Computational Fluid Dynamics based on the Unified Coordinates

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In Memory of Professor Jaw-Yen Yang

National Taiwan University
3 December, 2016

I am privileged to be a friend of Prof. Jaw-Yen Yang for the past 26 years beginning in 1990 when I was a visiting Professor to the Institute of Applied Mechanics, NTU. As a new comer to CFD I gave a graduate course on the theory and computation of supersonic flow, exploring my New Lagrangian Coordinate method. With his sharp scientific mind, J Y immediately extended the method and we completed 3 joint papers in a short time.

Since then I have generalized the method to “Unified Coordinates”, mostly done during 2005-6 when I was invited by Prof CC Chang to work at the Academia Sinica. Prof. K Xu and I have since published a monograph in 2012 to summarize its achievements.

Wai-How Hui
Kun Xu
Computational Fluid Dynamics Based on the Unified Coordinates

Computational Fluid Dynamics Based on the Unified Coordinates reviews the relative advantages and drawbacks of Eulerian and Lagrangian coordinates as well as the Arbitrary Lagrangian-Eulerian (ALE) and various moving mesh methods in Computational Fluid Dynamics (CFD) for one- and multi-dimensional flows. It then systematically introduces the unified coordinate approach to CFD, illustrated with numerous examples and comparisons to clarify its relation with existing approaches. The book is intended for researchers and practitioners in the field of Computational Fluid Dynamics.

Emeritus Professor Wai-Hou Hui and **Professor Kun Xu** both work at the Department of Mathematics of the Hong Kong University of Science & Technology, Hong Kong, China.

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Computational Fluid Dynamics Based on Unified Coordinates

Wai-How Hui and Kun Xu

(统一坐标系下的计算流体力学方法)

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In memory of Prof. Yang, it is fitting to give a brief and up-to-date report on the achievements of the Unified Coordinate method in CFD.

Content

1. Role of Coordinates in CFD

Two coordinate systems

for describing fluid motion have existed:

Eulerian coordinates (mesh) are fixed in space

Lagrangian coordinates (mesh) move with fluid

Are they equivalent theoretically?

"Yes" for 1-D flow (DH Wagner, *J. Diff. Eq.*, **64,** 118-136, 1987)

"**No**" for 2- and 3-D flow (Hui, Li and Li, *J. Comput Phys.,* **153**, 596-637, 1999)

Computationally,

they are not equivalent, even for 1-D flow

In general, for 2-D and 3-D Flow

Eulerian method is relatively **simple**, because the Euler equations of gas dynamics can be written in **conservation** *PDE* **form**, which is the basis for shock-capturing methods, but

(a) it smears contacts badly, and

(b) it needs generating a body-fitted mesh

Lagrangian method **resolves contacts sharply**, but

(a) the gas dynamics equations could not be written in conservation PDE form, and
(b) it breaks down due to cell deformation.

2. The Unified Corrdinates (λ,ξ,η,ζ)

are given by the transformation from Eulerian (*t, x, y, z*)

$$dt = d\lambda$$
$$dx = Ud\lambda + Ad\xi + Ld\eta + Pd\varsigma$$
$$dy = Vd\lambda + Bd\xi + Md\eta + Qd\varsigma$$
$$dz = Wd\lambda + Cd\xi + Nd\eta + Rd\varsigma$$

We get

$$\frac{D_Q}{Dt}\begin{pmatrix}\xi \\ \eta \\ \varsigma\end{pmatrix} = 0 \qquad \frac{D_Q}{Dt} \equiv \frac{\partial}{\partial t} + Q \cdot \nabla_{\vec{x}}$$

So (ξ,η,ζ), and hence the computational mesh, move with arbitrary velocity $Q = (U, V, W)$ -- the mesh velocity.

