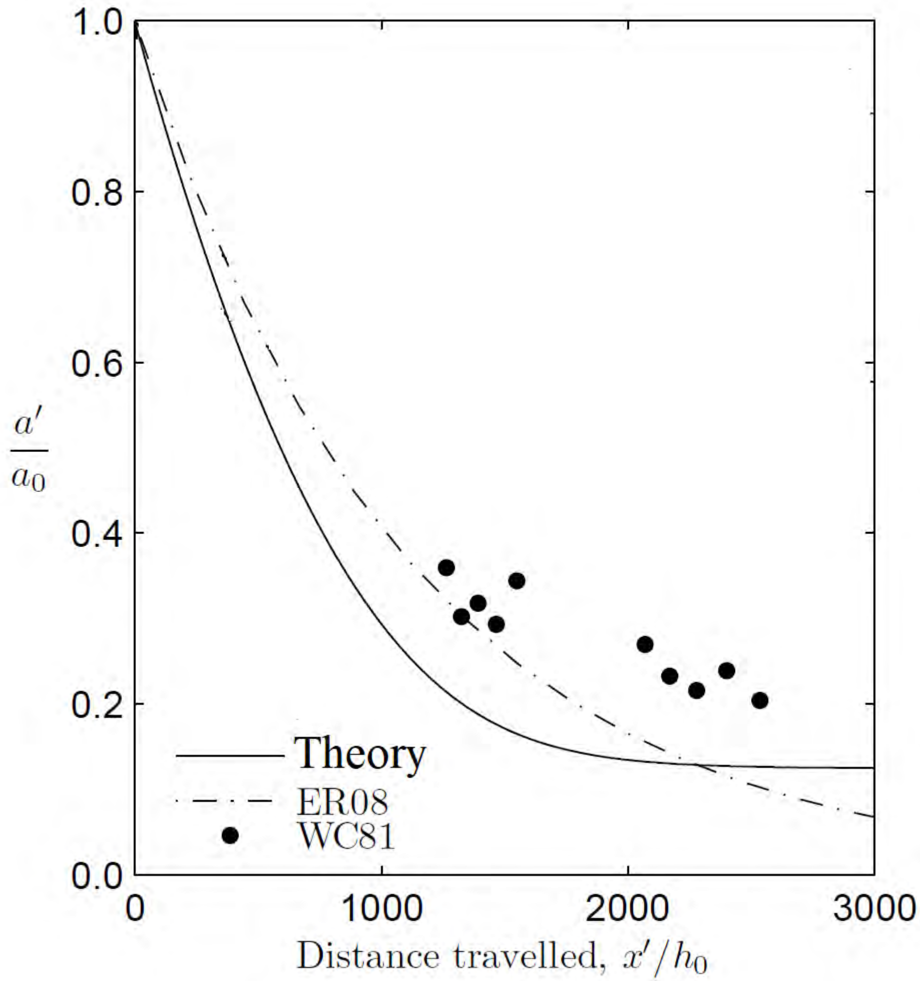


Weissenberg number: $\mathrm{Wi} = \dfrac{\mu_m / E_m}{L_0 \sqrt{g h_0}}$

$$\frac{1}{\epsilon}\frac{\partial H}{\partial t}+\nabla\cdot\left[\left(H+\gamma\frac{\alpha}{\mu}\overline{d}\right)\overline{\boldsymbol{u}}\right]-\gamma\frac{\alpha}{\mu}\int_0^t\frac{\partial\nabla\cdot\overline{\boldsymbol{u}}}{\partial\tau}I(t-\tau)\,\mathrm{d}\tau=O(\mu^4)$$

$$\frac{\partial\overline{\boldsymbol{u}}}{\partial t}+\epsilon\overline{\boldsymbol{u}}\cdot\nabla\overline{\boldsymbol{u}}+\frac{1}{\epsilon}\nabla H-\frac{\mu^2}{3}\nabla\nabla\cdot\left(\frac{\partial\overline{\boldsymbol{u}}}{\partial t}\right)=O(\mu^4)$$

