2017年環境流體力學短期講座
Short Course on Environmental Flows

數學、海浪、與沿海動態過程
Mathematics, ocean waves and coastal dynamic processes

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September 2017

Ocean waves in deep water

(Typhoon Morakot)

Swells at California coast



Tsunami experiment at OSU

Network for Earthquake Engineering Simulation (NEES)

Ocean Waves

- What are ocean waves?

  Ocean waves are perturbations of sea surface from an equilibrium state

  - **Equilibrium state refers to a “steady” and smooth sea**
  - **Perturbations mean disturbances of that state**
- There are several sources of ocean waves
  - **Energy transferred to ocean surface by wind (capillary waves; wind waves)**
  - **Seismic activity, landslides, volcanism (tsunamis)**
  - **Gravitational attraction of the moon and sun acting on the rotating earth (tides; internal waves; planetary waves)**
- Different restoring forces for different waves
  - **Capillary waves: surface tension and gravity**
  - **Wind waves: gravity**
  - **Tsunamis: gravity and earth rotation**
  - **Internal waves: density stratification and gravity**
  - **Tides: gravity and earth rotation**
  - **Planetary waves: gravity and earth rotation**

Ocean wave energy distribution

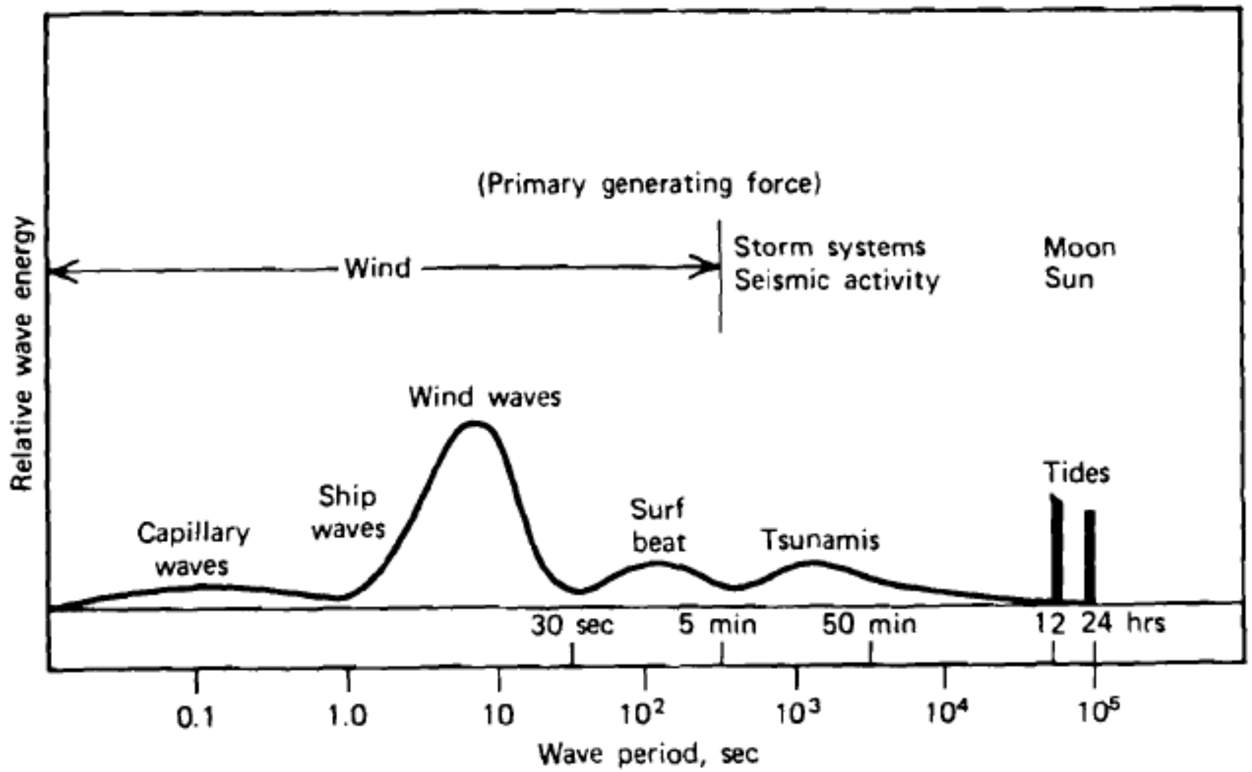
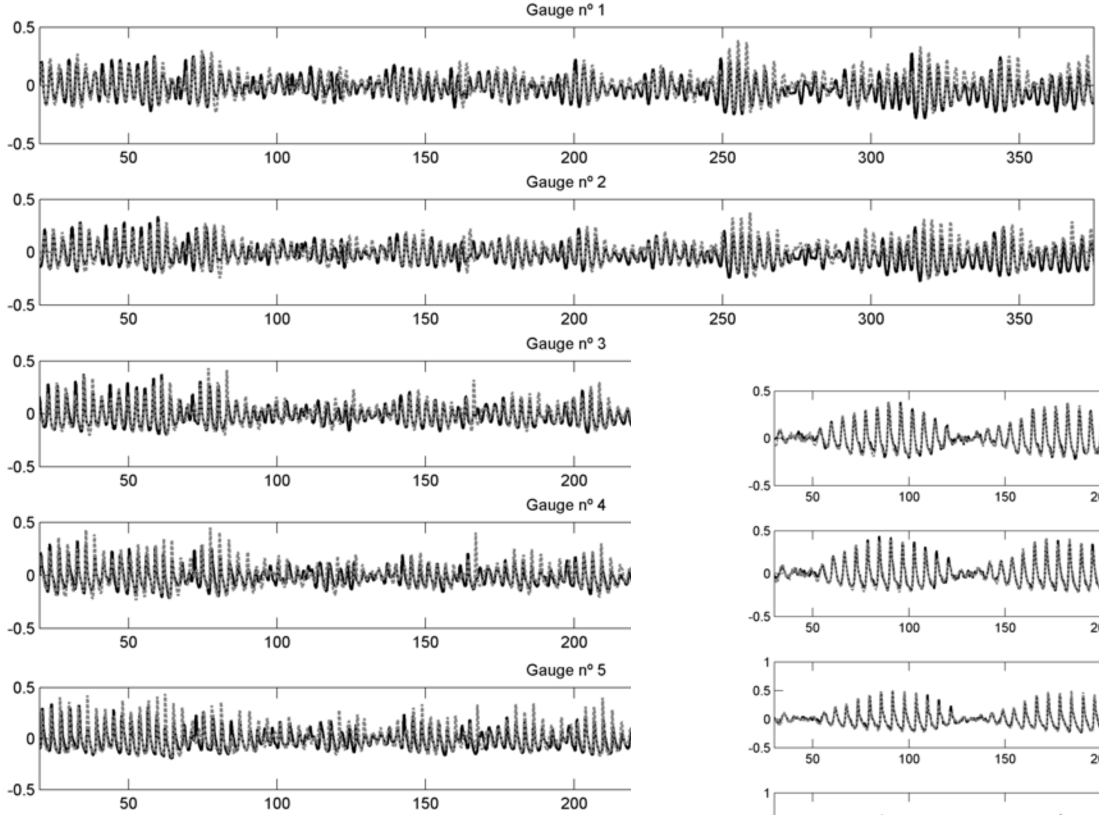
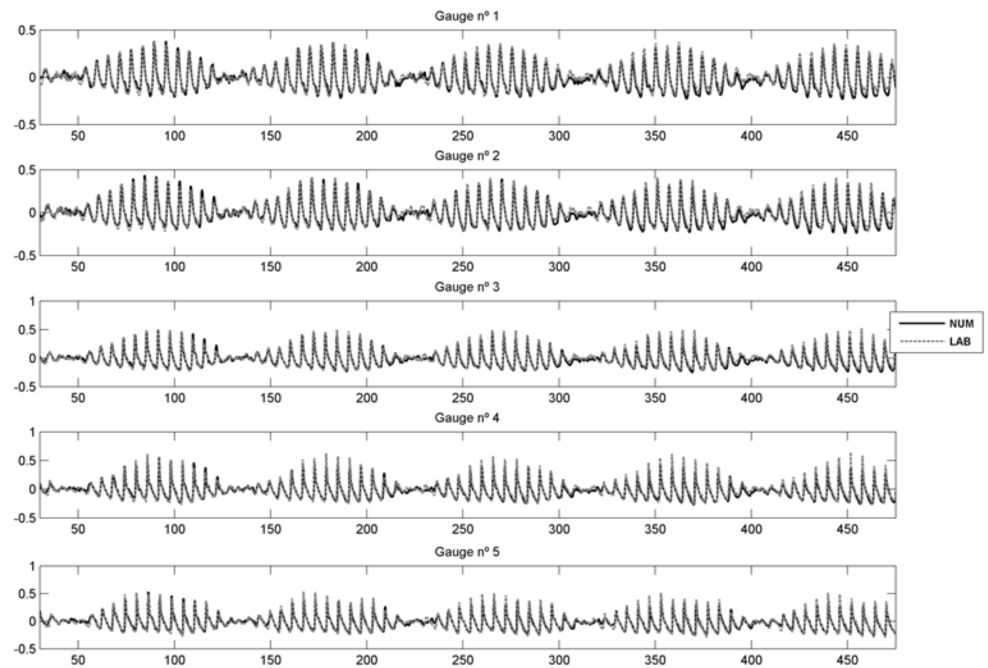


Figure 2.1 Estimated relative ocean wave energy and primary generating forces (adapted from Munk, 1950).

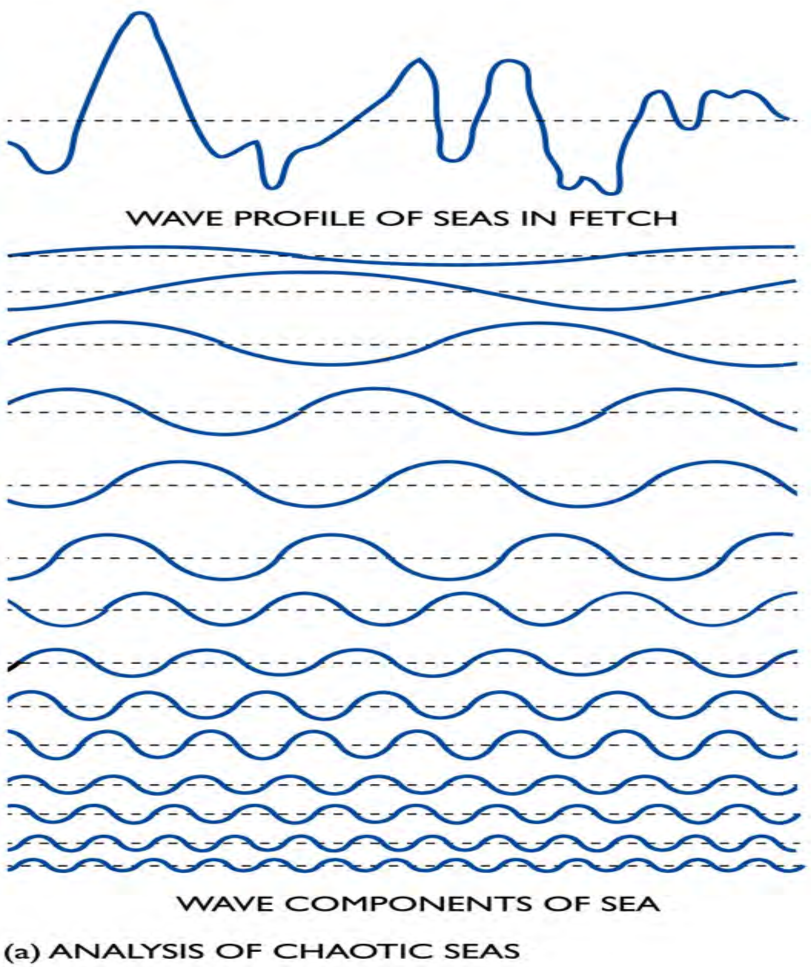
Typical records of free Surface fluctuations

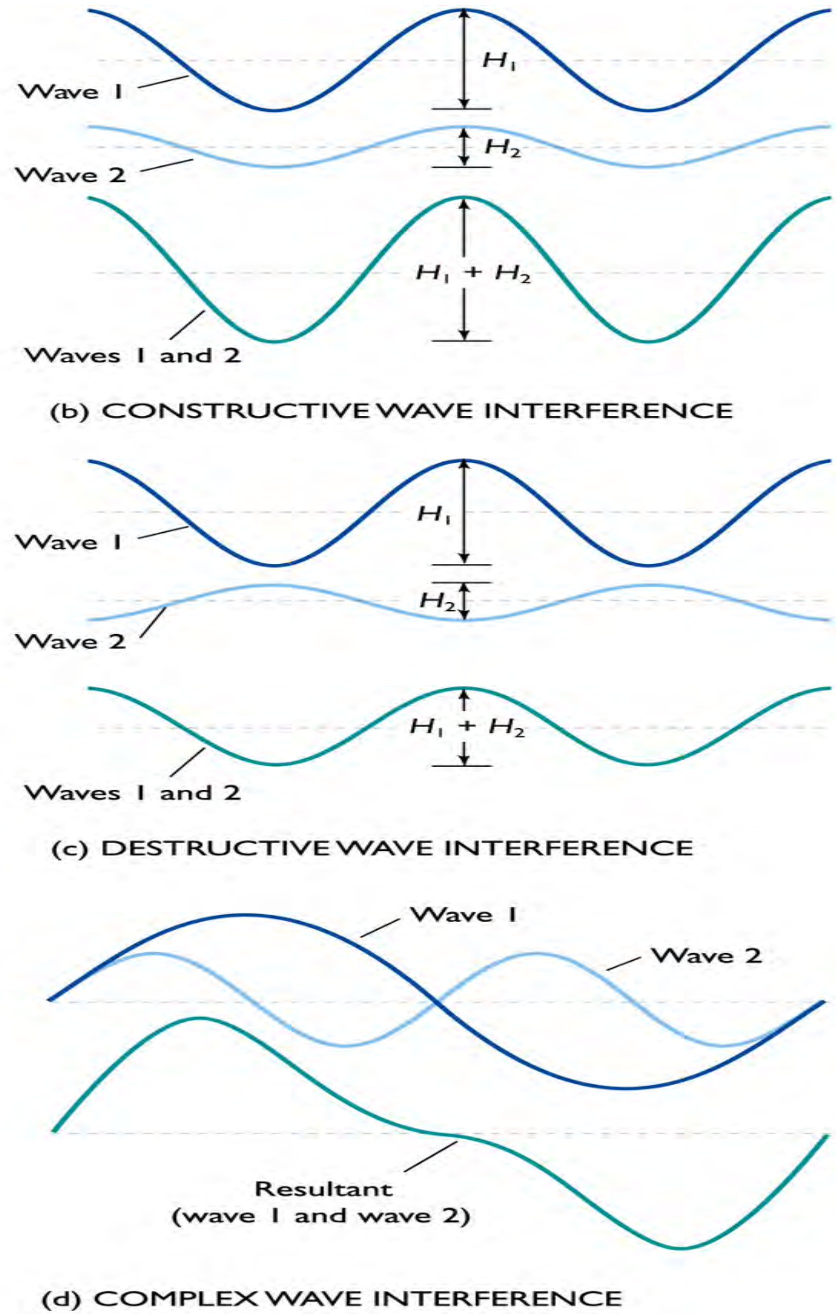




Free surface displacement at a fixed location

$$\eta(t) = \sum_{i=1}^{\infty} (A_i \cos \omega_i t + B_i \sin \omega_i t) = \sum_{i=1}^{\infty} \tilde{A}_i \cos(\omega_i t + \theta_i)$$



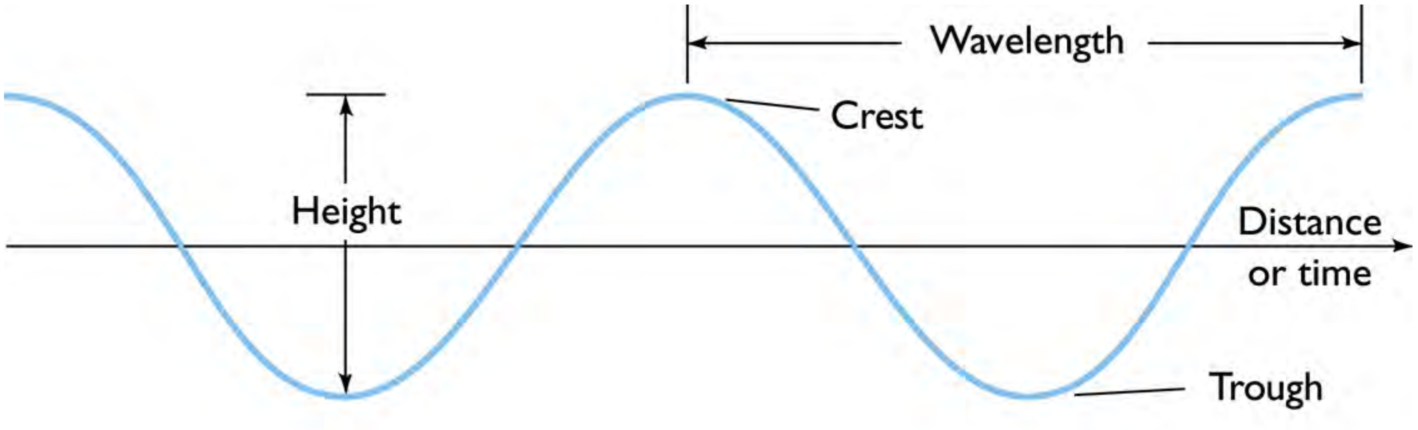


A sea state can be viewed as the superposition of simple harmonic waves

Basic wave form:

$$\eta(t) = A\cos\omega t$$

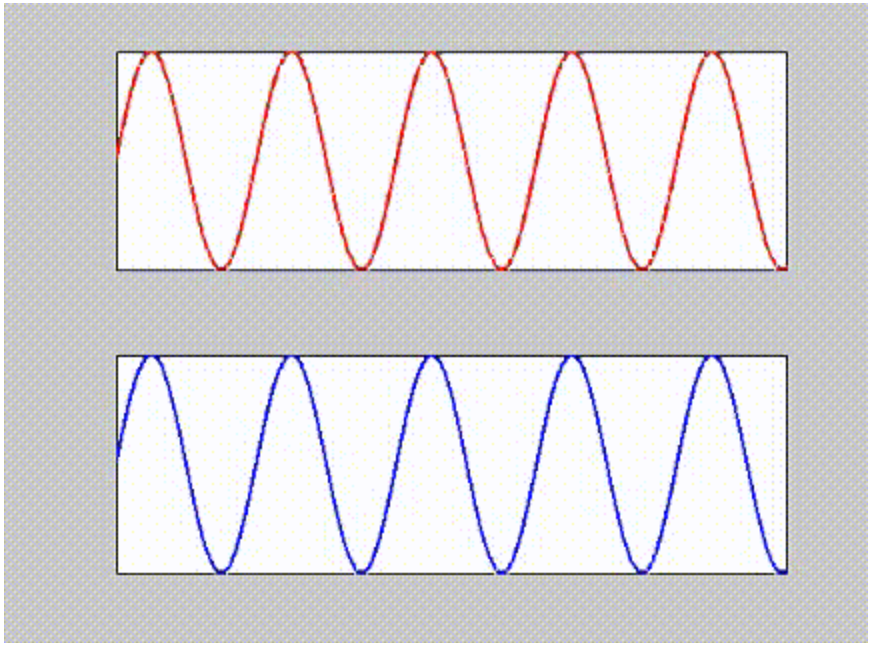
$$\eta(x) = A\cos kx$$



(a) WAVE PARAMETERS

- Wave crest
- Wave trough
- Wave amplitude
- Wave height
- Wavelength; Wave period
- Wave number; Wave frequency
- Wave steepness