Compatibility Conditions

(only nine of them are independent)

$$\frac{\partial A}{\partial \lambda}=\frac{\partial U}{\partial \xi},\qquad \frac{\partial L}{\partial \lambda}=\frac{\partial U}{\partial \eta},\qquad \frac{\partial P}{\partial \lambda}=\frac{\partial U}{\partial \varsigma},$$

$$\frac{\partial B}{\partial \lambda}=\frac{\partial V}{\partial \xi},\qquad \frac{\partial M}{\partial \lambda}=\frac{\partial V}{\partial \eta},\qquad \frac{\partial Q}{\partial \lambda}=\frac{\partial V}{\partial \varsigma},$$

$$\frac{\partial C}{\partial \lambda}=\frac{\partial W}{\partial \xi},\qquad \frac{\partial N}{\partial \lambda}=\frac{\partial W}{\partial \eta},\qquad \frac{\partial R}{\partial \lambda}=\frac{\partial W}{\partial \varsigma}.$$

Time evolution

(Geometric conservation laws)

$$\frac{\partial A}{\partial \eta}=\frac{\partial L}{\partial \xi},\qquad \frac{\partial A}{\partial \varsigma}=\frac{\partial P}{\partial \xi},\qquad \frac{\partial L}{\partial \varsigma}=\frac{\partial P}{\partial \eta},$$

$$\frac{\partial B}{\partial \eta}=\frac{\partial M}{\partial \xi},\qquad \frac{\partial B}{\partial \varsigma}=\frac{\partial Q}{\partial \xi},\qquad \frac{\partial M}{\partial \varsigma}=\frac{\partial Q}{\partial \eta},$$

$$\frac{\partial C}{\partial \eta}=\frac{\partial N}{\partial \xi},\qquad \frac{\partial C}{\partial \varsigma}=\frac{\partial R}{\partial \xi},\qquad \frac{\partial N}{\partial \varsigma}=\frac{\partial R}{\partial \eta}.$$

Free divergence constraints

Special Cases:

Eulerian $\boldsymbol{Q} = \boldsymbol{0}$

Lagrangian $\boldsymbol{Q} = \boldsymbol{q}$, $\boldsymbol{q}$ being fluid velocity

In the general case, we have a unified (Euler-Lagrangian) coordinate system with 3 degrees of freedom: U, V and W can be chosen freely.

- It is cruicial that the transformation be written in **differential form** so that the mesh velocity (U, V, W) can be chosen freely.

- Most people seek the transformation in **finite form**, e.g.
- $x = F(t, \xi, \eta, \zeta)$
- $y = G(t, \xi, \eta, \zeta)$
- $z = H(t, \xi, \eta, \zeta)$
- Then it is impossible to choose the mesh velocity freely
- and to write down these transformation functions.

- For example, if we choose the mesh velocity to be the fluid velocity, ie Lagrangian coordinates, then it is impossible to write down the transformation functions F, G and H. They are part of the solution: the fluid trajectories, which are of course to be determined by the unknown flow as well as the boundary and initial conditions.

- The **beauty of the differential transformation** is that you can freely choose the mesh velocity without working out the finite transformation functions, whose existence is guaranteed by the compatibility conditions; yet the governing equations can be written in closed conservation PDE form.

One way to determine the mesh velocity is**:**

For 1-D **choose the coordinate as a material one**

$\frac{D_q \xi}{Dt} = 0$ plus **adaptive Godunov scheme** (Lepage & Hui, *JCP*, 1995)

For 2-D **choose one coordinate as a material one**

$\frac{D_q \xi}{Dt} = 0$ plus **mesh-orthogonality preserving**

For 3-D **choose two coordinates as material ones**

plus **mesh-skewness preserving**

In this way, *UC* resolves contacts sharply without mesh tangling.

(There are other ways to choose the mesh velocity, see $ 7)

3. One-Dimensional Flow

It is shown that

UC (*Lagrangian coordinate* +

shock-adaptive Godunov scheme)