A Bingham-plastic seabed



Rationale

- “Better” physics
- Derivation of bathymetry by aerial imagery
- Shoreline protection by a synthetic seabed
- Wave energy extraction: seafloor carpet

Theoretical analysis

- Applicable model equations
  - Approximations
  - Perturbation expansions

- Physical quantities
  - Amplitude attenuation
  - Wavelength variation

Simplifications

- Constant water depth
- Non-breaking waves
- Idealized model
- Focus on the fate of surface waves

Bottom boundary layer

- Inviscid
  - No-flux condition

$$u(z=-h)=u_p \neq 0$$

- Viscous fluid
  - No-slip condition

$$u(z=-h)=u_p+u_r=0$$

Viscous effect: Linear waves

- Dimensionless equation

$$\frac{\partial u'}{\partial t'} = -\frac{gk}{\omega^2}\frac{\partial p'}{\partial x'} + \frac{\nu k}{\omega}\frac{\partial^2 u'}{\partial x'^2} + \frac{\nu}{\omega\delta^2}\frac{\partial^2 u'}{\partial z'^2}$$

$$\Rightarrow \begin{cases} \dfrac{\partial u_p}{\partial t} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} \\ \dfrac{\partial u_r}{\partial t} = \nu \dfrac{\partial^2 u_r}{\partial z^2} \end{cases}, \quad \delta \propto \sqrt{\frac{\nu}{\omega}}$$

$$u_p = \frac{gka}{\omega}\frac{\cosh k(z+h)}{\cosh kh}\cos(kx-\omega t)$$

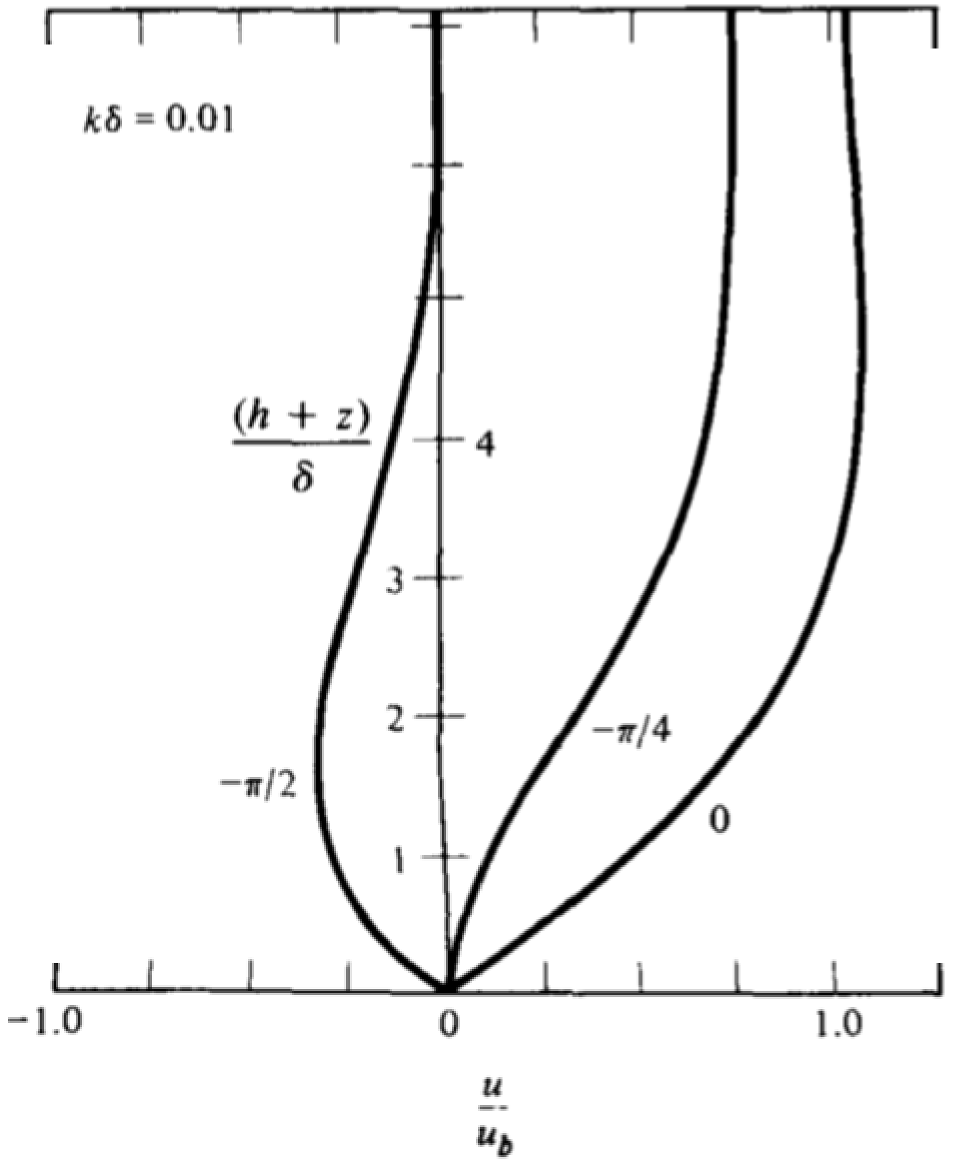
$$\frac{\partial u_r}{\partial t} = \nu\frac{\partial^2 u_r}{\partial z^2}, \qquad \begin{cases} u_r(-h) = -u_p \\ u_r(\infty) \to 0 \end{cases}$$