Progressive waves and Standing waves



$$\eta(x,t) = A\cos(kx - \omega t)$$

$$c = \omega / k = celerity$$

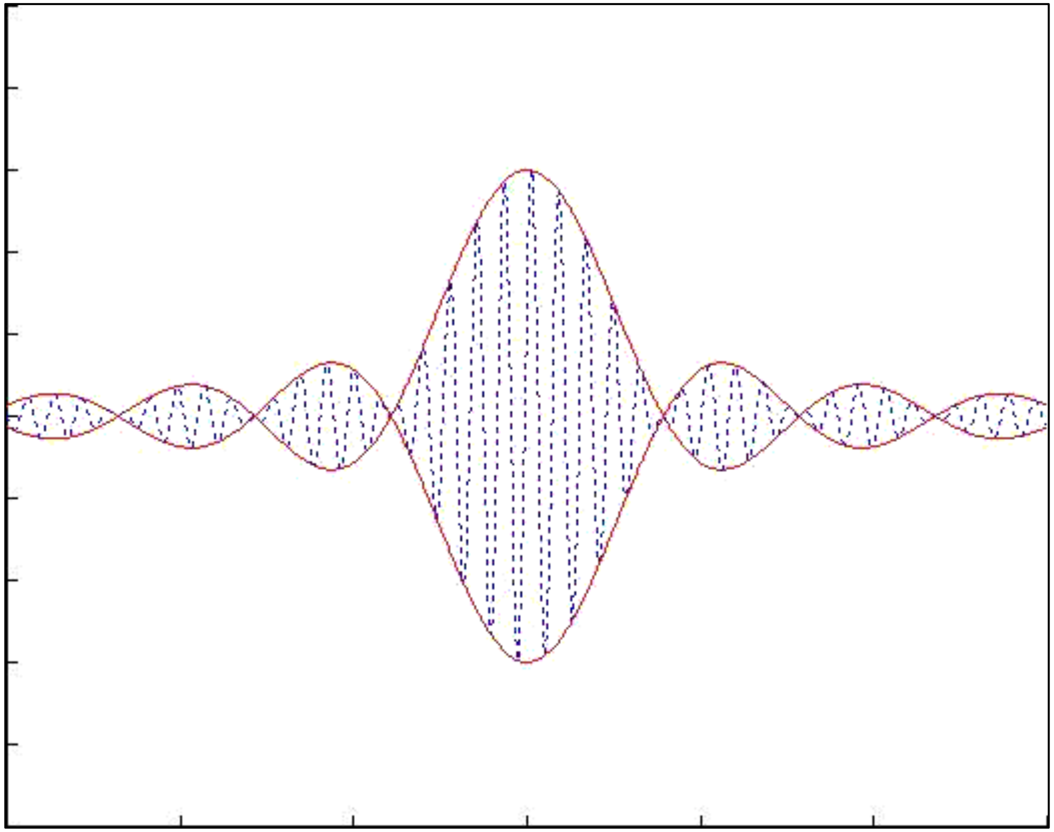
$$\eta(x,t) = A\cos(kx)\cos(\omega t)$$

The wave form of **progressive waves** moves with a constant speed in a prescribed direction. The speed of wave propagation is also called **celerity** or **phase speed**.
The wave form of the progressive waves will appear as stationary to an observer moving with the same speed as the phase speed.

The wave form of **standing waves** does not propagate in space. The water surface oscillates vertically between fixed positive and negative elevations. There are nodes (zero elevation fluctuation) and antinodes (maximum elevation fluctuation.).

Wave group

Superposition of progressive waves with slightly different wave numbers and wave frequencies.



$$\eta(x,t) = A\{\cos[kx - \omega t] + \cos[(k+\Delta k)x - (\omega + \Delta\omega)t]\}$$

$$= B[(\Delta k)x - (\Delta\omega)t]\cos(kx - \omega t) + C[(\Delta k)x - (\Delta\omega)t]\sin(kx - \omega t)$$

$$C_g = \Delta\omega / \Delta k = \textit{group velocity}$$

Mathematical formulation of 2D small amplitude wave problem

$$\boxed{\vec{\Omega} = \nabla \times \vec{u} = 0 \quad \Rightarrow \quad \text{irrotational flow (potential flow)}}$$

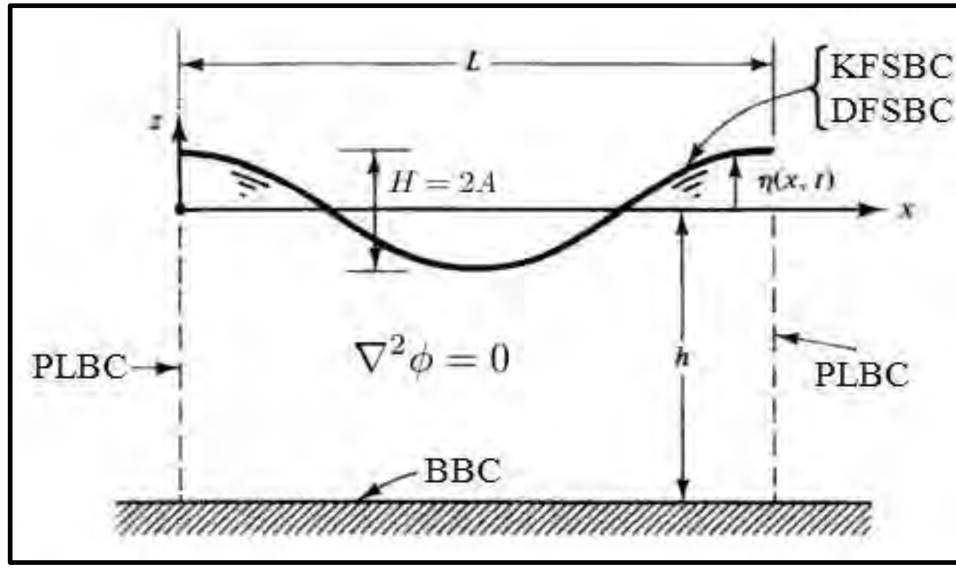
$$\eta(x,t) = A\cos(kx - \omega t)$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \quad \begin{matrix} -\infty < x < \infty \\ h < z < 0 \end{matrix}$$

$$\frac{\partial \phi}{\partial z} = 0, \;\; z = -h$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t}, \;\; z = 0$$

$$\frac{\partial \phi}{\partial t} = -g\eta, \;\; z = 0$$



By observation: $\boxed{\phi(x,z,t) = f(z)\sin(kx - \omega t)}$

Since $\eta(x,t)$ is defined, we only need to solve for $f(z)$, $-h < z < 0$

We only need two boundary conditions in z-dir. The third BC determines the (DISPERSION) relationship between:

$$k\left(= \frac{2\pi}{L}\right) \text{ and } \omega\left(= \frac{2\pi}{T}\right)$$

Linear wave theory: problem & solution

$$\eta(x,t) = A\cos(kx-\omega t)$$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=0, \quad -\infty < x < \infty, \quad h < z < 0$$

$$\frac{\partial\phi}{\partial z}=0, \; z=-h$$

$$\frac{\partial\phi}{\partial z}=\frac{\partial\eta}{\partial t}, \; z=0$$

$$\frac{\partial\phi}{\partial t}=-g\eta, \; z=0$$

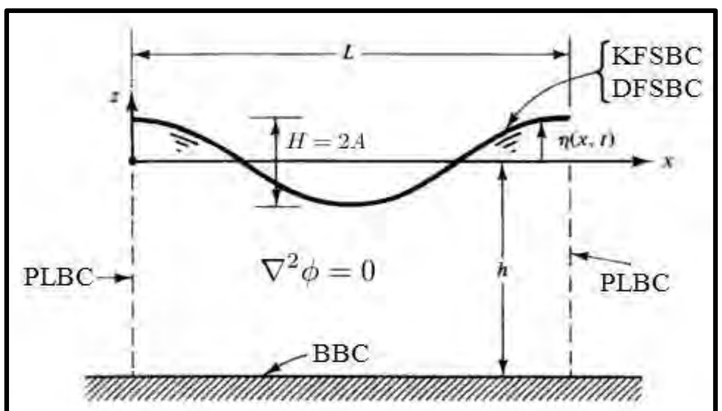
$$\phi=\frac{A\omega}{k}\frac{\cosh k(z+h)}{\sinh kh}\sin(kx-\omega t)$$

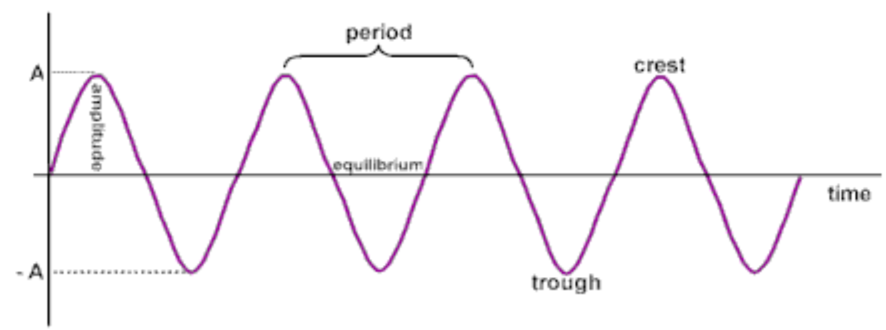
$$\omega^2 = gk\tanh kh$$

(dispersion relationship)

Note: $A=\frac{H}{2}, \quad k=\frac{2\pi}{L}, \quad \omega=\frac{2\pi}{T}$

$$u=\frac{\partial\phi}{\partial x}, \quad w=\frac{\partial\phi}{\partial z}$$





Progressive waves in a constant depth – Phase velocity

6.10
Sequence of photographs showing a plane progressive wave system advancing into calm water. The water is darkened with dye, and the lower half of the water depth is not shown. The wave energy is contained within the heavy diagonal lines, and propagates with the group velocity. (The boundaries of the wave group diffuse slowly with time, due to dispersion.) The position of one wave crest is connected in successive photographs by the light line, which advances with the phase velocity. Each wave crest moves with the phase velocity, equal to twice the group velocity of the boundaries. Thus each wave crest vanishes at the front end and, after the wavemaker is turned off, arises from calm water at the back. The interval between successive photographs is 0.25 s and the wave period is 0.36 s. The wavelength is 0.23 m and the water depth is 0.11m.

Free surface displacement

$$\eta(x,t) = A\cos(kx - \omega t)$$
$$= A\cos\left[k(x - ct)\right];$$

Phase velocity (wave celerity)

$$c = \frac{\omega}{k} = \frac{L}{T}$$

Dispersion relation for small amplitude waves

$$\omega^2 = gk\tanh kh$$

Phase velocity (wave celerity)

$$c = \frac{\omega}{k} = \sqrt{\frac{gh}{kh}\tanh kh}$$

Dispersive wave syatem: $c(\omega)$ *or* $c(k)$

Taylor series of tanh *kh*

For small *kh*, the Taylor series of tanh *kh* is

$$\tanh kh = kh - \frac{1}{3}(kh)^3 + O((kh)^5).$$

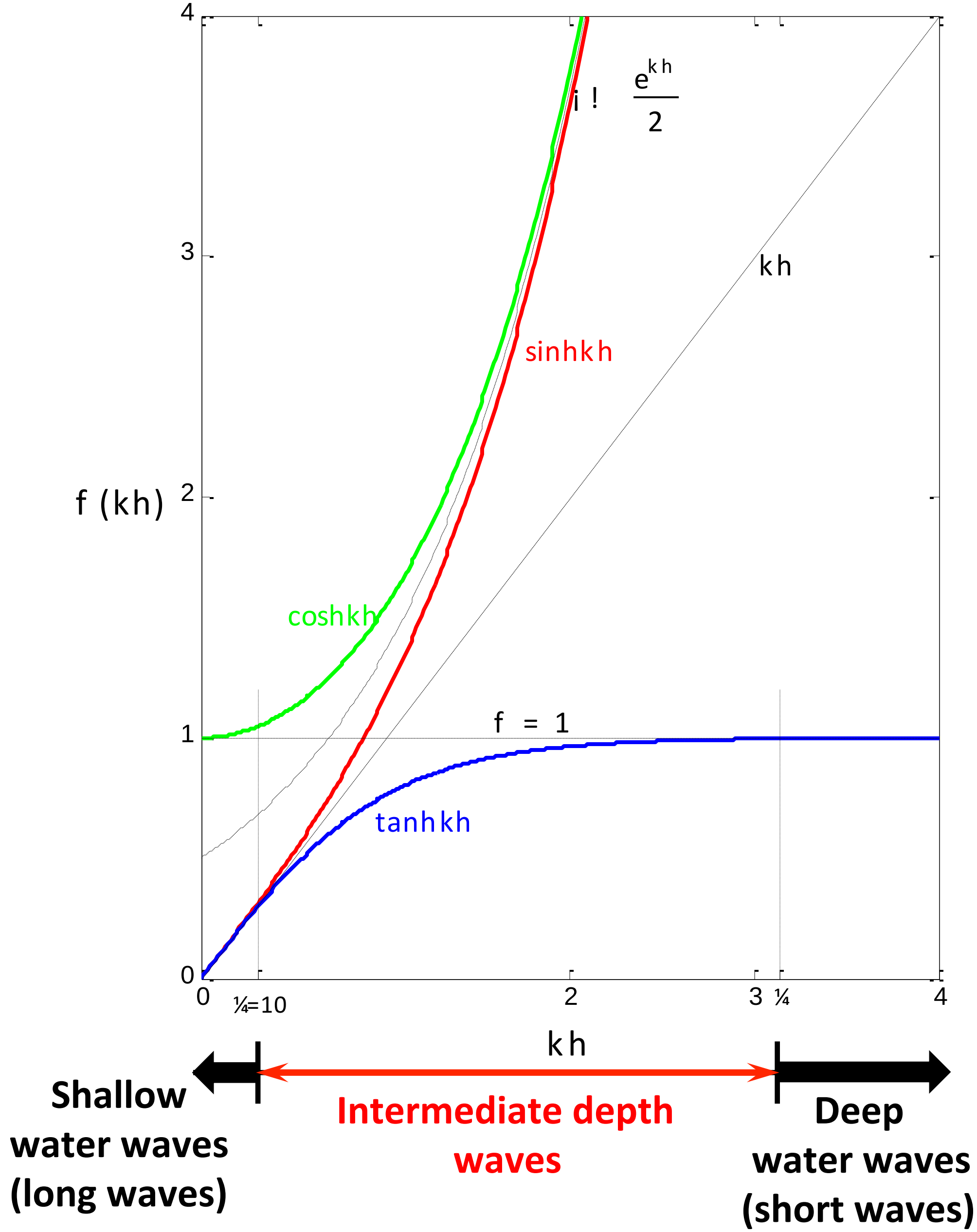
The dispersion relationship can be expaned as

$$\omega^2 = gk\tanh kh = k^2\ (\mathrm{gh})[1 - \frac{1}{3}(kh)^2 + O((kh)^4)]$$

Phase velocity (wave celerity) can be approximated as

$$c = \frac{\omega}{k} = \sqrt{gh}[1 - \frac{1}{6}(kh)^2 + O((kh)^4)]$$

Different wave systems at two limits



$$\begin{cases} \omega^2 = gk\tanh kh \\ c = \dfrac{L}{T} = \dfrac{\omega}{k} \end{cases}$$

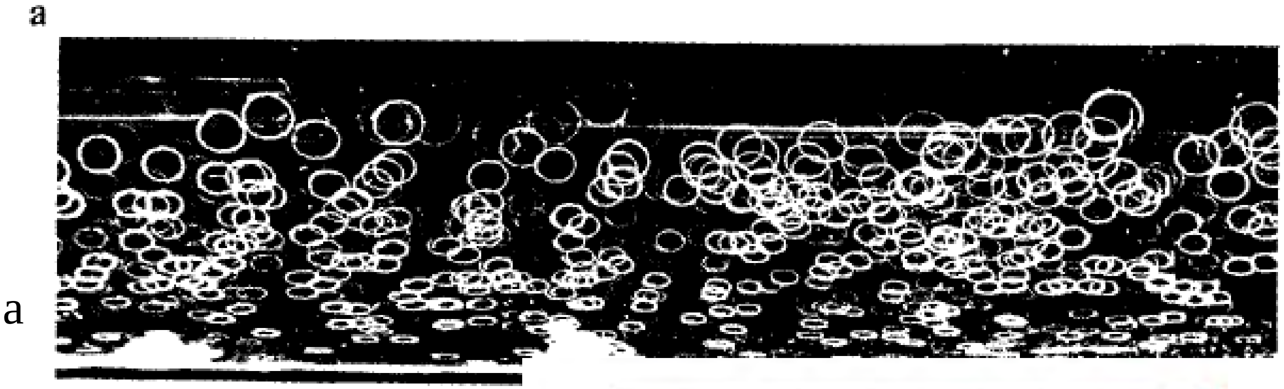
Shallow water limit (small kh)

- $kh < \dfrac{\pi}{10}$ or $\dfrac{h}{L} > \dfrac{1}{20}$
- $\cosh kh \to 1$; $\sinh kh$, $\tanh kh \to kh$
- $\omega^2 = gk^2 h \Rightarrow k \to \dfrac{\omega}{\sqrt{gh}}$
- $c \to \sqrt{gh}$ (**nondispersive**)

Deep water limit (large kh)

- $kh > \pi$ or $\dfrac{h}{L} > \dfrac{1}{2}$
- $\cosh kh$, $\sinh kh \to \dfrac{e^{kh}}{2}$; $\tanh kh \to 1$
- $\omega^2 = gk \Rightarrow k \to \dfrac{\omega^2}{g}$
- $c \to \dfrac{g}{\omega} = \sqrt{\dfrac{g}{k}}$ (**dispersive**)

Fluid Particle Velocity and Particle trajectories

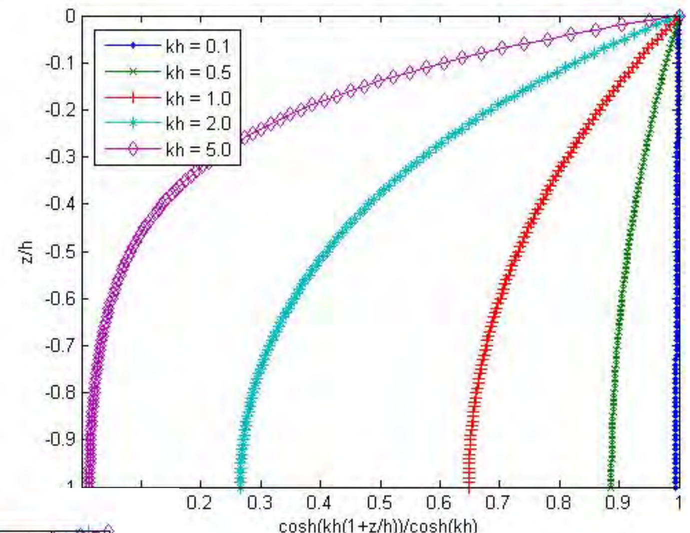


Free surface displacement for a progressive wave train over a constant water depth