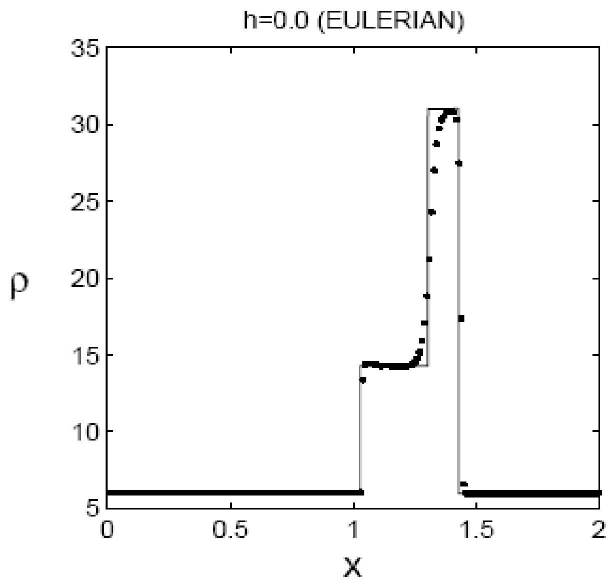
is superior to Lagrangian system

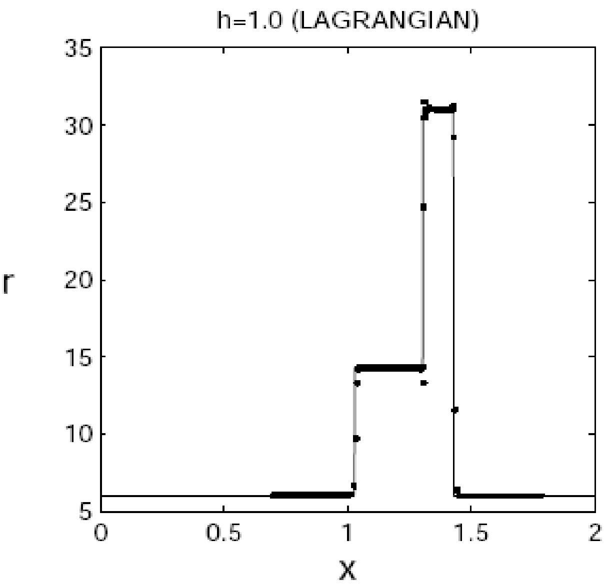
which, in turn, is superior to the

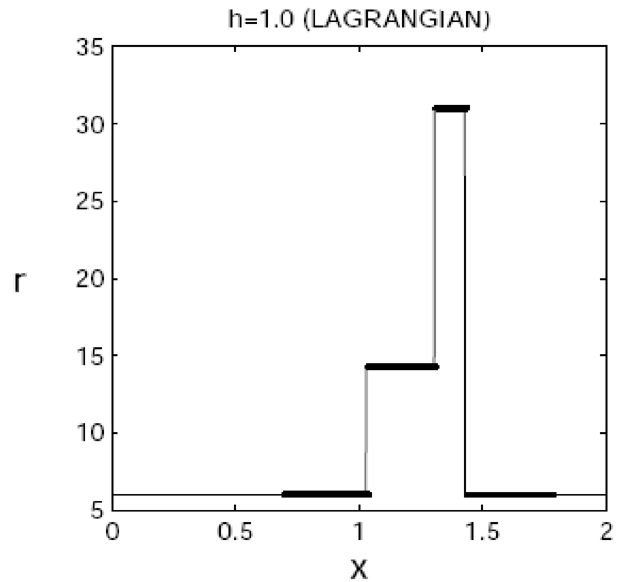
Eulerian system

(WH Hui and S Kudriakov, *JCP*, 2008)

Example 1. A Riemann Problem (Godunov-MUSCL)







UC (Lagrangian + Adaptive Godunov scheme)

_UC_ can cure all known defects of Eulerian and Lagrangian shock-capturing methods:

contact smearing
slow moving shocks
sonic-point glitch
wall-overheating*
start-up errors*
low-density flow*
strong rarefaction waves*
(*also shared in Lagrangian Computation)

UC is superior to Lagrangian and Eulerian system and is completely satisfactory.

The same holds for 2-D supersonic flow: the algorithm is the same using UC.

4. Multi-Dimensional Flow

For simplicity, we consider 2-D flow

2-D Gas Dynamic Equations

In Eulerian Coordinates

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \qquad \text{(E)}$$

where

$$E = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho e \end{pmatrix}, \quad F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u \left(e + \frac{p}{\rho} \right) \end{pmatrix}, \quad G = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v \left(e + \frac{p}{\rho} \right) \end{pmatrix}$$

$$e = \frac{1}{2}(u^2 + v^2) + \frac{1}{\gamma - 1}\frac{p}{\rho}$$

Eq. (E) is: (a) in conservation form,
(b) hyperbolic in *t*

Transforming to the unified coordinates, (E) become

$$\frac{\partial E}{\partial \lambda} + \frac{\partial F}{\partial \xi} + \frac{\partial G}{\partial \eta} = 0 \qquad \text{(U)}$$