$$\Rightarrow u_r = -\frac{gka}{\omega}\frac{e^{-\theta}}{\cosh kh}\cos(kx-\omega t+\theta)$$

$$\theta = \sqrt{\frac{\omega}{2\nu}}(z+h) = \frac{z+h}{\delta}$$

- Bed shear stress

$$\tau_{xz} \approx \rho\nu \left.\frac{\partial u_r}{\partial z}\right|_{z=-h}$$



$$u_r = -\frac{gka}{\omega}\frac{e^{-\frac{z+h}{\delta}}}{\cosh kh}\cos\left(kx - \omega t + \frac{z+h}{\delta}\right)$$

$$\tau_{xz} = \rho\nu \left.\frac{\partial u_r}{\partial z}\right|_{z=-h}$$

$$= \rho\sqrt{\frac{\nu}{\omega}}\frac{gka}{\cosh kh}\cos\left(kx - \omega t - \frac{\pi}{4}\right)$$

$$\Rightarrow 45° \text{ phase lag, } \overline{\tau}_{xz} = 0$$

- Conventional bottom friction model

$$\tau_{xz} = \rho \sqrt{\frac{\nu}{\omega}} \frac{gka}{\cosh kh} \cos\left(\theta - \frac{\pi}{4}\right)$$

$$u_b = u_p(z=-h) = \frac{gka}{\omega} \frac{1}{\cosh kh} \cos\theta$$

$$\Rightarrow \tau_{xz} = f u_b |u_b|$$

- Wave energy dissipation

$$\frac{dE}{dt} = -\varepsilon_D = -\int_0^\lambda \int_{-h}^0 \tau_{xz} \frac{\partial u_r}{\partial z} dz dx$$

$$\Rightarrow \frac{1}{2}\rho g \frac{da^2}{dt} = \overline{\rho \nu \int_{-h}^0 \left(\frac{\partial u_r}{\partial z}\right)^2 dz}$$

$$\Rightarrow a = a_0 e^{-\alpha t}, \quad \alpha = \frac{1}{2} \frac{\nu k}{\delta \sinh 2kh}$$

Viscous effect: Nonlinear long waves

- Energy dissipation mechanism
  - Quadratic model

$$\frac{\partial \boldsymbol{u}}{\partial t} + \cdots = \cdots + \boldsymbol{R}_f, \quad \boldsymbol{R}_f = f \boldsymbol{u}_b \left| \boldsymbol{u}_b \right|$$