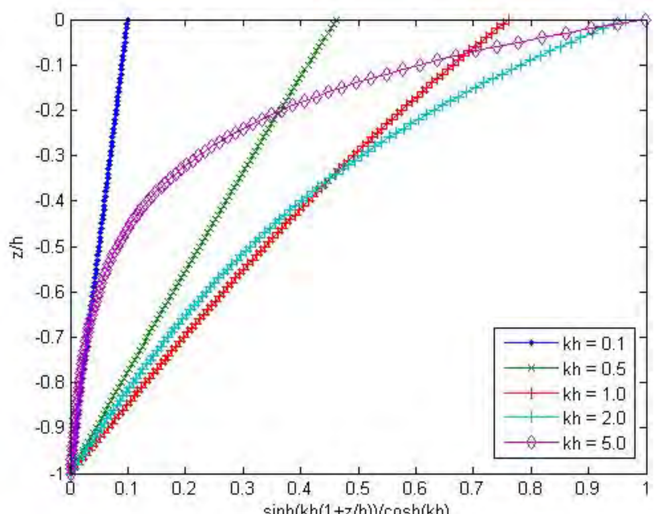
$$\eta(x,t) = A\cos(kx - \omega t)$$

Horizontal and vertical velocity components are

$$u(x,z,t) = \frac{gkA}{\omega}\frac{\cosh k(z+h)}{\cosh kh}\cos(kx - \omega t);$$



$$w(x,z,t) = \frac{gkA}{\omega}\frac{\sinh k(z+h)}{\cosh kh}\sin(kx - \omega t)$$



Fluid Particle Velocity and Particle trajectories

The instantaneous position of a water particle can be denoted as $(x_1+\zeta, z_1+\xi)$. In other word, the water particle is moving in an orbit around (x_1, z_1). The position of the particle can be expressed as

$$\vec{x} = \vec{x}_0 + \int_{t_0}^{t} \vec{u}_L(\vec{x}_0, t')dt',$$

Since at any given time, t, the Lagrangian velocity is the same as the Eulerian velocity

$$\vec{u}_L(\vec{x}_0, t) = \vec{u}(\vec{x}, t) = \vec{u}\left(\vec{x}_0 + \int_{t_0}^{t} \vec{u}_L(\vec{x}_0, t')dt', t\right)$$

$$\approx \vec{u}\left(\vec{x}_1, t\right).$$

Thus,

$$\zeta = x - x_1 = -A\frac{\cosh k(z_1+h)}{\sinh kh}\sin(kx_1-\omega t)$$

$$\xi = z - z_1 = A\frac{\sinh k(z_1+h)}{\sinh kh}\cos(kx_1-\omega t)$$

and

$$\left(\frac{\zeta}{C}\right)^2 + \left(\frac{\xi}{D}\right)^2 = 1 \quad ===\succ \quad \text{an ellipse}$$

where

$$C = A\frac{\cosh k(z_1+h)}{\sinh kh}, \; D = A\frac{\sinh k(z_1+h)}{\sinh kh}$$

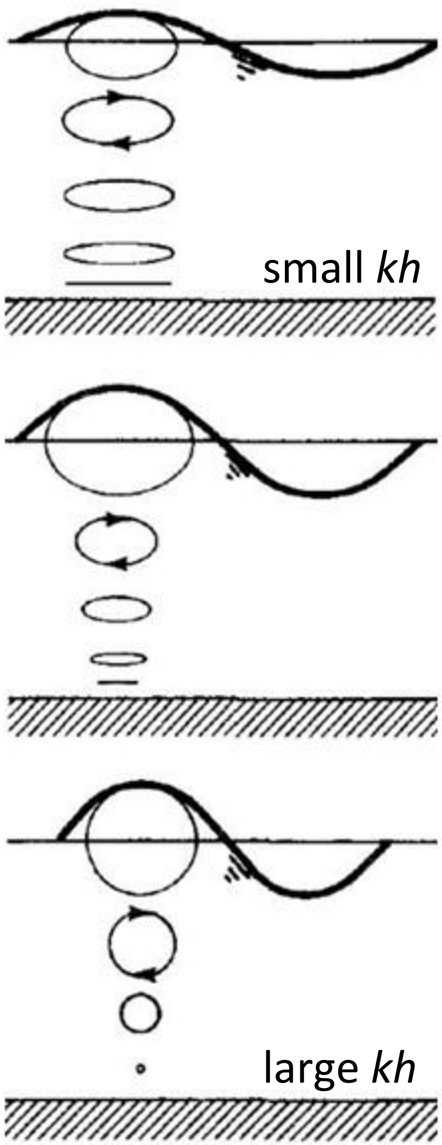
In deep water $kh >> 1$

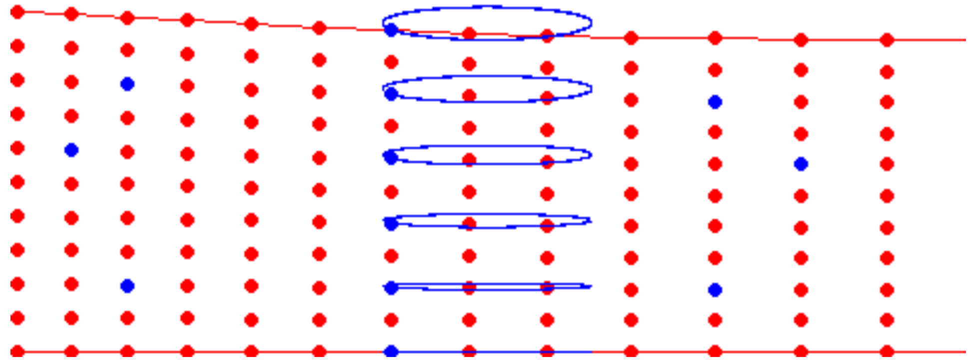
$$C = A\exp(kz_1), \; D = A\exp(kz_1)$$

In shallow water $kh << 1$

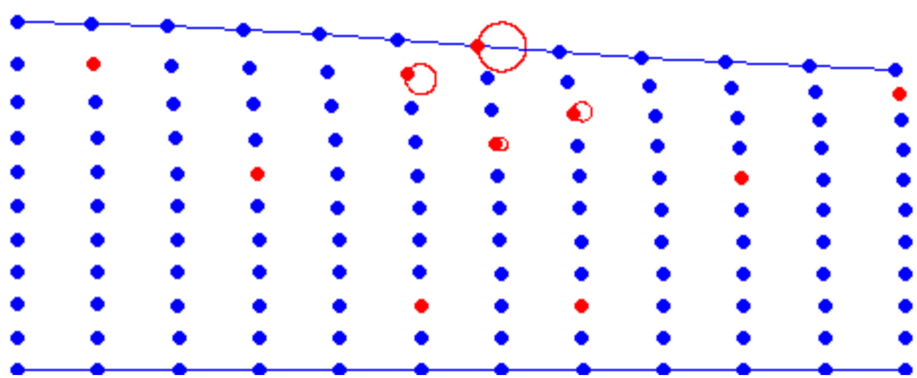
$$C = \frac{A}{kh}, \; D = A\left(1+\frac{z_1}{h}\right)$$

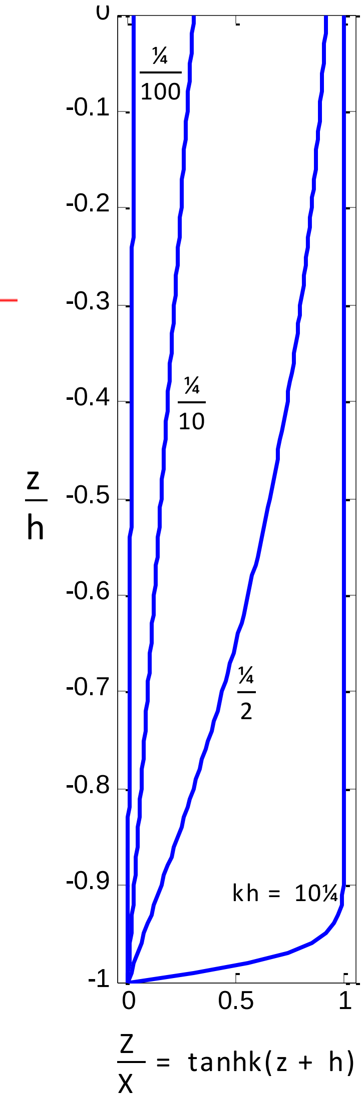
Water particle path



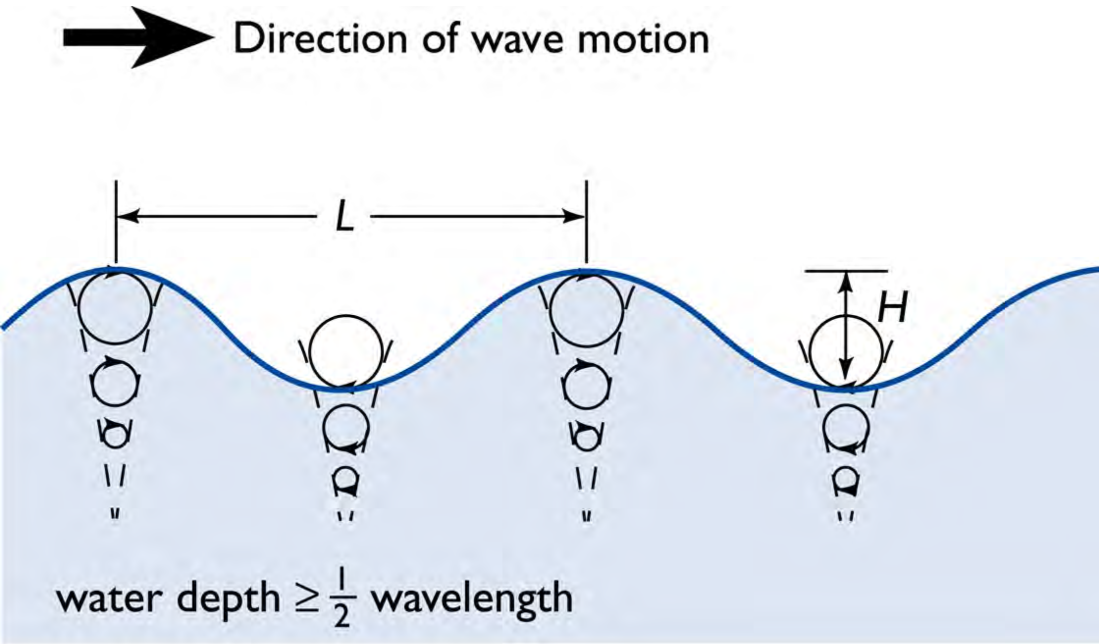


$$\left(\frac{\zeta}{C}\right)^2+\left(\frac{\xi}{D}\right)^2=1 \quad ===\succ \quad \text{an ellipse}$$





Fluid Particle Velocities in shallow and deep water limits

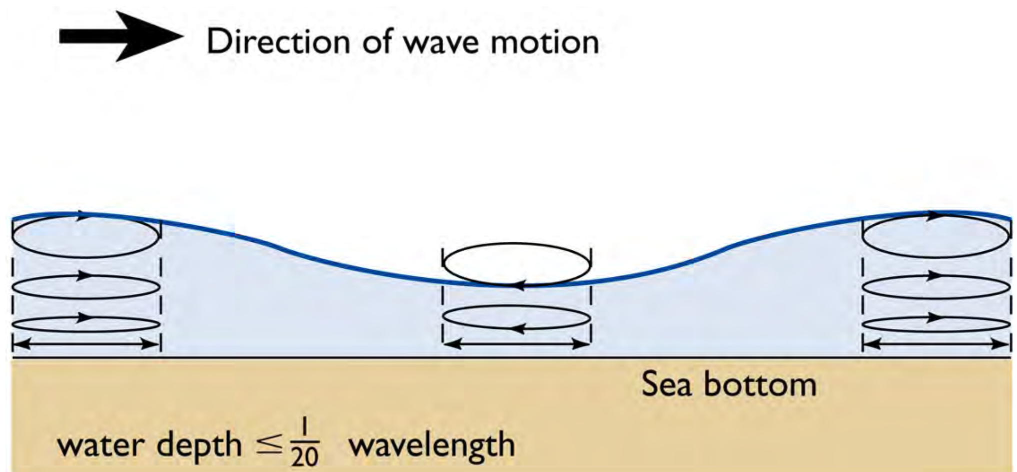


(a) DEEP-WATER WAVE $h/L > 1/2$

Horizontal and vertical velocities become

$$u(x,z,t) = \frac{gkA}{\omega}\exp(kz)\cos(kx-\omega t);$$

$$w(x,z,t) = \frac{gkA}{\omega}\exp(kz)\sin(kx-\omega t)$$



(b) SHALLOW-WATER WAVE $h/L < 1/20$

Horizontal and vertical velocities become

$$u(x,z,t) = \frac{gkA}{\omega}\cos(kx-\omega t);$$

$$w(x,z,t) = 0$$

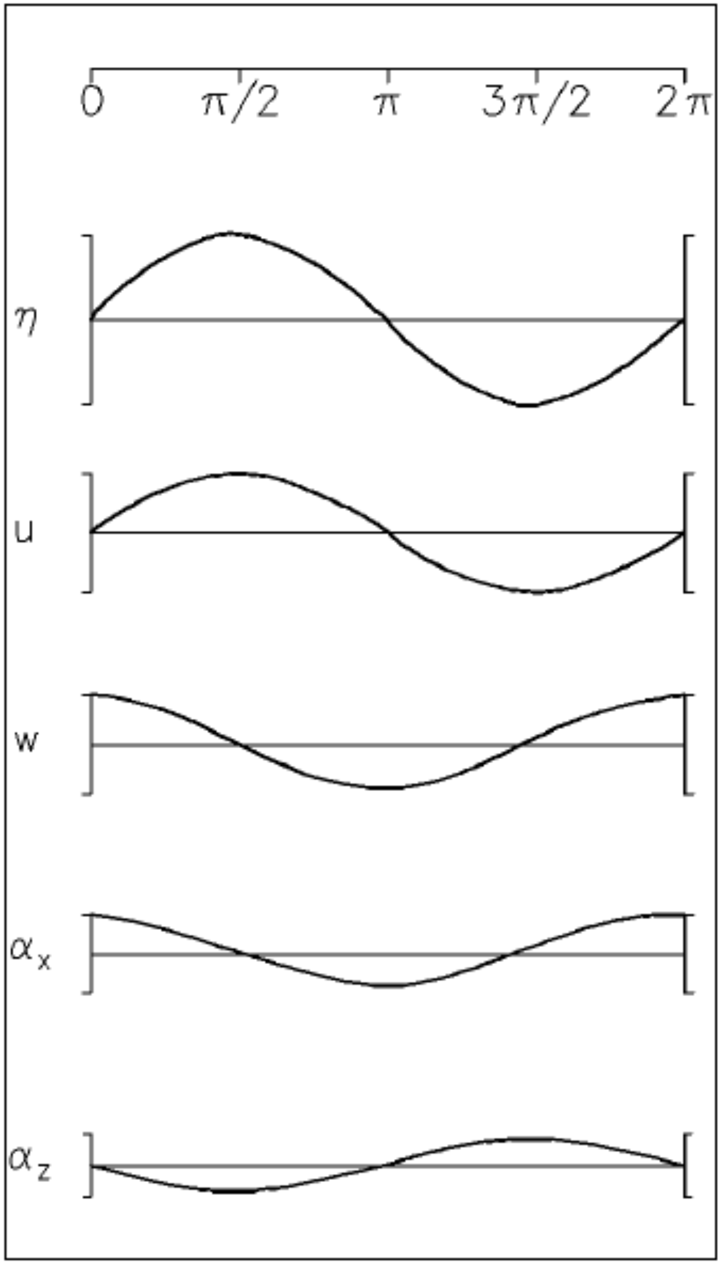


Figure II-1-3. Profiles of particle velocity and acceleration by Airy theory in relation to the surface elevation

Horizontal and vertical velocity components are

$$u(x,z,t) = \frac{gkA}{\omega}\frac{\cosh k(z+h)}{\cosh kh}\cos(kx-\omega t);$$

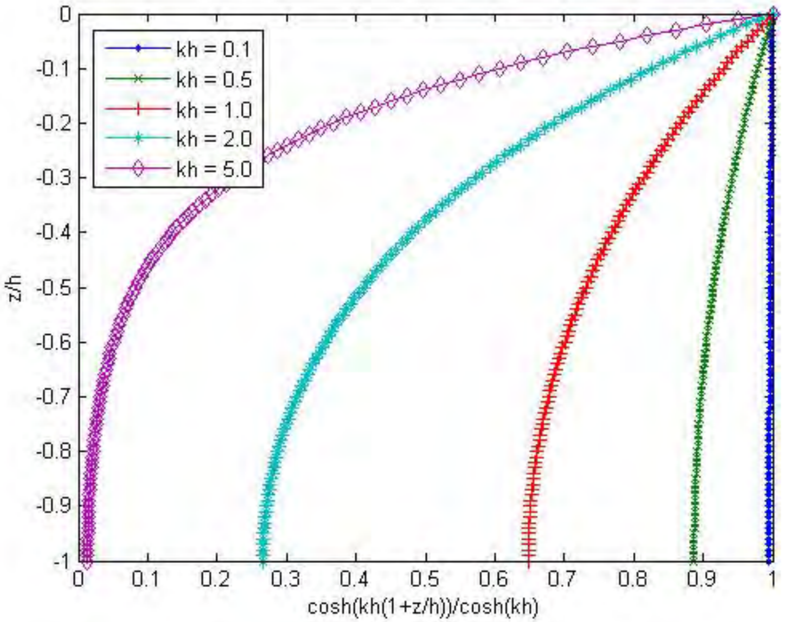
$$w(x,z,t) = \frac{gkA}{\omega}\frac{\sinh k(z+h)}{\cosh kh}\sin(kx-\omega t)$$

Horizontal and vertical acceleration components are

$$\alpha_x(x,z,t) = \frac{\partial u}{\partial t} = gkA\frac{\cosh k(z+h)}{\cosh kh}\sin(kx-\omega t);$$

$$\alpha_z(x,z,t) = \frac{\partial w}{\partial t} = -gkA\frac{\sinh k(z+h)}{\cosh kh}\cos(kx-\omega t)$$

Pressure field



Hydrostatic pressure:

$$P_s = -\rho g z$$

Dynamic pressure:

$$P_d = \rho g A \frac{\cosh k(z+h)}{\cosh kh} \cos(kx - \omega t)$$

Total pressure

$$P = P_s + P_d$$

For shallow water waves,

the total pressure becomes

$$P = -\rho g (z + \eta)$$

Wave energy in small amplitude progressive wave

$$\eta(x,t) = A\cos(kx - \omega t)$$

- Total wave energy (mechanical energy) = potential energy + kinetic energy
- Potential energy per wave per unit width:

$$PE = \int_x^{x+L} \left\{ \int_0^\eta \rho g z dz \right\} dx = \int_x^{x+L} \frac{1}{2} \rho g \eta^2 dx$$

$$= \frac{1}{8} \rho g H^2 \int_x^{x+L} \cos^2(kx - \omega t) dx = \frac{L}{16} \rho g H^2$$

$$\boxed{\int_0^\pi \cos^2 \theta d\theta = \frac{\pi}{2}}$$

- Kinetic energy per wave per unit width:

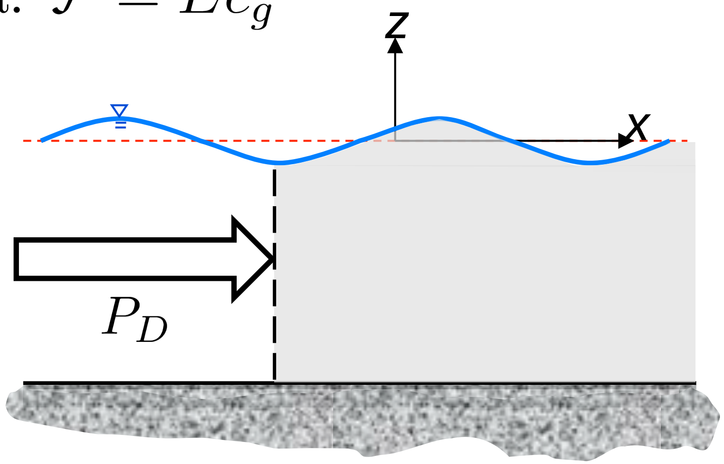
$$KE = \int_x^{x+L} \left\{ \int_{-h}^\eta \frac{u^2 + w^2}{2} dz \right\} dx = \frac{L}{16} \rho g H^2$$

- Total wave energy per wave per unit width: $E = \dfrac{L}{8} \rho g H^2$
- Total wave energy per unit surface area: $\overline{E} = \dfrac{E}{L} = \dfrac{1}{8} \rho g H^2$

$$\boxed{\begin{array}{l}\text{Average wave energy} \propto H^2 \\ \quad (\text{energy density, } \overline{E})\end{array}}$$

Energy flux across a water column

- Energy flux, $\mathcal{F}$, is the rate at which energy is transferred
- In our linear wave problem, energy flux equals the rate of work being done by the pressure: $\mathcal{F} = \dfrac{\Delta E}{\Delta t} = \dfrac{W}{\Delta t} = \dfrac{P_D \Delta x}{\Delta t}$
- Average wave energy flux per unit area: $\overline{\mathcal{F}} = \overline{E} c_g$
- $\overline{\mathcal{F}} =$ rate of work done by the dynamic pressure



$$= \frac{1}{T}\int_t^{t+T}\left[\int_{-h}^{\eta}(\rho g z + p)\, u dz\right] dt$$

$$\approx \frac{1}{T}\int_t^{t+T}\int_{-h}^{0}(\rho g z + p)\, u dz dt$$

$$= \overline{E}\frac{\omega}{k}\left[\frac{1}{2}\left(1+\frac{2kh}{\sinh 2kh}\right)\right] \Rightarrow \boxed{c_g = c\left[\frac{1}{2}\left(1+\frac{2kh}{\sinh 2kh}\right)\right]} \leq c$$

Characteristics of wave energy flux

- $\overline{\mathcal{F}} = \overline{E}c_g, \quad c_g = c\left[\frac{1}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right)\right]$
- c_g is the rate at which the energy is transmitted
- Recall $\omega^2 = gk\tanh kh \Rightarrow 2\omega\frac{\partial\omega}{\partial k} = \frac{\omega^2}{k} + \frac{\omega^2 h}{\sinh kh\cosh kh}$

$$\Rightarrow \boxed{c_g = \frac{\partial\omega}{\partial k}} \quad \left(c = \frac{\omega}{k}\text{: phase speed}\right)$$

- c_g is called the <u>group velocty</u>
- For deep water waves, $c_g \approx \frac{1}{2}c = \frac{gT}{4\pi}$
- For shallow water waves, $c_g \approx c = \sqrt{gh}$

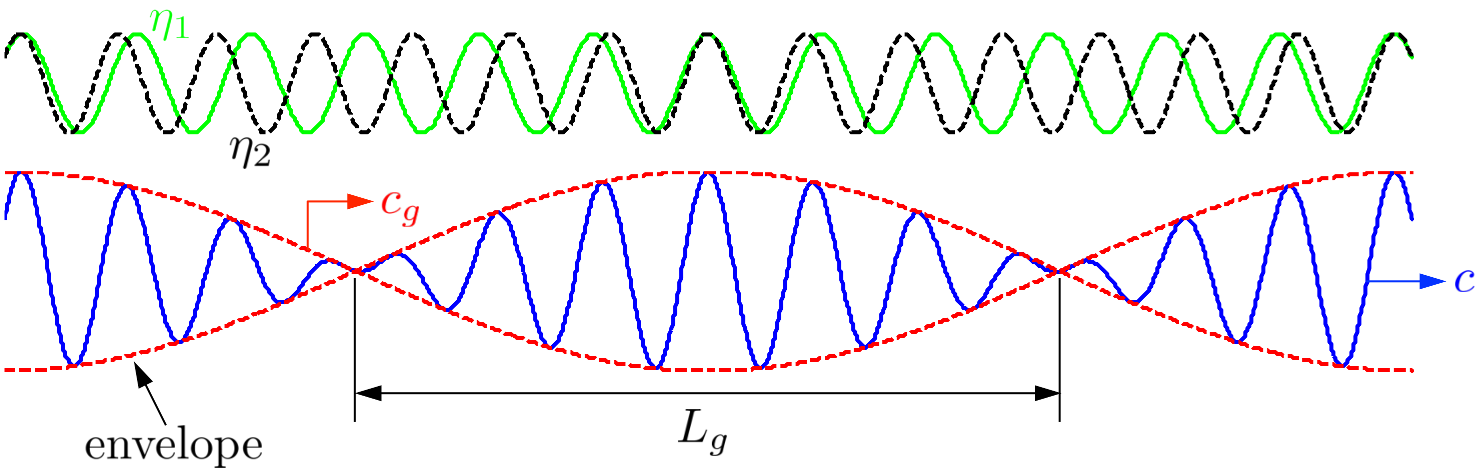
Surf beats and Group velocity

Superimposing two small amplitude waves with same amplitude, but slightly different wave number and frequency results in surf beats

$$\eta(x,t) = A\{\cos[kx - \omega t] + \cos[(k+\Delta k)x - (\omega + \Delta\omega)t]\}$$

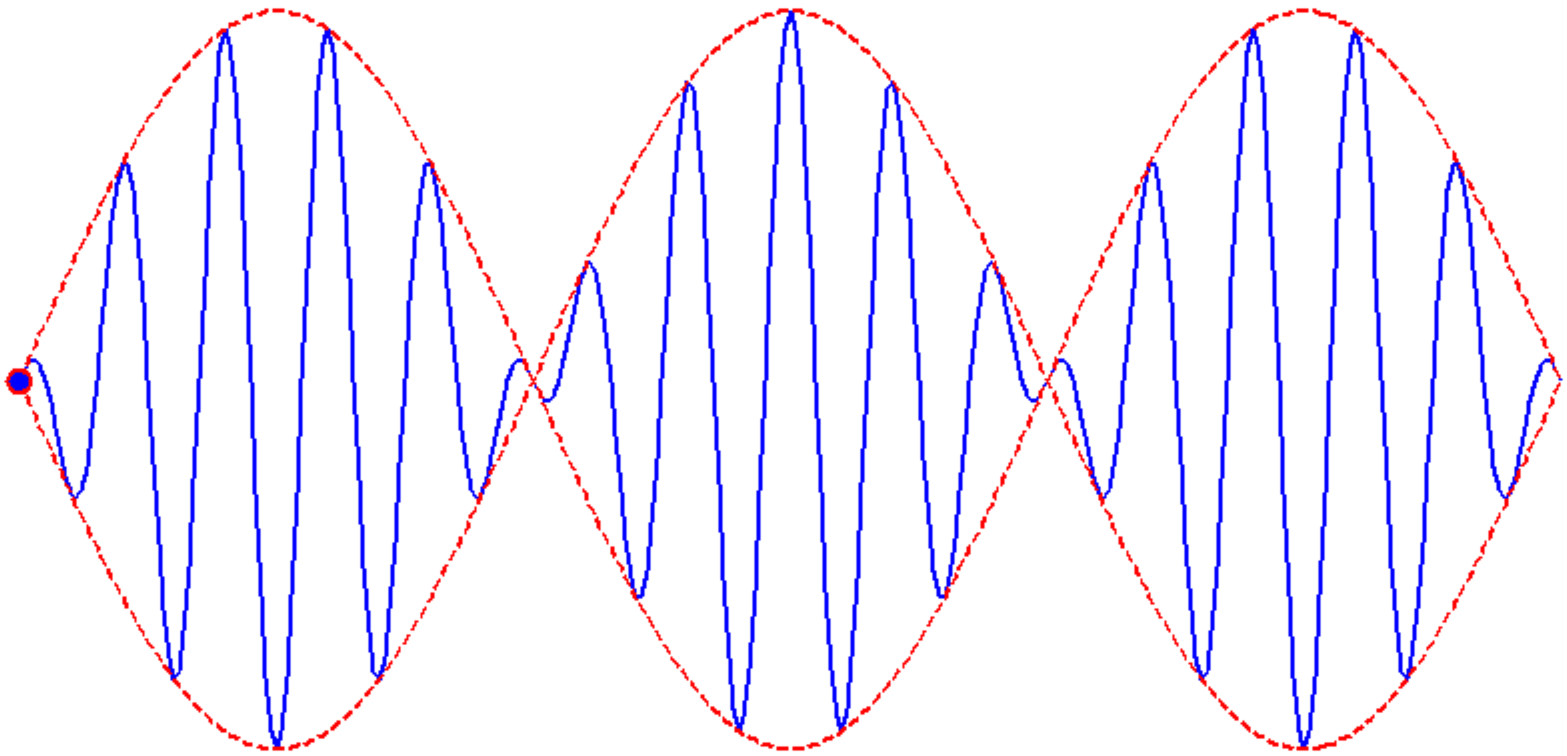
$$= B[(\Delta k)x - (\Delta\omega)t]\cos(kx - \omega t) + C[(\Delta k)x - (\Delta\omega)t]\sin(kx - \omega t)$$

$$C_g = \Delta\omega / \Delta k = group\ velocity; \qquad c = \omega / k = phase\ velosccty$$



Group velocity and phase velocity

$C_g = \Delta\omega / \Delta k =$ *group velocity*; $c = \omega / k =$ *phase veloscty*



For simple harmonic wave, $\Delta\omega$ *and* Δk *approach zero*
Group velocity $= \partial\omega / \partial k$

$$\frac{c_g}{c} = \frac{1}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right) = n, \quad \frac{1}{2} \le n \le 1$$

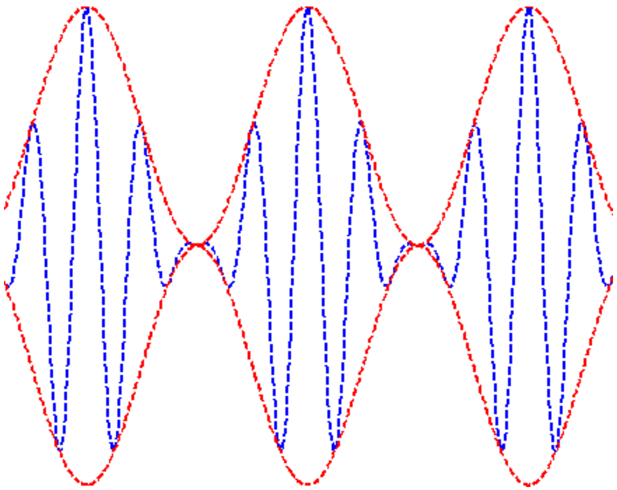
Frequency dispersion effects

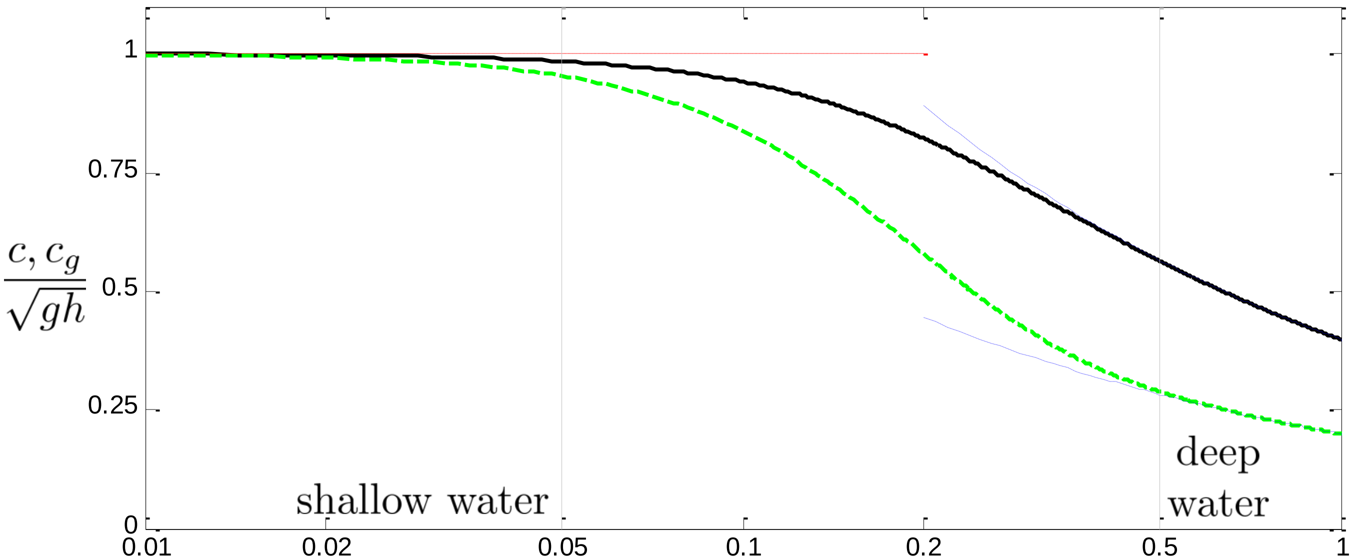
- $\omega^2 = gk\tanh kh$ specifies the relation between k and ω
- $c = \dfrac{\omega}{k} \begin{cases} \text{Deep water waves } (kh > \pi)\text{: } c = \dfrac{g}{\omega} < \sqrt{gh} \text{ (dispersive)} \\ \text{Shallow water waves } (kh < \pi/10)\text{: } c = \sqrt{gh} \text{ (nondispersive)} \end{cases}$
- kh: frequency dispersion parameter
- $c_g = \dfrac{\partial\omega}{\partial k} \begin{cases} \text{Deep water waves } (kh > \pi)\text{: } c_g = \dfrac{c}{2} \\ \text{Shallow water waves } (kh < \pi/10)\text{: } c_g = c \end{cases}$
- The wave energy must travel with the speed of the group of waves, c_g
- For a group of nondispersive waves: $c_g = c = \sqrt{gh}$

$\Rightarrow$ the wave group remains the same form

- For a group of dispersive waves: $c_g = \dfrac{c}{2} = \dfrac{gT}{4\pi}$

$\Rightarrow$ individual wave crests travel from the tail toward the front of the wave group

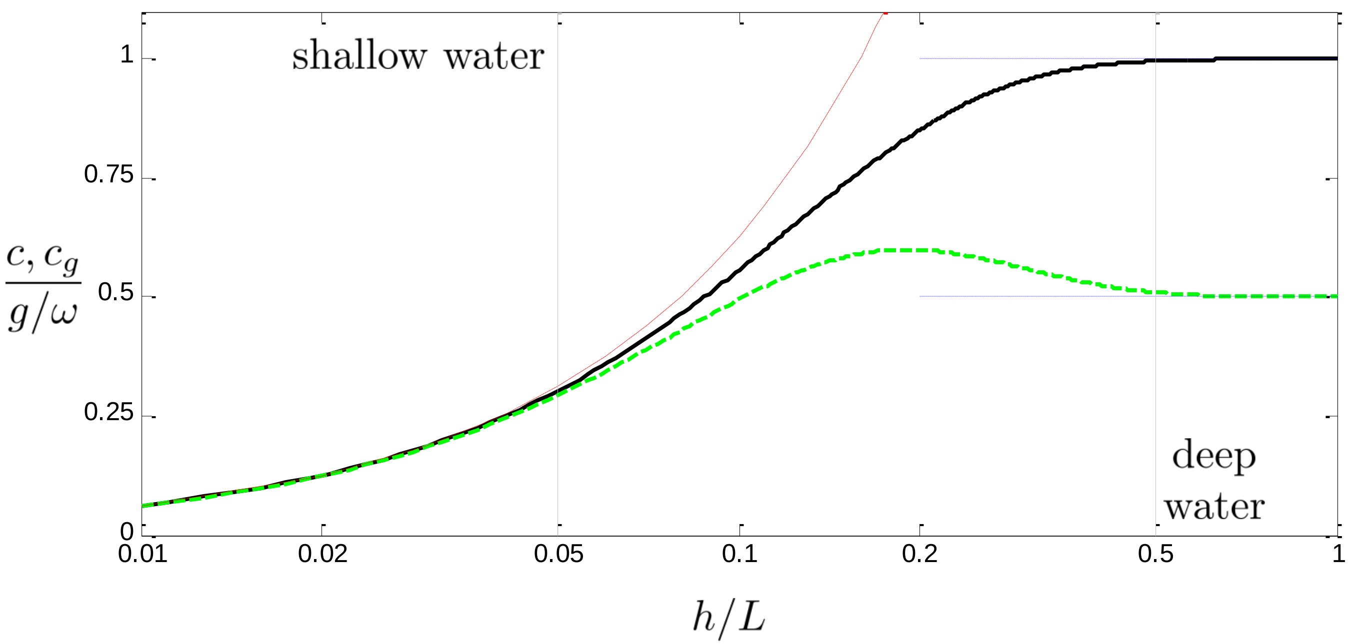




$$\frac{c}{\sqrt{gh}} = \sqrt{\frac{\tanh kh}{kh}}$$

$$\frac{c_g}{\sqrt{gh}} = n\sqrt{\frac{\tanh kh}{kh}}$$

$$n = \frac{1}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right)$$



$$\frac{c}{g/\omega} = \tanh kh$$

$$\frac{c_g}{g/\omega} = n \tanh kh$$

$$n = \frac{1}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right)$$

Characteristics of shallow water waves

Free surface displacement

$$\eta(x,t) = A\cos(kx - \omega t)$$

Dispersion relationship

$$\omega^2 = (gh)k^2$$

Phase velocity

$$c = \sqrt{gh}$$

Horizontal and vertical velocities

$$u(x,z,t) = \frac{gkA}{\omega}\cos(kx - \omega t) = \frac{gk}{\omega}\eta = \frac{\eta}{h}c;$$

$$w(x,z,t) = 0$$

The total pressure

$$P = -\rho g(z - \eta)$$

Conservation of mass

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left(uh\right) = 0.$$

Conservation of momentum

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}$$

These equations can be combined as

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2}{\partial x^2}\left(ghu\right) = 0.$$

If h = constant,

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0.$$

$$\left[\frac{\partial^2}{\partial t^2} \propto (-\omega^2);\ \frac{\partial^2}{\partial x^2} \propto (-k^2)\right]$$

Limitations on the linear theory

For all *kh:* $\left(\dfrac{u\partial u/\partial x}{\partial u/\partial t}\right)_{x=0} \sim \left(\dfrac{uk}{\omega}\right)_{x=0} \sim \left(\dfrac{u}{c}\right)_{x=0} = \dfrac{kA}{\tanh kh} << 1$

For $kh = O(1)$: $kA << 1$

For $kh << 1$: $A/h << 1$

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Short Course on Environmental Flows