where

$$E = \begin{pmatrix} \rho J \\ \rho J u \\ \rho J v \\ \rho J e \\ A \\ B \\ L \\ M \end{pmatrix}, \quad F = \begin{pmatrix} \rho X \\ \rho X u + pM \\ \rho X v - pL \\ \rho X e + p(uM - vL) \\ -U \\ -V \\ 0 \\ 0 \end{pmatrix}, \quad G = \begin{pmatrix} \rho Y \\ \rho Y u - pB \\ \rho Y v + pA \\ \rho Y e + p(vA - uB) \\ 0 \\ 0 \\ -U \\ -V \end{pmatrix}$$

With $J = AM - BL$, $X = (u - U)M - (v - V)L$ and $Y = (v - V)A - (u - U)B$.

Eq (U) is: (a) closed system in conservation form,

(b) hyperbolic in **λ**, except in Lagrangian

when $U = u$ and $V = v$

Remarks

Consequences of having a closed system of Governing equations in conservation form are:

(a) Effects of moving mesh on the flow are fully accounted for;

(b) The system can by solved as easily as the Eulerian one.

5. Lagrangian Case

For $U = u$ *and* $V = v$, we get

$$\frac{\partial E}{\partial \lambda} + \frac{\partial F}{\partial \xi} + \frac{\partial G}{\partial \eta} = 0 \qquad \text{(L)}$$

where

$$E = \begin{pmatrix} \rho J \\ \rho J u \\ \rho J v \\ \rho J e \\ A \\ B \\ L \\ M \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ pM \\ -pL \\ p(uM - vL) \\ -u \\ -v \\ 0 \\ 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ -pB \\ pA \\ p(vA - uB) \\ 0 \\ 0 \\ -u \\ -v \end{pmatrix}$$

Remarks

(1) The gas dynamics equations in Lagrangian coordinates **are written in conservation form for the first time**. (WH Hui, et al, *J. Comput. Phys.*, ***153***, 1999)

(2) Lagrangian GD equation is only **weakly hyperbolic**: all eigenvalues are real, but there is no complete set of linearly independent eigenvectors. This is also true for 3-D case. Hence, **Lagrangian GD is not equivalent to Eulerian GD.**

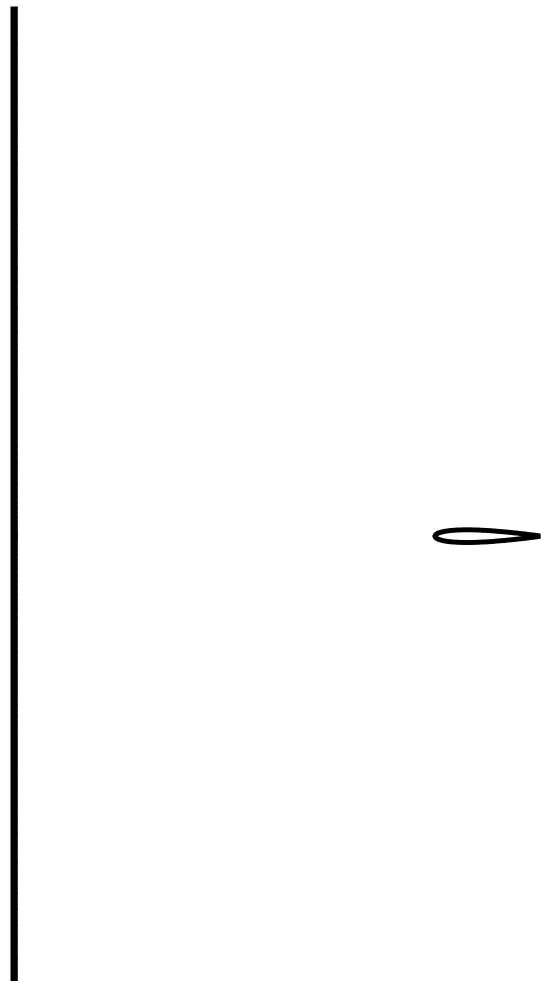
B. Despres & C. Mazeran, (*Arch. Rational Mech. Anal.*,***178***, 2005) **reached same conclusions**.