- Dimensionless equation

$$\mu^2 \nabla \cdot \boldsymbol{u} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \epsilon \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{\epsilon}{\mu^2} w \frac{\partial \boldsymbol{u}}{\partial z} = -\nabla p + \alpha^2 \left[\nabla^2 \boldsymbol{u} + \frac{1}{\mu^2} \frac{\partial^2 \boldsymbol{u}}{\partial z^2} \right]$$

$$\epsilon \frac{\partial w}{\partial t} + \epsilon^2 \boldsymbol{u} \cdot \nabla w + \frac{\epsilon^2}{\mu^2} w \frac{\partial w}{\partial z} = -\epsilon \frac{\partial p}{\partial z} - 1 + \epsilon \alpha^2 \left[\nabla^2 w + \frac{1}{\mu^2} \frac{\partial^2 w}{\partial z^2} \right]$$

$$\epsilon = \frac{a_0'}{h_0'}, \quad \mu = \frac{h_0'}{\ell_0'}, \quad \alpha^2 = \frac{\nu}{\ell_0' \sqrt{g h_0'}} \left(= \frac{1}{\mathrm{Re}} \right)$$

- Boussinesq approximation

$$O(\epsilon) \sim O(\mu^2) < 1 \quad \Rightarrow \quad \epsilon = 10^{-1}$$

- Eddy viscosity

$$\alpha^2 = \mu \frac{\nu}{h_0' \sqrt{g h_0'}} = 10^{-0.5} \frac{10^{-3}}{10^0 \cdot 10^{0.5}}$$

$$\Rightarrow O(\alpha) \sim O(\epsilon^2) \sim O(\mu^4)$$

- Perturbation expansion

$$O(\alpha) \sim O(\epsilon^2) \sim O(\mu^4)$$

$$\boldsymbol{u} = \nabla \Phi(\boldsymbol{x}, z, t) + \boldsymbol{u}_0^r(\boldsymbol{x}, z, t) + \alpha \boldsymbol{u}_1^r(\boldsymbol{x}, z, t) + \cdots$$

$$w = \frac{\partial \Phi}{\partial z} + \alpha \mu w_1^r + \cdots$$

We shall require the irrotational part to be corrected at $O(\alpha)$

- Boussinesq equations with zero viscosity
  - Expand the velocity potential in the vertical coordinate
  - Obtain a recursive relation from the continuity
  - Apply the bottom no-flux condition to get rid of odd order terms
  - Express the velocity potential in terms of the horizontal velocity at the bottom
  - KFSBC & DFSBC become the depth-integrated conservation laws

- With consideration of viscosity
  - Expand the velocity potential in the vertical coordinate
  - Obtain a recursive relation from the continuity
  - ~~Apply the bottom no-flux condition to get rid of odd order terms~~ ➔ w_b = ?
  - Express the velocity potential in terms of the horizontal velocity at the bottom
  - KFSBC & DFSBC become the depth-integrated conservation laws

- Rotational part: boundary layer analysis

$$\nabla \cdot \boldsymbol{u}_0^r + \frac{\partial w_1^r}{\partial \eta} = 0, \quad \eta = \frac{z+1}{(\alpha/\mu)}$$

$$\frac{\partial \boldsymbol{u}_0^r}{\partial t} = \frac{\partial^2 \boldsymbol{u}_0^r}{\partial \eta^2}, \qquad \frac{\partial p}{\partial z} = 0$$

$$\boldsymbol{u} = \nabla \Phi(\boldsymbol{x}, z, t) + \boldsymbol{u}_0^r(\boldsymbol{x}, z, t) + \alpha \boldsymbol{u}_1^r(\boldsymbol{x}, z, t) + \cdots$$

$$w = \frac{\partial \Phi}{\partial z} + \alpha \mu w_1^r + \cdots$$

$$\frac{\partial \boldsymbol{u}_0^r}{\partial t} = \frac{\partial^2 \boldsymbol{u}_0^r}{\partial \eta^2}$$