數學、海浪、與沿海動態過程
Mathematics, ocean waves and coastal dynamic processes

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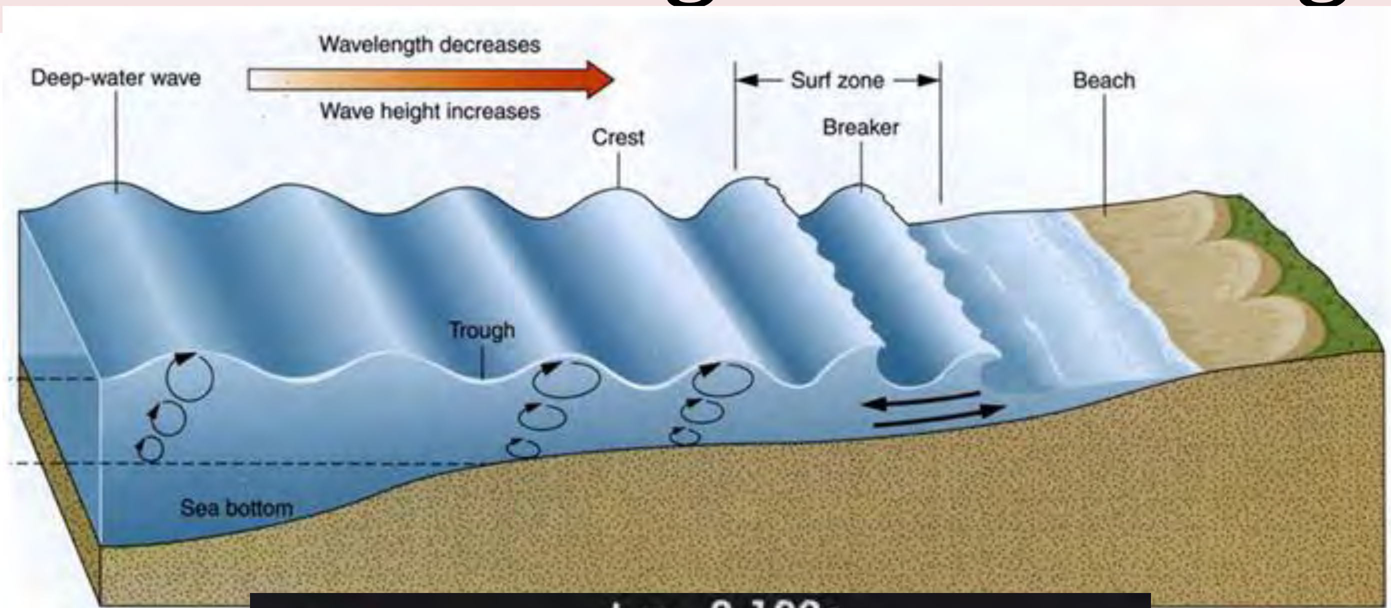
National University of Singapore
Cornell University

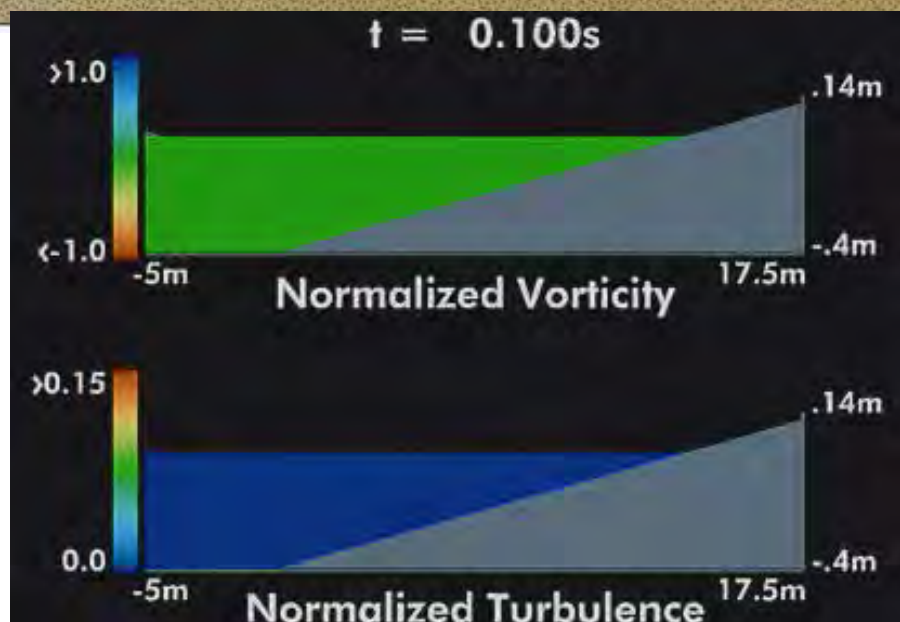
September 2017

Wave transformations – effects of bathymetry

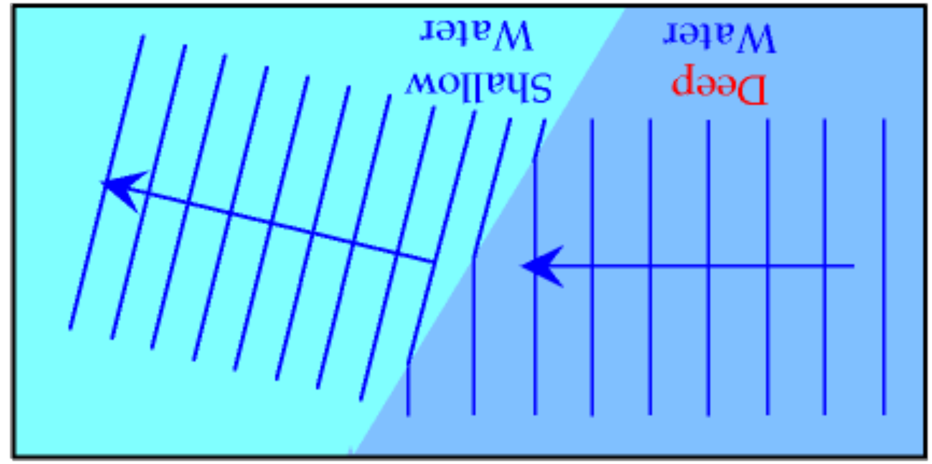
- Wave shoaling
- Wave breaking
- Wave reflection
- Wave refraction
- Wave diffraction

Wave shoaling and breaking



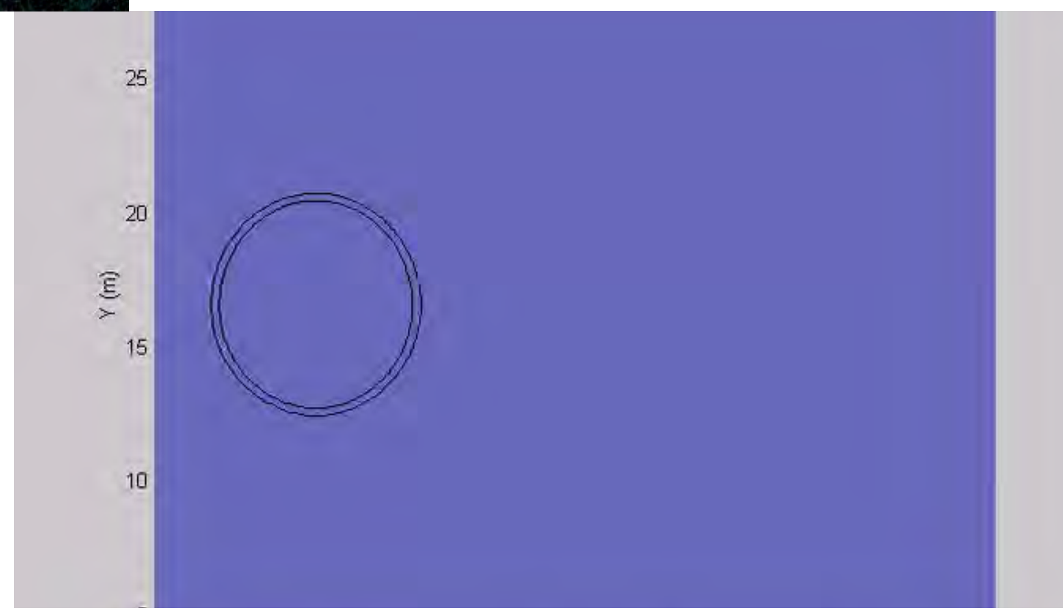


Wave refraction



Wave changes direction as water depth changes

Long crest waves refract over a submerged shoal



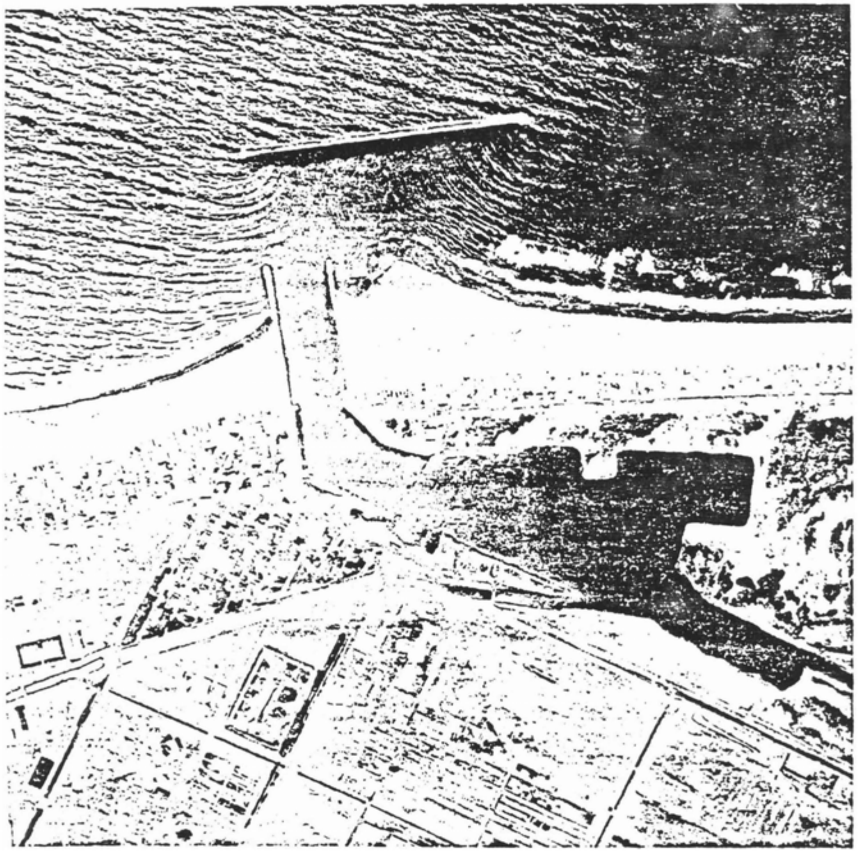
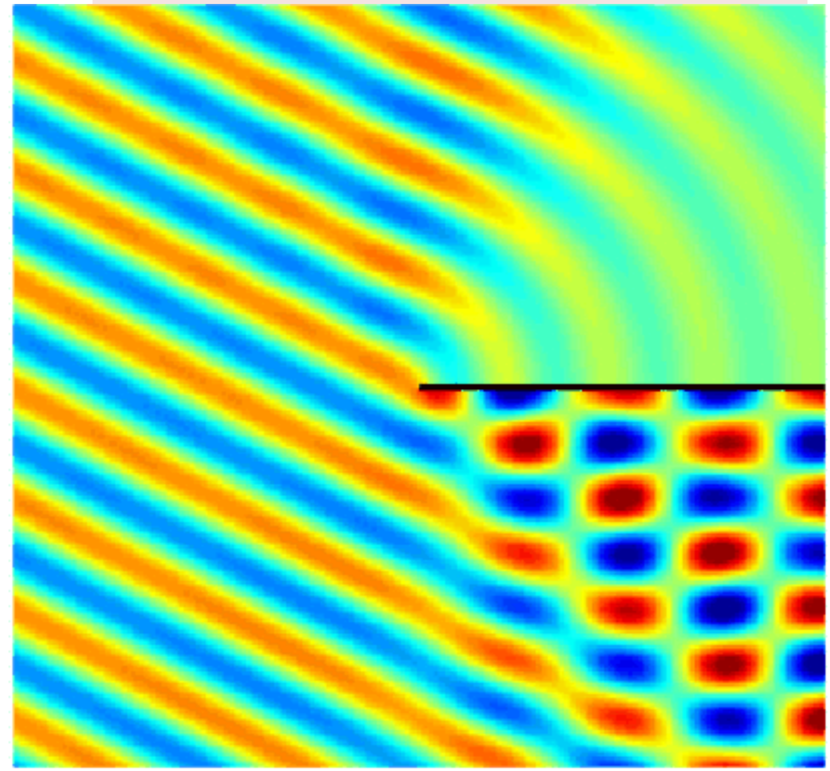
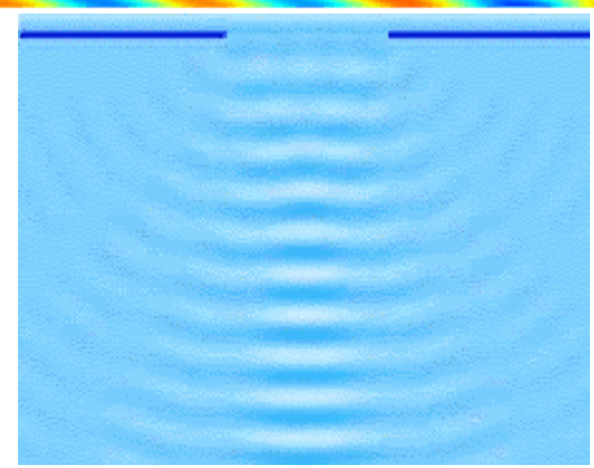


FIGURE 1-49 WAVE DIFFRACTION AT CHANNEL ISLANDS HARBOR BREAKWATER, CALIFORNIA

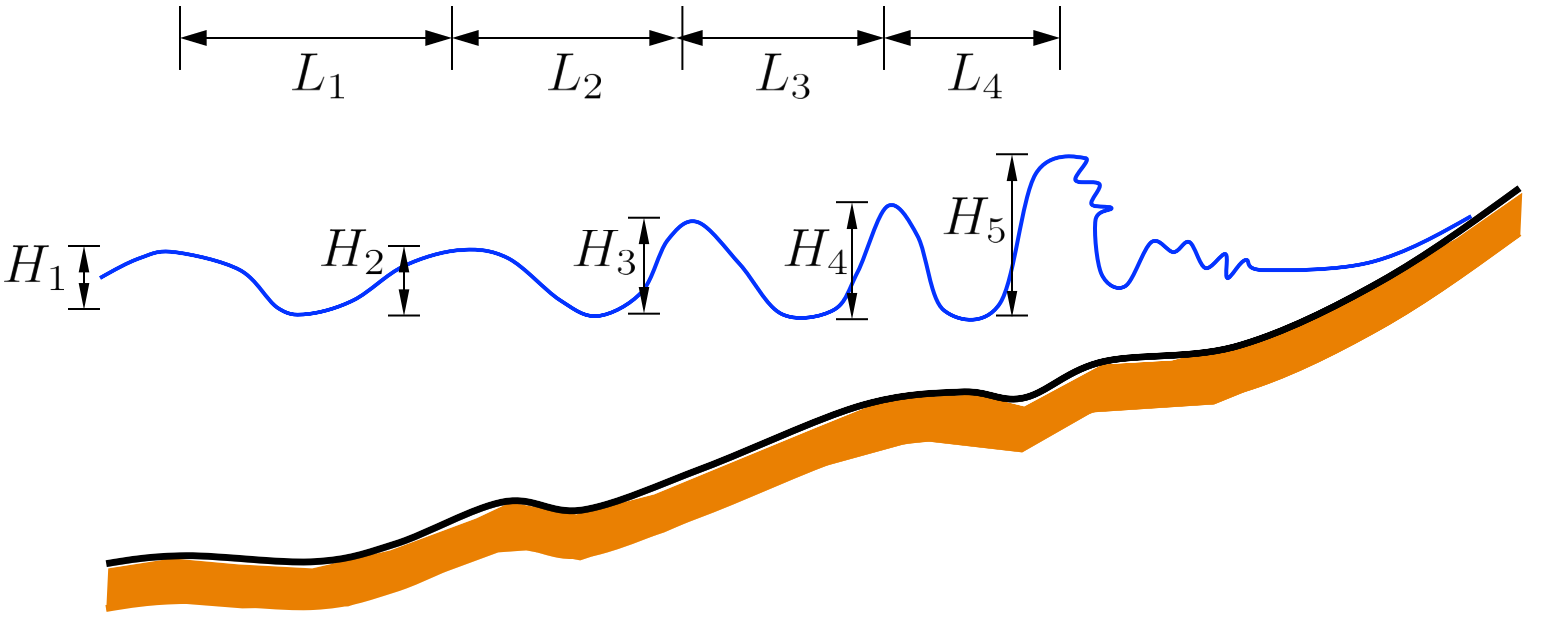
Wave diffraction by An offshore breakwater



Wave diffraction by a pair of breakwaters



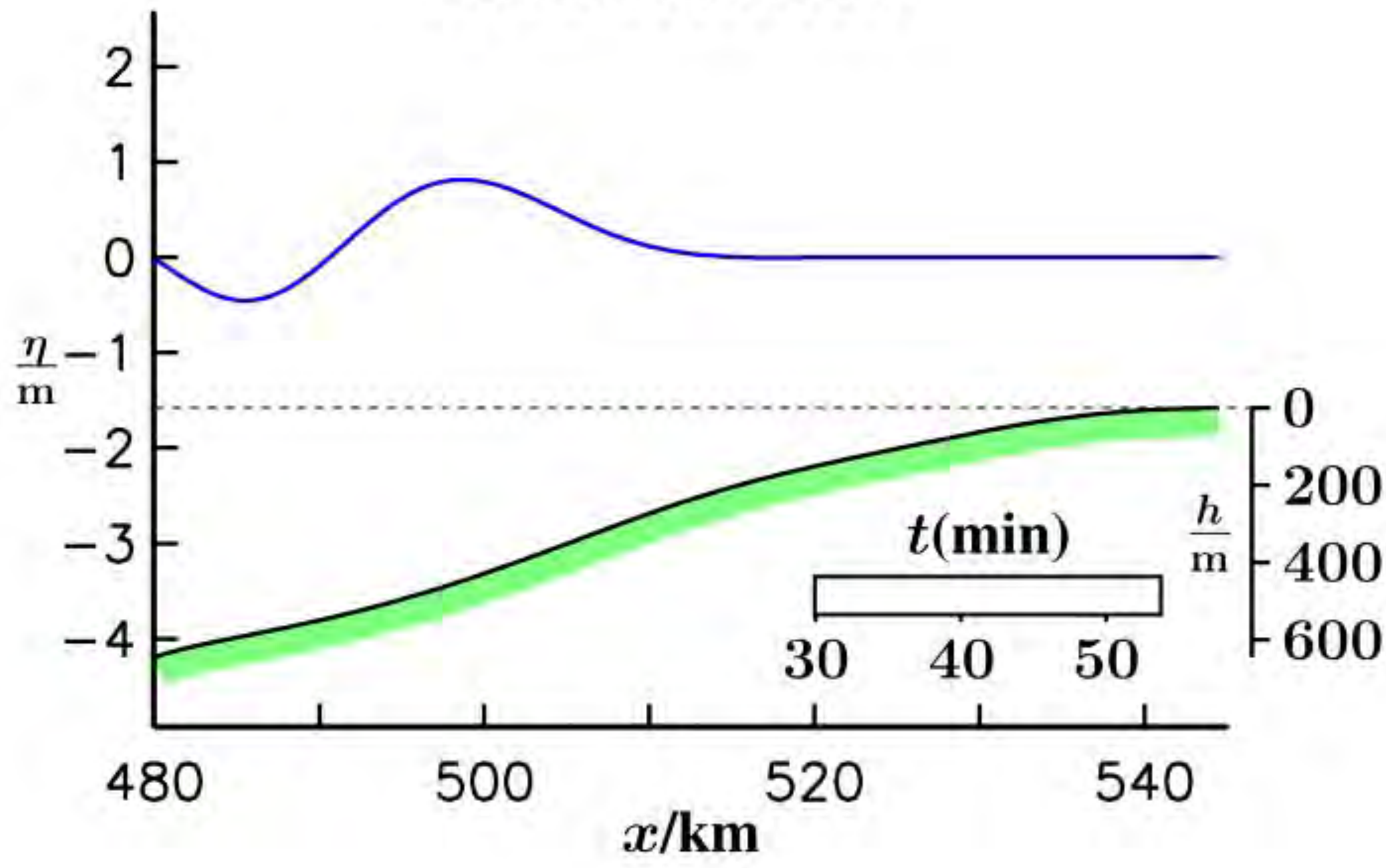
Wave shoaling in shallow waters



As waves reach shallower water, intuitively

- phase speed, c, decreases
- wavelength, $L = Tc$, decreases
- group velocity, c_g, decreases
- wave height, H, increases (assuming no energy loss)