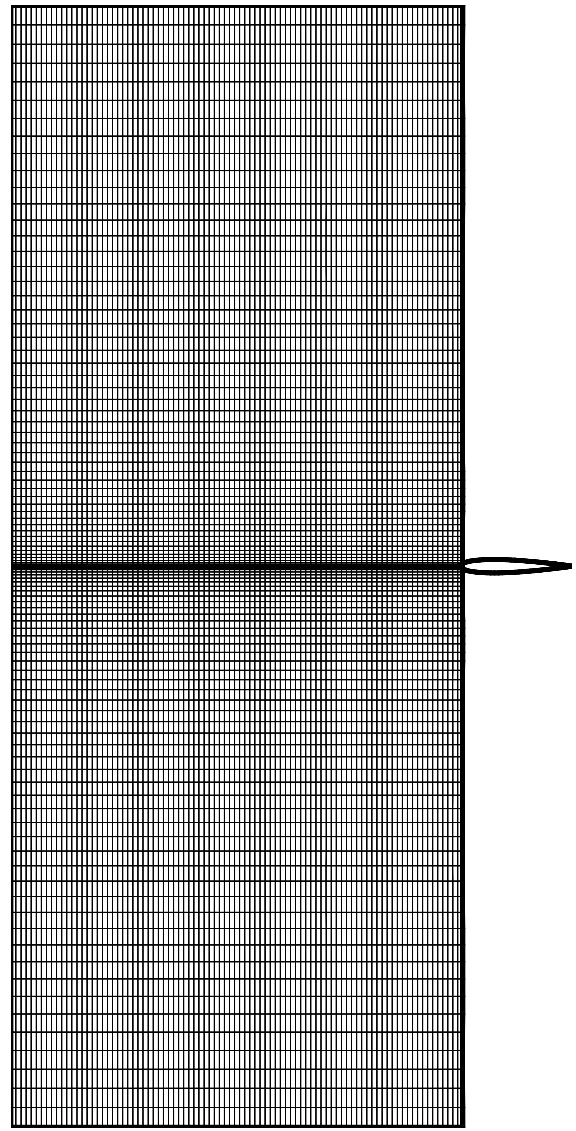
6. Automatic Mesh Generation

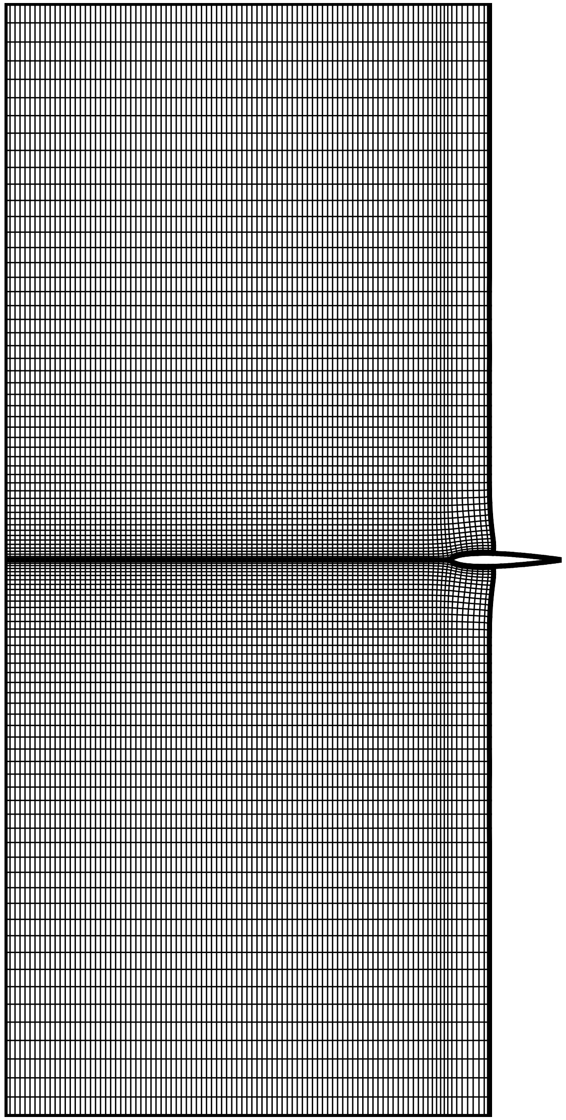
Example 2. Steady subsonic flow