$$\boldsymbol{u}_0^r = -\nabla \Phi, \qquad \frac{\partial \Phi}{\partial z} = -\alpha \mu w_1^r, \qquad \eta = 0$$

$$\boldsymbol{u}_0^r \to 0, \qquad w_1^r \to 0, \qquad \eta \to \infty$$

$$\boldsymbol{u}_0^r(\boldsymbol{x}, \eta, t) = -\frac{\eta}{\sqrt{4\pi}} \int_0^t \frac{\nabla \Phi(\boldsymbol{x}, z = -1, T)}{\sqrt{(t-T)^3}} \mathrm{e}^{-\eta^2/4(t-T)}\, \mathrm{d}T$$

$$w_1^r(\boldsymbol{x}, 0, t) = -\frac{1}{\sqrt{\pi}} \int_0^t \frac{\nabla^2 \Phi(\boldsymbol{x}, z = -1, T)}{\sqrt{(t-T)}}\, \mathrm{d}T$$

- Boussinesq equations

$$\Phi(\boldsymbol{x}, z, t) = \sum_{n=0}^{\infty} (z+1)^n \phi_n(\boldsymbol{x}, t)$$

$$\phi_{n+2} = \frac{-\mu^2 \nabla^2 \phi_n}{(n+1)(n+2)}, \quad n = 0, 1, 2, \ldots$$

$$\phi_1 = \frac{\alpha \mu}{\sqrt{\pi}} \int_0^t \frac{\nabla^2 \phi_0(\boldsymbol{x}, T)}{\sqrt{(t-T)}} \, \mathrm{d}T$$

$$\Phi = \phi_0 + (z+1)\frac{\alpha\mu}{\sqrt{\pi}}\int_0^t \frac{\nabla^2\phi_0(\boldsymbol{x},T)}{\sqrt{(t-T)}}\,\mathrm{d}T$$

$$-\frac{\mu^2}{2}(z+1)^2\nabla^2\phi_0 + \frac{\mu^4}{24}(z+1)^4\nabla^2\nabla^2\phi_0 + O(\mu^6)$$

$$\mu^2\left[\frac{\partial\zeta}{\partial t} + \epsilon\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\zeta\right] = \frac{\partial\Phi}{\partial z}, \quad z = \epsilon\zeta$$

$$\mu^2\left(\frac{\partial\Phi}{\partial t} + \zeta\right) + \frac{1}{2}\epsilon\left[\mu^2|\boldsymbol{\nabla}\Phi|^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right] = 0, \quad z = \epsilon\zeta$$

$$\boxed{\boldsymbol{u}_{\mathrm{b}} = \nabla\phi_0, \qquad H = 1 + \epsilon\zeta}$$

$$\frac{1}{\epsilon}\frac{\partial H}{\partial t} + \nabla\cdot(H\boldsymbol{u}_{\mathrm{b}}) - \frac{\mu^2}{6}\nabla^2(\nabla\cdot\boldsymbol{u}_{\mathrm{b}}) - \frac{1}{\mu^2}\frac{\alpha\mu}{\sqrt{\pi}}\int_0^t \frac{\nabla\cdot\boldsymbol{u}_{\mathrm{b}}}{\sqrt{(t-T)}}\,\mathrm{d}T = O(\mu^4)$$

$$\frac{\partial\boldsymbol{u}_{\mathrm{b}}}{\partial t} + \epsilon\boldsymbol{u}_{\mathrm{b}}\cdot\nabla\boldsymbol{u}_{\mathrm{b}} + \frac{1}{\epsilon}\nabla H - \frac{\mu^2}{2}\nabla\left[\nabla\cdot\frac{\partial\boldsymbol{u}_{\mathrm{b}}}{\partial t}\right] = O(\mu^4)$$

$$\boxed{\overline{\boldsymbol{u}} = \frac{1}{H}\int_{-1}^{\epsilon\zeta} \nabla\Phi \, \mathrm{d}z = \boldsymbol{u}_{\mathrm{b}} - \frac{\mu^2}{6} H^2 \nabla^2 \boldsymbol{u}_{\mathrm{b}} + O(\mu^4)}$$