Wave shoaling



Mild slope assumption

- For general uneven bottom, $h = h(x)$, $\Rightarrow$ solve the Laplace equation
- For slowly varying depth $\Rightarrow$ approximate theory

Slowly varying depth:

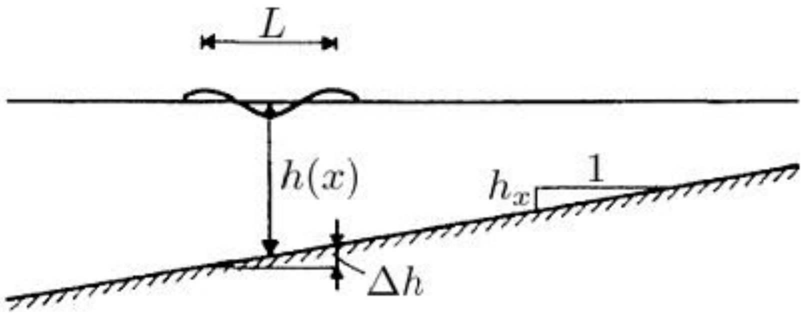
- Over a wavelength, the relative depth variation is small:

$$\frac{\Delta h}{h} \ll 1 \Rightarrow \frac{L\frac{dh}{dx}}{h} \ll 1 \quad \text{or} \quad \frac{1}{kh}\frac{dh}{dx} \ll 1$$

- Locally, we have constant water depth

$\Rightarrow$ the constant depth solutions can be applied locally

$$\begin{cases} \eta = A(x)\cos\left[\int^x k(x')dx' - \omega t\right] \\ c(x) = \dfrac{\omega}{k(x)} \\ \omega^2 = gk(x)\tanh k(x)h(x) \end{cases}$$



Wave shoaling over slowly varying depth

- "0": known quantities at some reference location offshore
- Energy conservation:

$$\overline{\mathcal{F}} = \overline{E} c_g = \text{constant} \Rightarrow \frac{A(x)}{A_0} = \sqrt{\frac{c_{g0}}{c_g(x)}} \quad \left(\because \overline{E} = \frac{1}{2}\rho g A^2 \right)$$

$$\text{Note: } c_g = \frac{1}{2}\frac{\omega}{k}\left[1 + \frac{2kh}{\sinh 2kh}\right]$$

- In shallow water, $kh \ll 1$: $\dfrac{A}{A_0} = \sqrt{\dfrac{c_{g0}}{\sqrt{gh}}}$, $\;L = cT = \sqrt{gh}T$

$$\boxed{\frac{A}{h} \sim h^{-\frac{5}{4}}} \quad \left(\text{if } k_0 h_0 \ll 1 \text{ is also true, } \frac{A}{A_0} = \left(\frac{h_0}{h}\right)^{\frac{1}{4}} \text{ (Green's law)} \right)$$

$$\Rightarrow \boxed{\frac{h}{L} \sim h^{\frac{1}{2}}}$$

$$\boxed{\frac{A}{L} \sim h^{-\frac{3}{4}}}$$

Note:

- Mild slope assumption: gentle slope
- Eventually, waves will breaks

Shoaling effects

- Assume the reference wave solution ("0") is deep water wave
- Energy conservation:

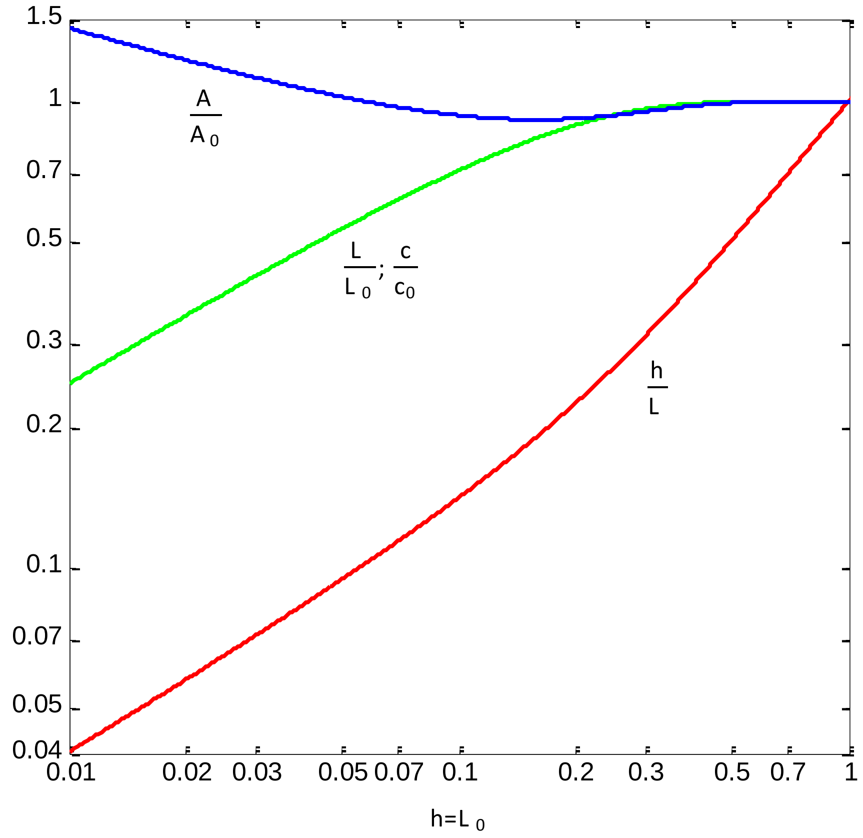
$$\frac{A}{A_0} = \sqrt{\frac{c_0}{c\left[1 + \frac{2kh}{\sinh 2kh}\right]}}$$

- Constant frequency:

$$\frac{L}{L_0} = \frac{c}{c_0} = \tanh kh$$

- $k_0 h = kh \dfrac{L}{L_0}$

$$\Rightarrow \left\{ \begin{array}{l} \frac{L}{L_0} \\ \frac{c}{c_0} \\ \frac{A}{A_0} \end{array} \right. \text{all functions of } k_0 h$$



Constancy of wave period

Wave period is independent of depth

(proof by contradiction)

- Consider two different locations:

$$\text{Point A: } h = h_A,\ T = T_A; \quad \text{Point B: } h = h_B,\ T = T_B$$

- Within a time intervel Δt :

$$\#\text{ of waves pass trhough point A: } N_A = \frac{\Delta t}{T_A}$$

$$\#\text{ of waves pass trhough point B: } N_B = \frac{\Delta t}{T_B}$$

- # of waves accumulated between two locations:

$$N = N_B - N_A = \Delta t \left(\frac{1}{T_2} - \frac{1}{T_1} \right)$$

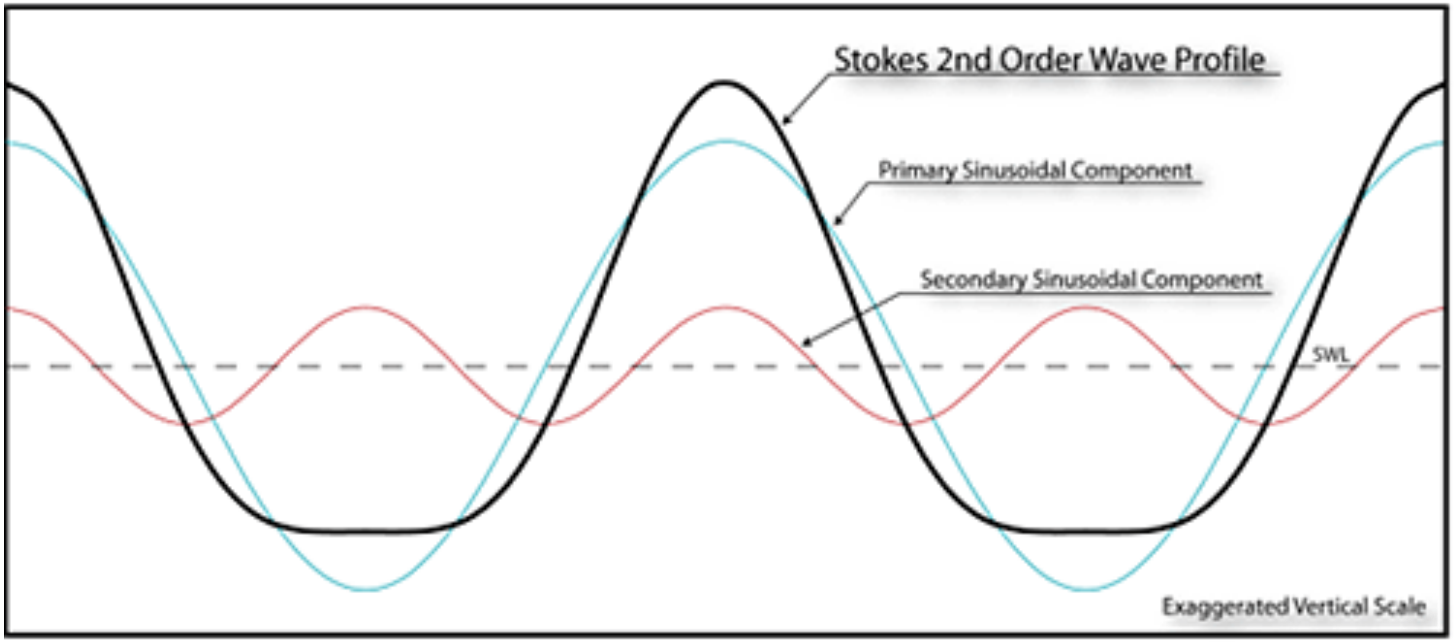
- Δt is arbitrary:

$$\lim_{\Delta t \to \infty} N = \lim_{\Delta t \to \infty} \Delta t \left(\frac{1}{T_2} - \frac{1}{T_1} \right) \to \pm\infty \to \text{can't be true!}$$

Effects of nonlinearity

- Quadratic nonlinearity generates higher and lower harmonics
- Wave crests become higher and wave trough become flatter

$$f(t) = \cos \omega t; \quad f^2(t) = \frac{1}{2}\left[\cos 2\omega t + 1\right]$$



Effects of nonlinearity

- **The phase velocity (wave celerity) depends on wave amplitude (wave heights)**

$$c = \sqrt{gh} \quad \rightarrow \quad c = \sqrt{g(\eta + h)}$$

This is also called amplitude dispersion

- **The wave crest propagates fastest. Wave front becomes steep and the wave back gets stretched. The surface takes a saw-tooth shape.**

- **Surface profile becomes "unstable" and wave breaking occurs**

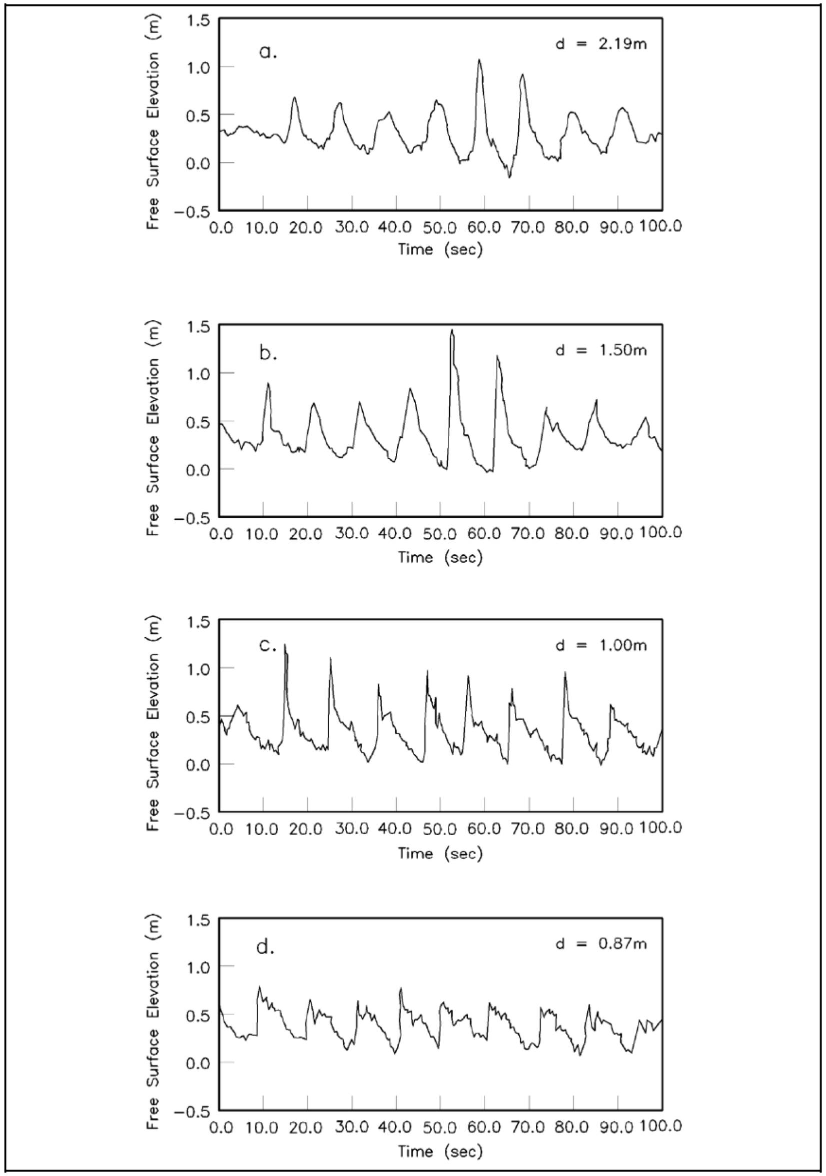


Figure II-4-3. Change in wave profile shape from outside the surf zone (a,b) to inside the surf zone (c,d). Measurements from Duck, NC (Ebersole 1987)

Wave breaking in shallow water

Breaking waves:

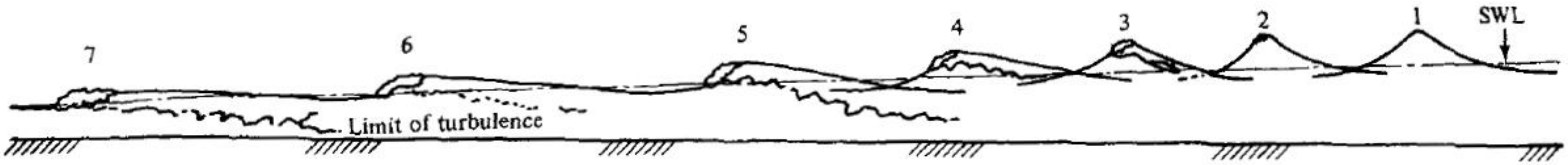
- $A \sim h^{-1/4}$, but A must be finite $\Rightarrow$ at some depth, the wave becomes unstable as the particle velocity exceeds the phase speed
- Wave energy is dissipated in the form of turbulence and work against bottom friction $\Rightarrow$ causes the decreasing of wave amplitude
- Bottom condition is important in the breaking waves
- Wave breaking can also occur in deep water when waves are too large
- Experimental approach is very important in the study of breaking waves

Typical types of wave breaking on beaches:

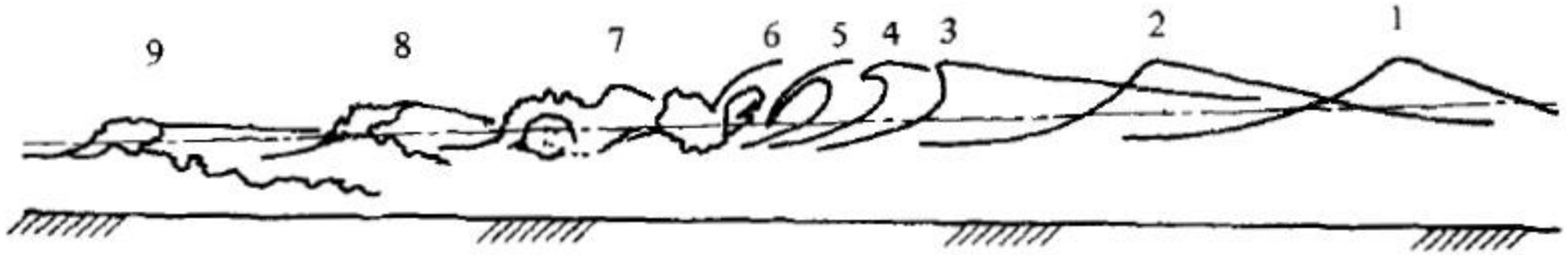
- Spilling breaker: on very mildly sloping beaches
- Plunging breaker: on steeper beaches
- Surging breaker: on very steep beacges

Breaking waves

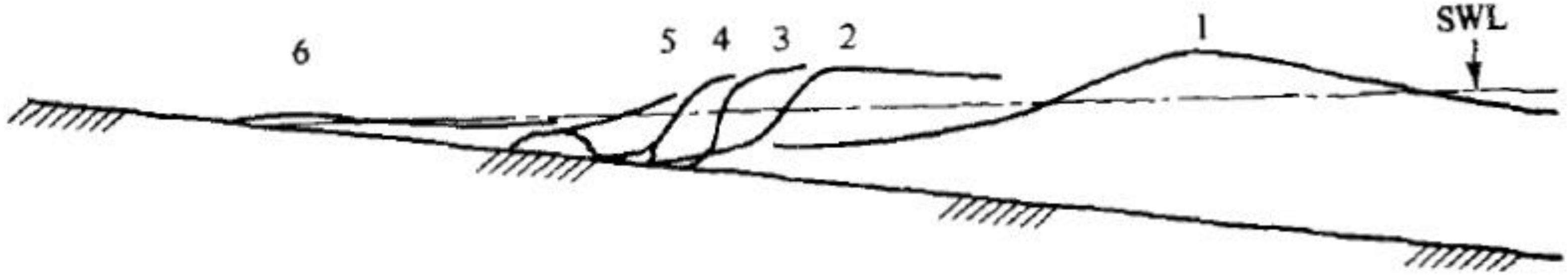
<u>Spilling breakers</u>: wave crests spill forward, creating foam and turbulent water, as wave fronts travel across a gently-sloped beach