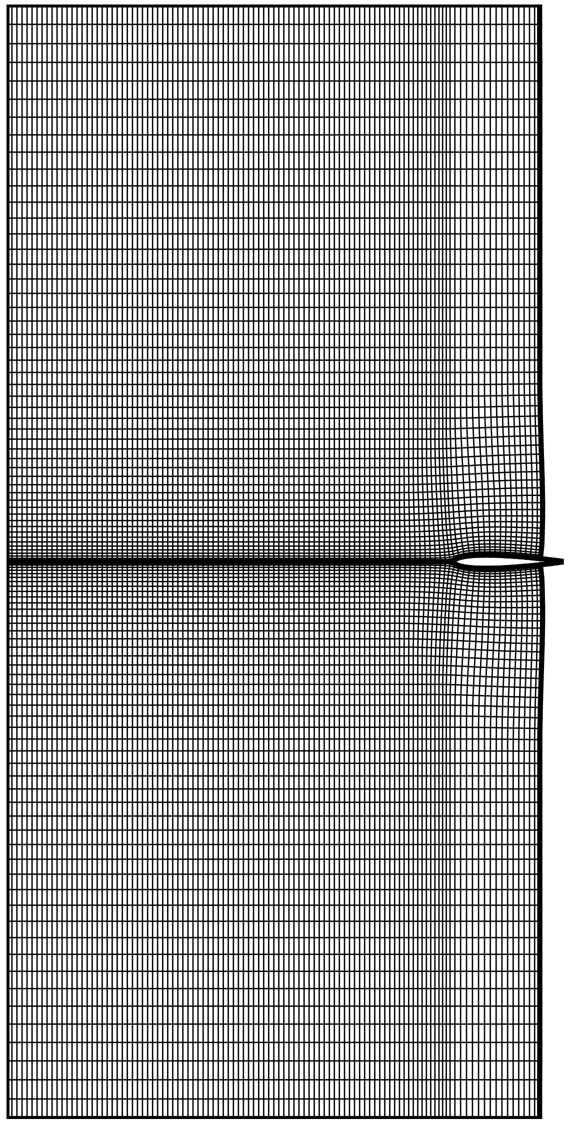
M = 0.8 past a NACA 0012 airfoil

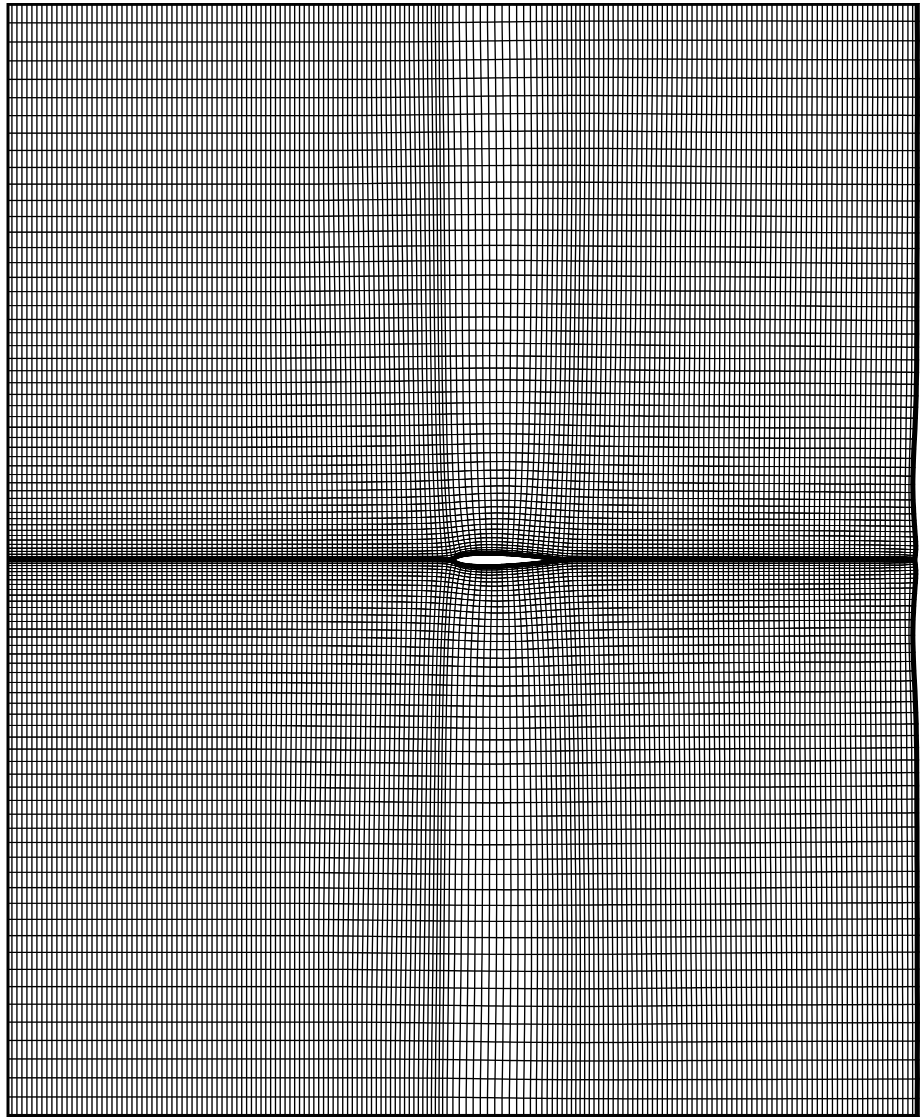
(Time-marching, WH Hui & JJ Hu)