$$\frac{1}{\epsilon}\frac{\partial H}{\partial t} + \nabla\cdot(H\overline{\boldsymbol{u}}) - \frac{\alpha}{\mu\sqrt{\pi}}\int_0^t \frac{\nabla\cdot\overline{\boldsymbol{u}}}{\sqrt{t-T}}\,\mathrm{d}T = O(\mu^4)$$

$$\frac{\partial\overline{\boldsymbol{u}}}{\partial t} + \epsilon\overline{\boldsymbol{u}}\cdot\nabla\overline{\boldsymbol{u}} + \frac{1}{\epsilon}\nabla H - \frac{\mu^2}{3}\nabla\nabla\cdot\left(\frac{\partial\overline{\boldsymbol{u}}}{\partial t}\right) = O(\mu^4)$$

- A set of dimensionless depth-averaged equations for conservation laws

- Bottom shear stress

$$\boldsymbol{\tau} = \frac{\partial \boldsymbol{u}_0^r}{\partial \eta}$$

$$\begin{aligned}\boldsymbol{\tau}_{\mathrm{b}} &= \boldsymbol{\tau}(\boldsymbol{x}, \eta = 0, t) \\ &= -\frac{1}{2\sqrt{\pi}} \int_0^t \frac{\overline{\boldsymbol{u}}}{\sqrt{(t-T)^3}} \,\mathrm{d}T\end{aligned}$$

- Damping of linear progressive waves

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \overline{u}}{\partial x} - \frac{\alpha}{\mu}\frac{1}{\sqrt{\pi}}\int_0^t \frac{\partial \overline{u}}{\partial x}\frac{1}{\sqrt{t-T}}\,\mathrm{d}T = 0$$

$$\frac{\partial \overline{u}}{\partial t} + \frac{\partial \zeta}{\partial x} = 0$$

➔ $$\frac{\partial \zeta}{\partial \xi} = \frac{1}{2\sqrt{\pi}}\int_0^t \frac{\partial \zeta}{\partial x}\frac{1}{\sqrt{t-T}}\,\mathrm{d}T, \qquad \xi = \left(\frac{\alpha}{\mu}\right)t$$

$$\boxed{\frac{\partial \zeta}{\partial \xi} = \frac{1}{2\sqrt{\pi}} \int_0^t \frac{\partial \zeta}{\partial x} \frac{1}{\sqrt{t-T}} \, \mathrm{d}T, \quad \xi = \left(\frac{\alpha}{\mu}\right) t}$$

In the moving coordinate,

$$\zeta = a(\xi)\mathrm{e}^{\mathrm{i}\sigma}, \quad \sigma = x - t$$

we have

$$\frac{\partial a}{\partial \xi}\mathrm{e}^{\mathrm{i}\sigma} = \left[\frac{\mathrm{i}}{2\sqrt{\pi}} \int_0^{\xi/(\alpha/\mu)} \frac{\mathrm{e}^{\mathrm{i}(x-T)}}{\sqrt{t-T}} \, \mathrm{d}T\right] a(\xi)$$

$$= \left[\frac{\mathrm{i}}{2\sqrt{\pi}}\mathrm{e}^{\mathrm{i}\sigma} \int_0^{\infty} \frac{\mathrm{e}^{\mathrm{i}\psi}}{\sqrt{\psi}} \, \mathrm{d}\psi\right] a(\xi) = a(\xi)\tfrac{1}{2}\mathrm{e}^{\mathrm{i}\sigma}\mathrm{e}^{-\mathrm{i}\pi/4}$$

$$\boxed{\frac{\partial a}{\partial \xi} e^{i\sigma} = a(\xi)\tfrac{1}{2} e^{i\sigma} e^{-i\pi/4}}$$

Now, let

$$a = a_0 e^{i\beta\xi}, \quad \beta = \beta_r + i\beta_i$$

we get

$$\beta_r = \beta_i = \frac{1}{2\sqrt{2}}$$

- TO BE CONTINUED -