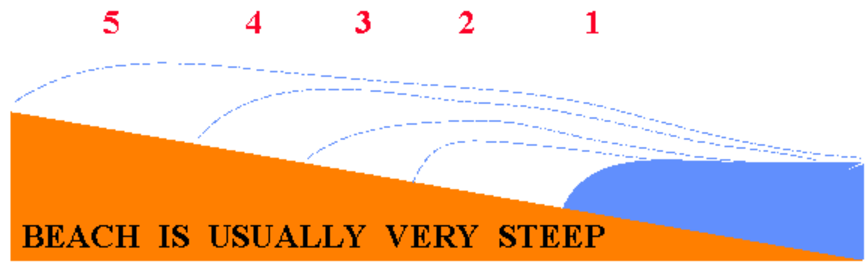
<u>Plunging breakers</u>: wave crests form spectacular open curl; crests fall forward with considerable force, dissipating energy in a well-defined area on a moderately-sloped beach



<u>Surging breakers</u>: long, relatively low waves whose front faces and crests remain relatively unbroken as waves slide up and down a steeply-sloped beach



SURGING BREAKER

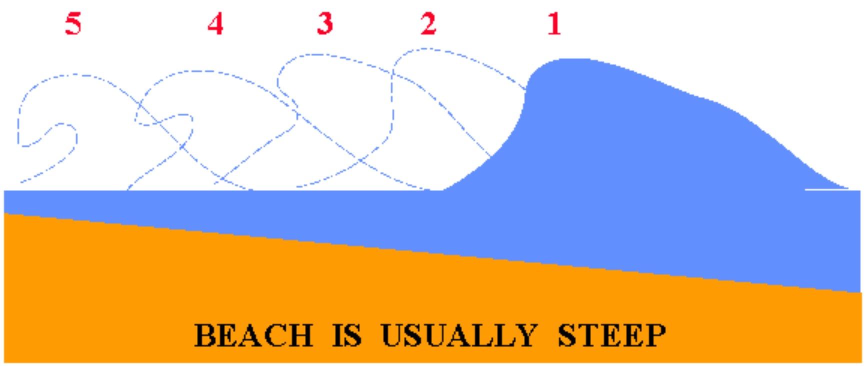


• Associated with very steep beaches

• Wave starts as a plunging, then the wave catches up with the crest, and the breaker surges up the beach face as a wall of water (with the wave crest and base traveling at the same speed)

•Results in a quickly rising and falling water level on the beach face

PLUNGING BREAKER



- Typical “surfer” waves
- Associated with steeper beaches
- Waves break very quickly and with substantial force
- Wave energy is released suddenly as the crest curls and then descends violently

SPILLING BREAKER

- **Most common breaking waves**

- **Associated with a moderate beach gradient (1:15 to 1:30 slope)**

- **Waves break slowly as they approach the shore**

- **The wave energy is gradually released over time and the beach**

Surf similarity parameter: Iribarren number

$$\begin{cases} I_{rb} = \dfrac{m}{\sqrt{H_b/L_0}} \\ I_{r0} = \dfrac{m}{\sqrt{H_0/L_0}} \end{cases},$$

m: beach slope

H_b: wave height at the breaking point

H_0: deep water wave height

L_0: deep water wavelength

I_{rb}	Breaker types	I_{r0}
$I_{rb} > 2.0$	Surging type	$I_{r0} > 3.3$
$0.4 < I_{rb} < 2.0$	Plunging type	$0.5 < I_{r0} < 3.3$
$I_{rb} < 0.4$	Spilling type	$I_{r0} < 0.5$

Empirical formulae

▷ Breaking condition:

- McCowan (1894): $\dfrac{H_b}{h_b} = 0.78$ (solitary wave)
- Miche (1944): $\dfrac{H_b}{L_b} = \dfrac{1}{7}\tanh\dfrac{2\pi h}{L}$
- Goda (1970): $\dfrac{H_b}{L_b} = 0.17\left\{1 - \exp\left[-1.5\dfrac{\pi h_b}{L_0}\left(1 + 15m^{4/3}\right)\right]\right\}$

▷ Breaker height and break depth:

- Weggel (1972): monochromatic waves, smooth plan beaches

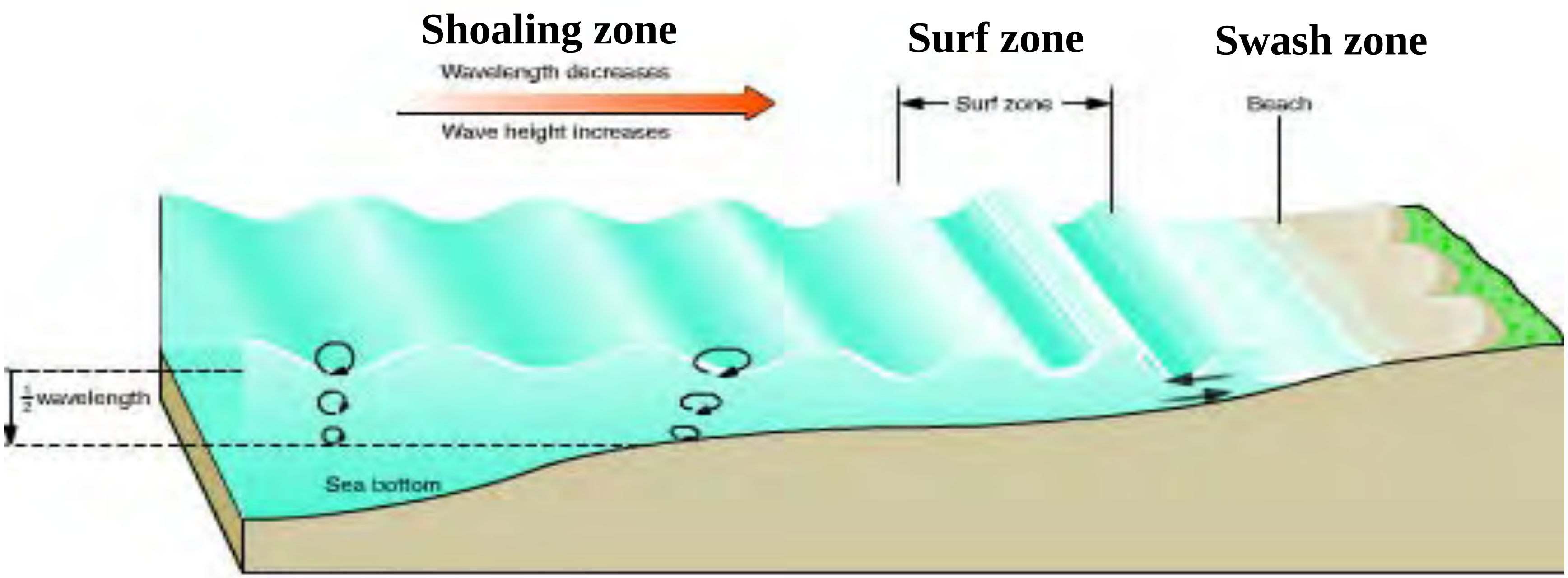
$$\frac{H_b}{h_b} = \kappa = \frac{1.56}{1 + e^{-19.5m}} - 43.8\left(1 - e^{-19m}\right)\frac{H_b}{gT^2}$$

- Shore Protection Manual (1984):

$$\frac{H_b}{h_b} = 0.75 + 25m - 112m^2 + 3870m^3 \quad \text{(solitary wave)}$$

- $H_b = \left(\dfrac{\kappa}{g}\right)^{1/5}\left(\dfrac{1}{2}c_0 H_0^2\right)^{2/5} \Rightarrow \dfrac{H_b}{H_0} = \left(\dfrac{\kappa}{16\pi^2}\right)^{1/5}\left(\dfrac{H_0}{gT^2}\right)^{-1/5}$

Surf zone and swash zone



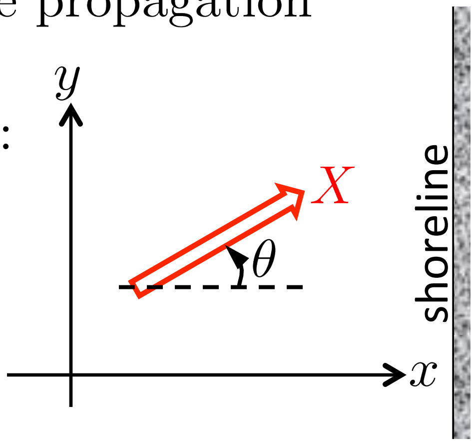
- In the surf zone wave breaking dominates and turbulence is significant;
- In the swash zone the beach face is dry and wet alternatively;
- Bottom friction effects are important in both zones;
- Significant sediment transport occurs in these regions.

Wave refraction

<u>Refraction</u>: change of wave propagation direction due to the depth variations

- Small amplitude progressive wave:

$$\begin{cases} \eta(X,t) = A\cos(kX - \omega t) \\ \omega^2 = gk\tanh kh \end{cases}, \quad X\text{-axis: direction of wave propagation}$$



- General expression in the Cartesian coordinate system:
  - ▷ $X = x\cos\theta + y\sin\theta$
  - ▷ $\eta(x,y,t) = A\cos\left((k\cos\theta)x + (k\sin\theta)y - \omega t\right)$
  - ▷ Define wavenumber vector, $\vec{k}$:

$$\vec{k} = (k_x, k_y) = (k\cos\theta, k\sin\theta), \ k = \left|\vec{k}\right| = \sqrt{k_x^2 + k_y^2}$$

$$\Rightarrow \eta(x,y,t) = A\cos\left(\vec{k}\cdot\vec{x} - \omega t\right), \ \vec{x} = (x,y)$$

  - ▷ Introduce phase function, S:

$$S(x,y,t) = S(\vec{x},t) = \vec{k}\cdot\vec{x} - \omega t \ \text{ (phase line: connection of constant } S)$$

$$\Rightarrow \boxed{\eta(\vec{x},t) = A\cos S}$$

Some important properties

- Oblique incident wave: $\eta(\vec{x},t) = A\cos S,\ S = k_x x + k_y y - \omega t$
- Properties of phase function:

$$\frac{\partial S}{\partial t} = -\omega,\ \nabla S = \left(\frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}\right) = (k_x, k_y) = (k\cos\theta, k\sin\theta)$$

- $\boxed{k^2 = \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2}$ (eikonal equation)

- $\dfrac{\partial(\nabla S)}{\partial t} = \nabla\left(\dfrac{\partial S}{\partial t}\right) \Rightarrow \boxed{\dfrac{\partial \vec{k}}{\partial t} + \left(\dfrac{\partial \omega}{\partial x} + \dfrac{\partial \omega}{\partial y}\right) = 0}$

(conservation of wavenumber)

- $\nabla \times \vec{k} = \nabla \times \nabla S = 0 \Rightarrow \boxed{\dfrac{\partial k\sin\theta}{\partial x} - \dfrac{\partial k\cos\theta}{\partial y} = 0}$

(Irrotational condition of wavenumber)

If $\dfrac{\partial}{\partial y} = 0 \Rightarrow \dfrac{d}{dx}(k\sin\theta) = 0 \Rightarrow \boxed{\dfrac{\sin\theta}{c} = \text{constant}}$ (Snell's law)

Wave refraction

In general the free surface profile of simple harmonic waves can be expressed as $\eta(\vec{x},t) = A\cos S(\vec{x},t)$, where the phase function $S(\vec{x},t)$ has the following important properties

$$\frac{\partial S}{\partial t} = -\omega, \quad \nabla S = \vec{k}.$$

$$\frac{\partial \vec{k}}{\partial t} + \nabla\omega = 0. \quad \Leftarrow \text{ Conservation of number of waves}$$

Furthermore

$$k^2 = \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 \Leftarrow \text{ Eikonal equation} \quad \text{and } \nabla\times\vec{k} = 0 \text{ or}$$

$$\frac{\partial k\cos\theta}{\partial y} - \frac{\partial k\sin\theta}{\partial x} = 0$$

For a given bathemetry $h(\vec{x})$ and wave frequency ω, one can find the value of the wave number from the dispersion relation

$$\omega^2 = gk\tanh kh$$

The equation above can be used to solve the θ. This is a nonlinear equation.

Conservation of Energy

$\bar{E} = \frac{1}{2}\rho g A^2,$ wave energy intensity averaged over one wave period

$\bar{E}\vec{C}_g = \frac{1}{2}\rho g A^2 \frac{d\omega}{dk}\frac{\vec{k}}{k},$ wave energy flux averaged over one wave period.

Conservation of energy averaged over one wave period

$$\nabla \bullet \left(\bar{E}\vec{C}_g\right) = 0.$$

or

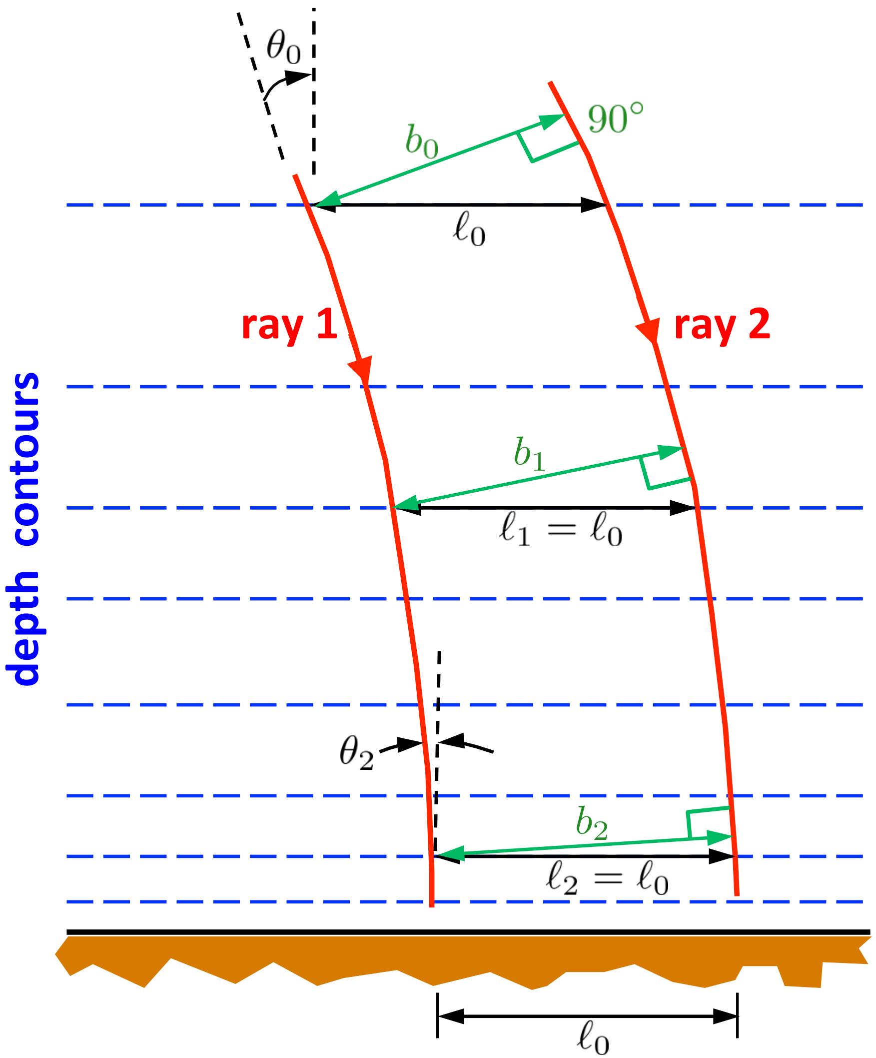
$$\frac{\partial}{\partial x}\left[\left(\frac{1}{2}\rho g \frac{d\omega}{dk}\cos\theta\right)A^2\right] + \frac{\partial}{\partial y}\left[\left(\frac{1}{2}\rho g \frac{d\omega}{dk}\sin\theta\right)A^2\right] = 0.$$

This is a linear PDE for A^2.

Determination of refraction pattern: ray tracing

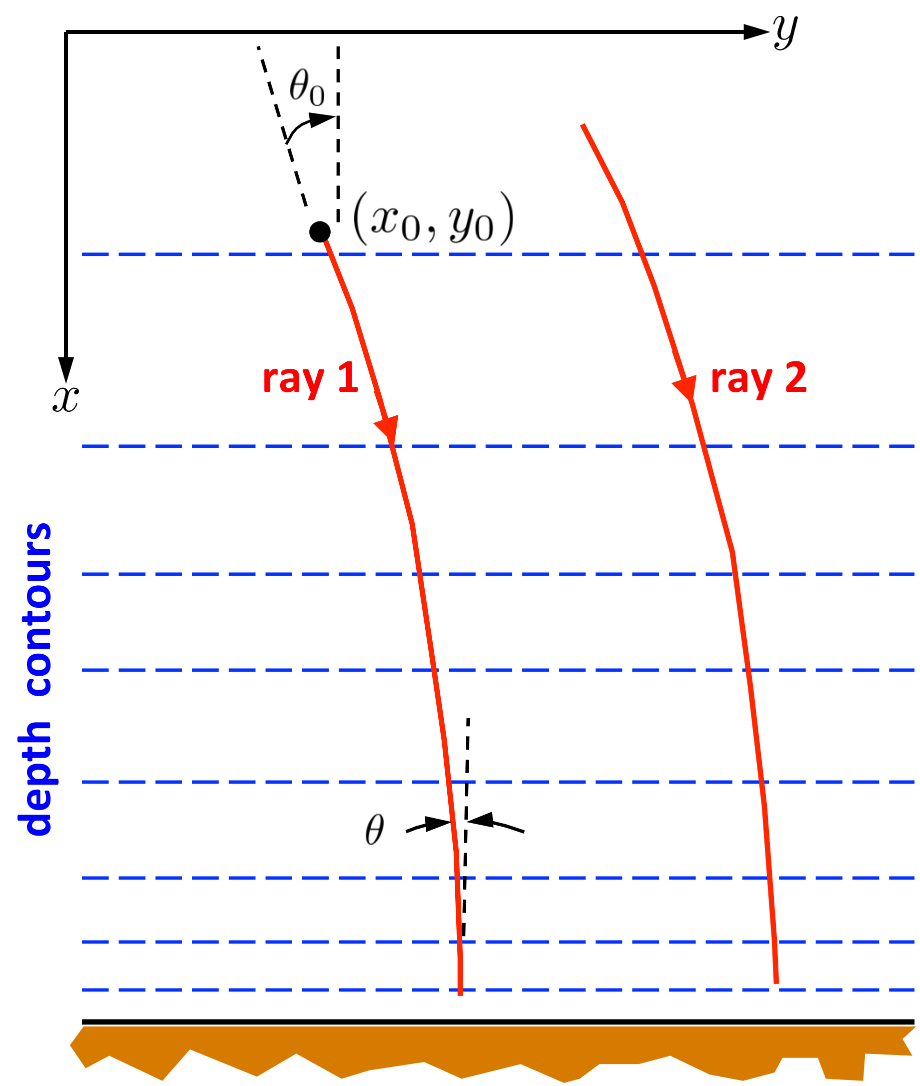
If $h(x)$, then $\partial/\partial y = 0$.

$$\frac{\partial k \sin\theta}{\partial x} = 0 \Rightarrow \frac{\sin\theta}{c} = \text{constant}$$



- Ray (orthogonal): a line drawn perpendicularly to the wave crests
- Depth contours are straight and parallel to to the shoreline
  $\Rightarrow$ ray 1 and ray 2 are parallel
- Conditions at "0" are given
- $\boxed{\text{Snell's law}}$ $\dfrac{\sin\theta_0}{c_0} = \dfrac{\sin\theta_1}{c_1} = \dfrac{\sin\theta_2}{c_2}$
  - $\triangleright$ $\theta \downarrow$ since h becomes shallower
  - $\triangleright$ $K_r = \sqrt{\dfrac{\cos\theta_0}{\cos\theta}}$ (refraction coefficient)
- $\boxed{\text{Amplitude variation}}$
  - $\triangleright$ Assume no energy loss, no reflection
  - $\triangleright$ $\dfrac{A}{A_0} = K_s K_r,\ K_s = \sqrt{\dfrac{c_{g0}}{c_g}}$ (shoaling coefficient)