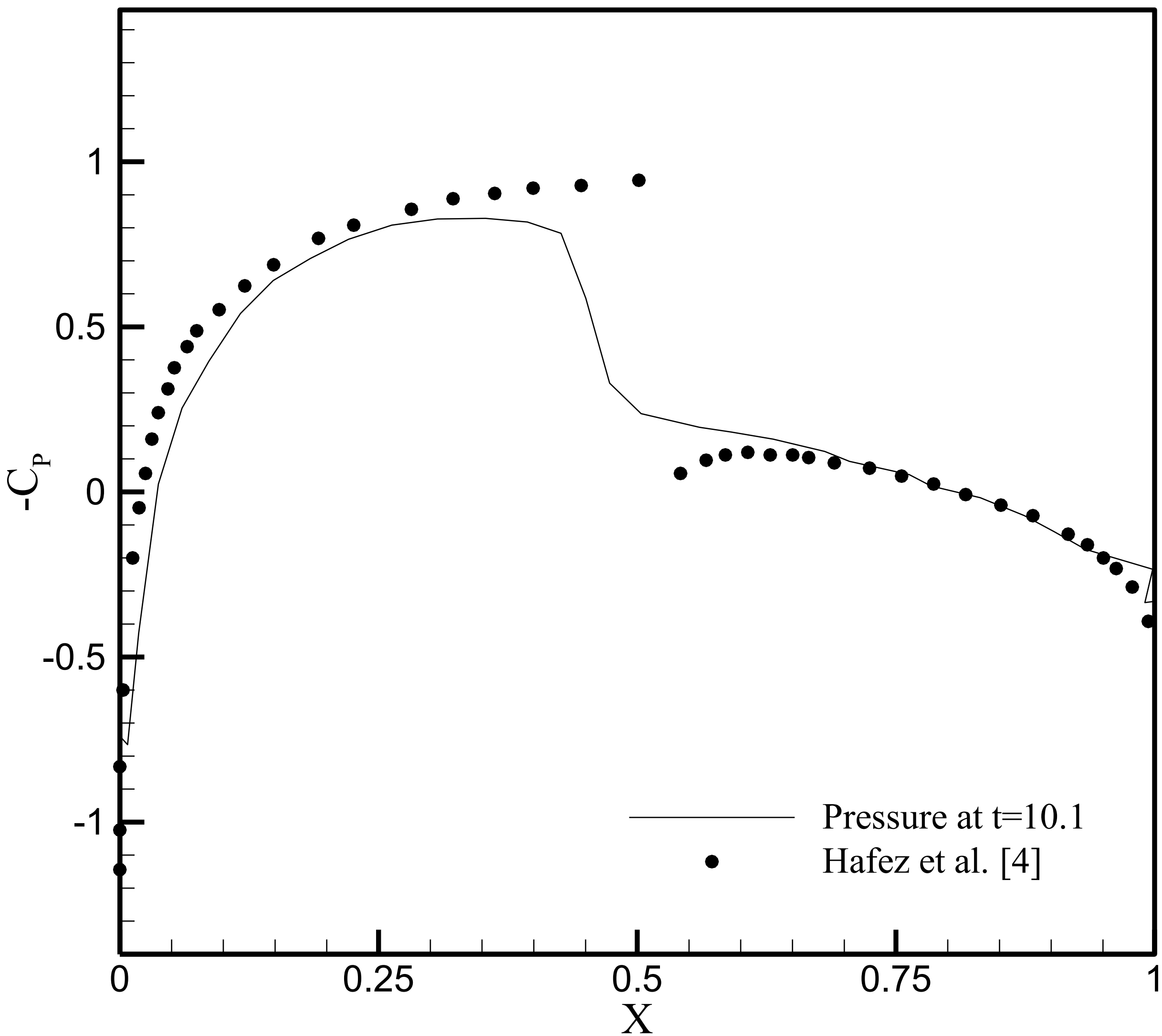