Wave ray tracing



- Start from (x_0, y_0): wave conditions are known at this location
- $\dfrac{dy}{dx} = \dfrac{k \sin\theta}{k \cos\theta}$
- Snell's law: $k \sin\theta = \text{constant } \mathcal{K}$

  $(\mathcal{K} = k_0 \sin\theta_0 \text{ at } (x_0, y_0))$

$$\Rightarrow k\cos\theta = \pm\sqrt{k^2 - \mathcal{K}^2}$$

$$\Rightarrow \frac{dy}{dx} = \pm\frac{\mathcal{K}}{\sqrt{k^2 - \mathcal{K}^2}}$$

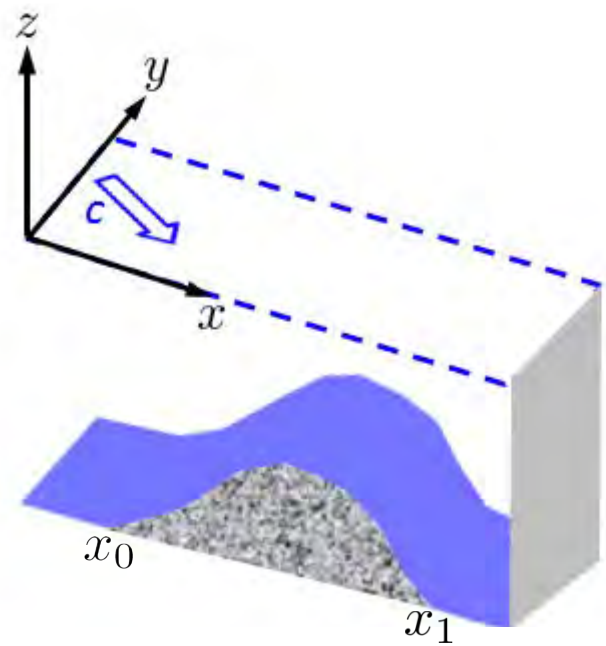
$$\Rightarrow y = y_0 \pm \int_{x_0}^{x} \frac{K}{\sqrt{k^2 - \mathcal{K}^2}} dx$$

Note:

$$\omega^2 = gk\tanh kh \Rightarrow \text{solve for } k$$

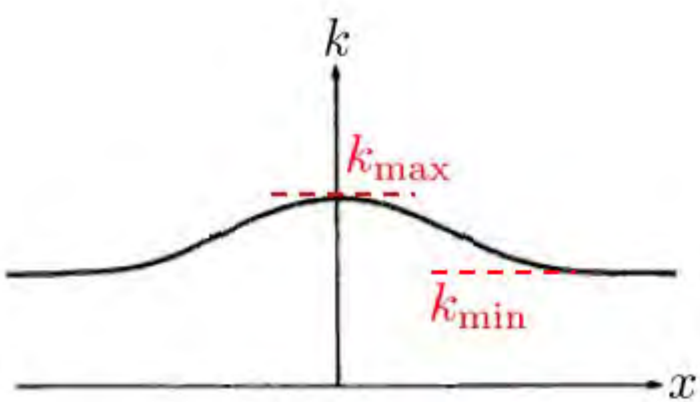
Question: can $k \leq \mathcal{K}$?

Waves over a submarine ridge



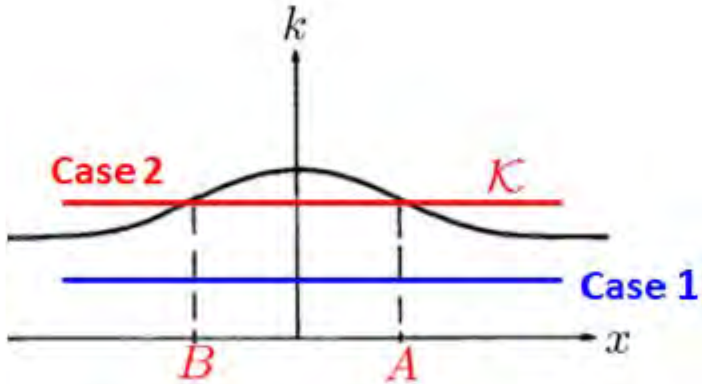
Plan wave passing over a submarine ridge

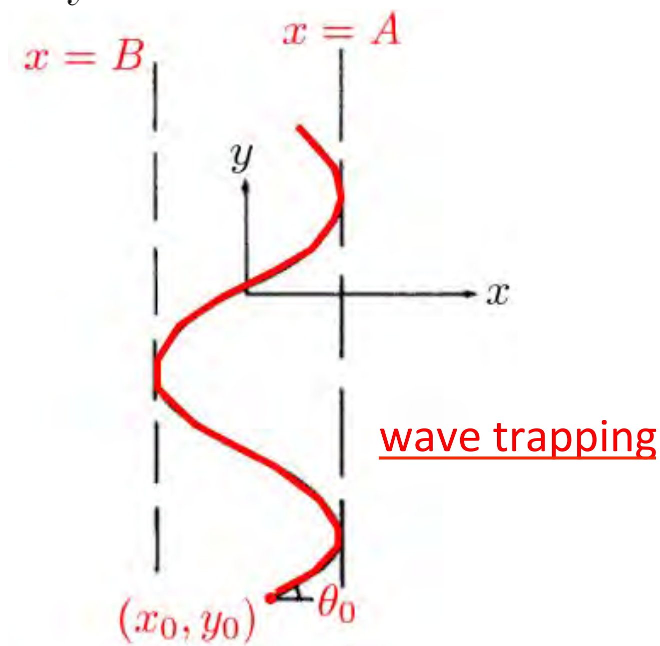
- $h = h(x)$
- Parallel incident wave rays enter the ridge or trough at $x = x_0$ with angle θ_0
- Wave conditions are given at $x = x_0$



Wavenumber: $k_{\min} \leq k \leq k_{\max}$

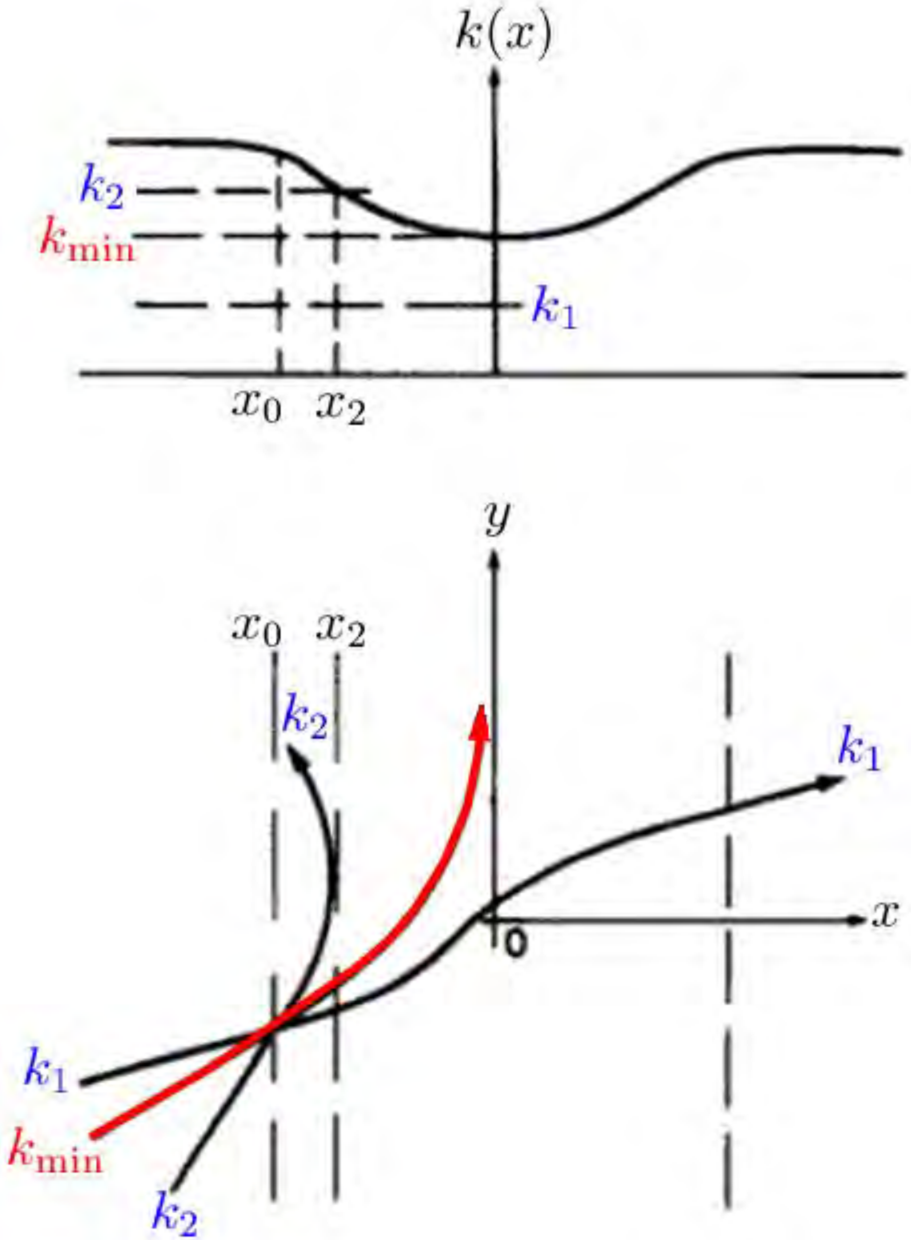
- Case 1: $\mathcal{K} = k_0 \sin\theta_0 < k_{\min}$ everywhere $\left(\text{i.e., } \sqrt{k^2 - \mathcal{K}^2} \text{ is real}\right)$
  - ▷ θ: decreasing $\to 0 \to$ increasing (due to the change of h)
  - ▷ For a limiting case where the summit of the ridge is above the mean water level $\Rightarrow$ wave rays would finally strike the shoreline perpendicularly ($\because k \to \infty$ as $h \to 0$)





- Case 2: $k_{\min} < \mathcal{K} < k_{\max}$
  - ▷ Rays can only exist in the region $k > \mathcal{K}$ (or, say, in the region $B < x < A$, where $k = \mathcal{K}$ at A and B)
  - ▷ Rays bounce back and forth within $B < x < A$ while advance in y-direction: this is called $\boxed{\text{wave trapping}}$
  - ▷ $\dfrac{dy}{dx} \to \pm\infty$ at A and B, respectively: A and B are called $\boxed{\text{caustics}}$

Waves over a submarine trough



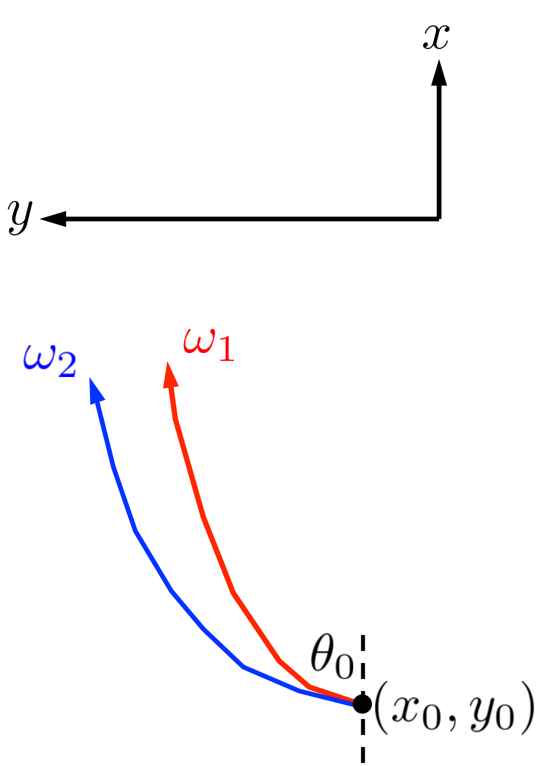
- $h = h(x)$
- $k_0 \sin\theta_0 = \mathcal{K} = k_1 < k_{\min}$

  Incident rays bend first toward, then away from the axis of the trough and pass the trough to the right side
- $\mathcal{K} = k_{\min}$

  Rays become parallel to the depth contours
- $\mathcal{K} = k_2 > k_{\min}$

  $x = x_2$ ($k = k_2$ at $x = x_2$) is a caustic

Directional spreading caused by refraction



Consider two wave trains with frequencies ω_1 and ω_2 propagating into shallower water. At point (x_0, y_0), they have the same incident angle, θ_0.

- Assumption: $\dfrac{d}{dy} = 0$
- The Snell's law requires:

$$\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_0}{c_{10}}, \qquad \frac{\sin\theta_2}{c_2} = \frac{\sin\theta_0}{c_{20}}$$

- If incident waves are deep water waves:

$$c_{10} = \frac{g}{\omega_1}, \qquad c_{20} = \frac{g}{\omega_2}$$

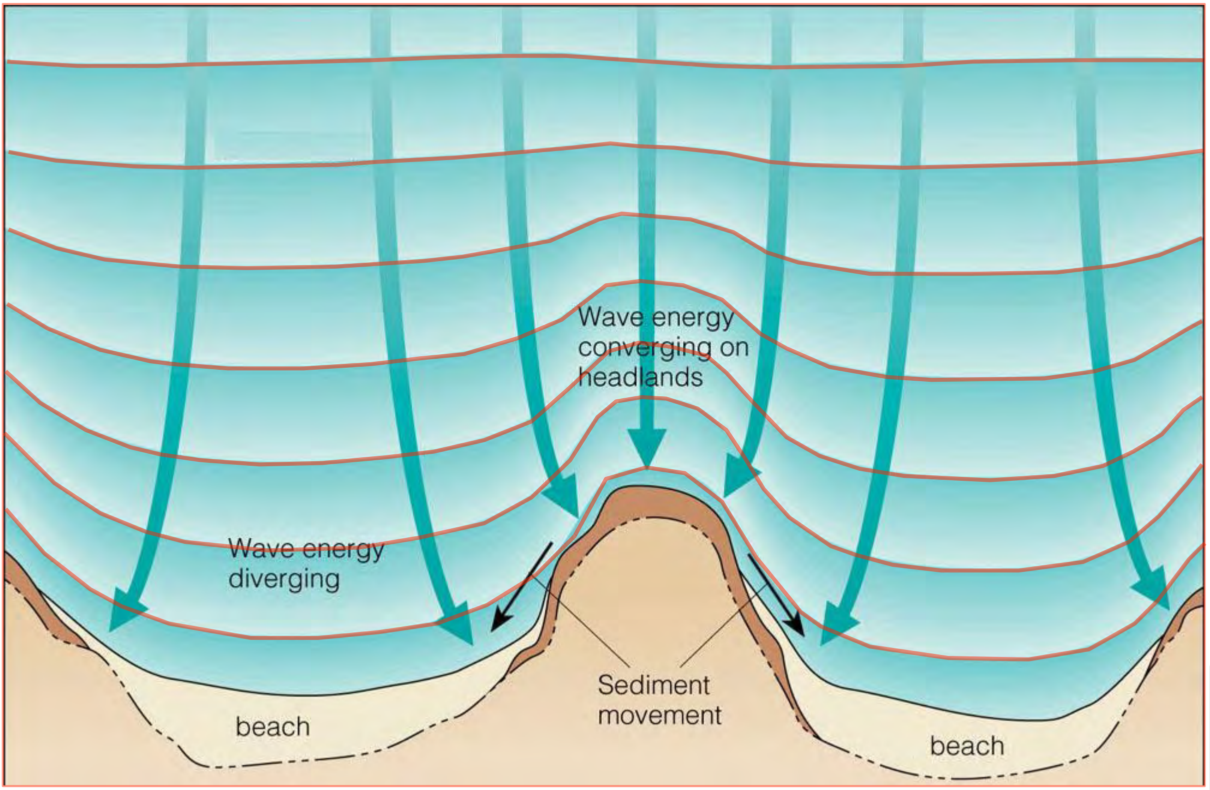
- In shallow water, waves become long waves:

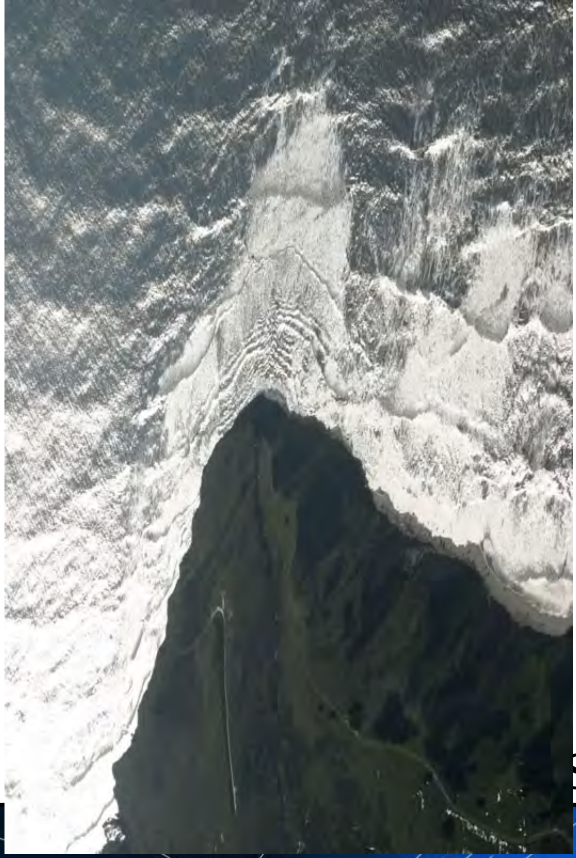
$$c_1 = c_2 = \sqrt{gh}$$

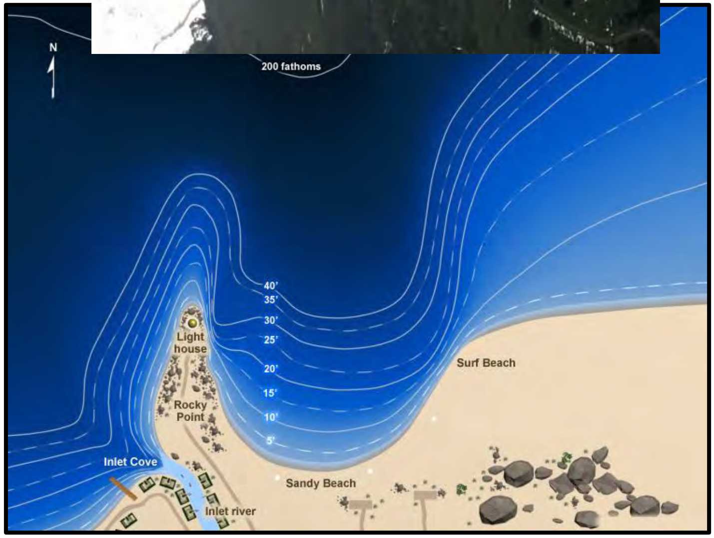
- Finally, we obtain:

$$\theta_1 > \theta_2 \text{ if } \omega_1 > \omega_2 \text{ (vice versa)}$$

Refraction







- Wave energy concentrates on headlands
- Wave energy spreads out in bays

Summary on the wave ray approach

- Provides a very good visualization on the wave propagation pattern
- The method breaks when rays cross each other (ray focusing) or tangential to a common curve (caustics)
- Can not deal with reflection and diffraction
  - Wave amplitude must vary slowly