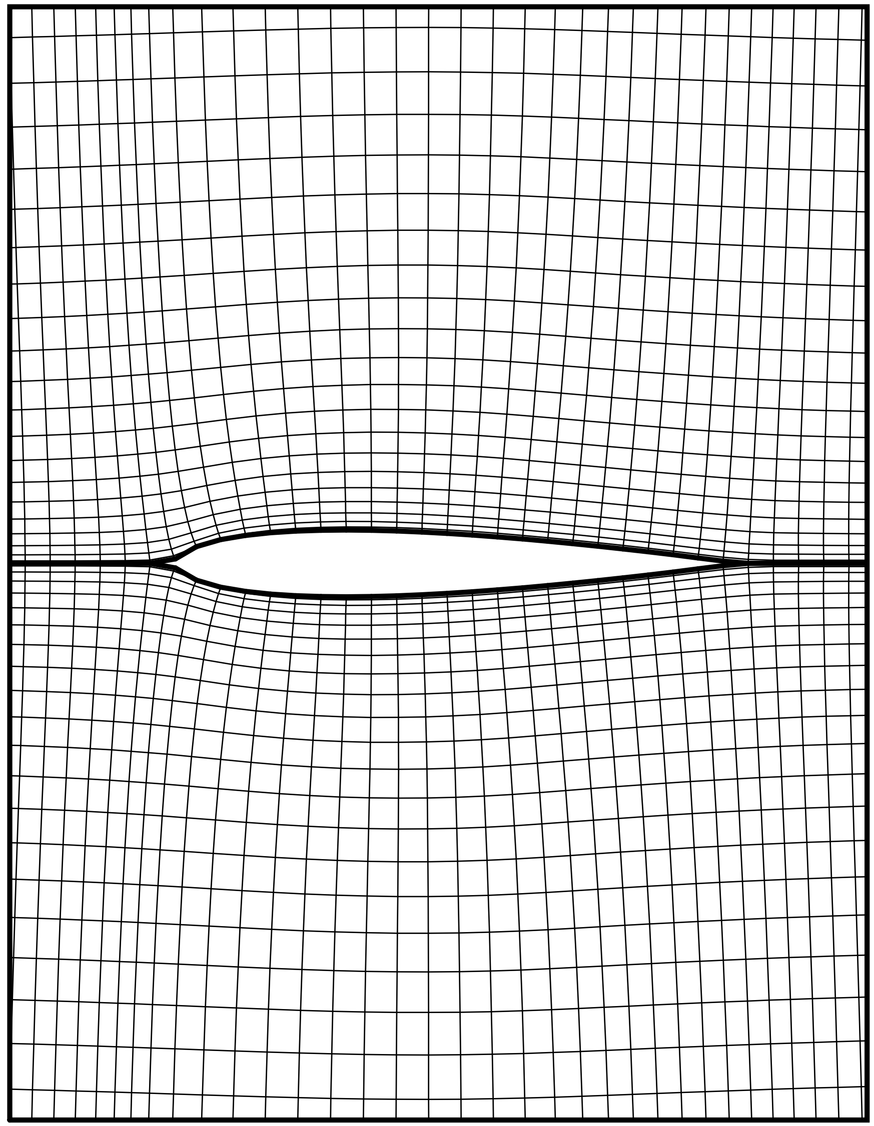


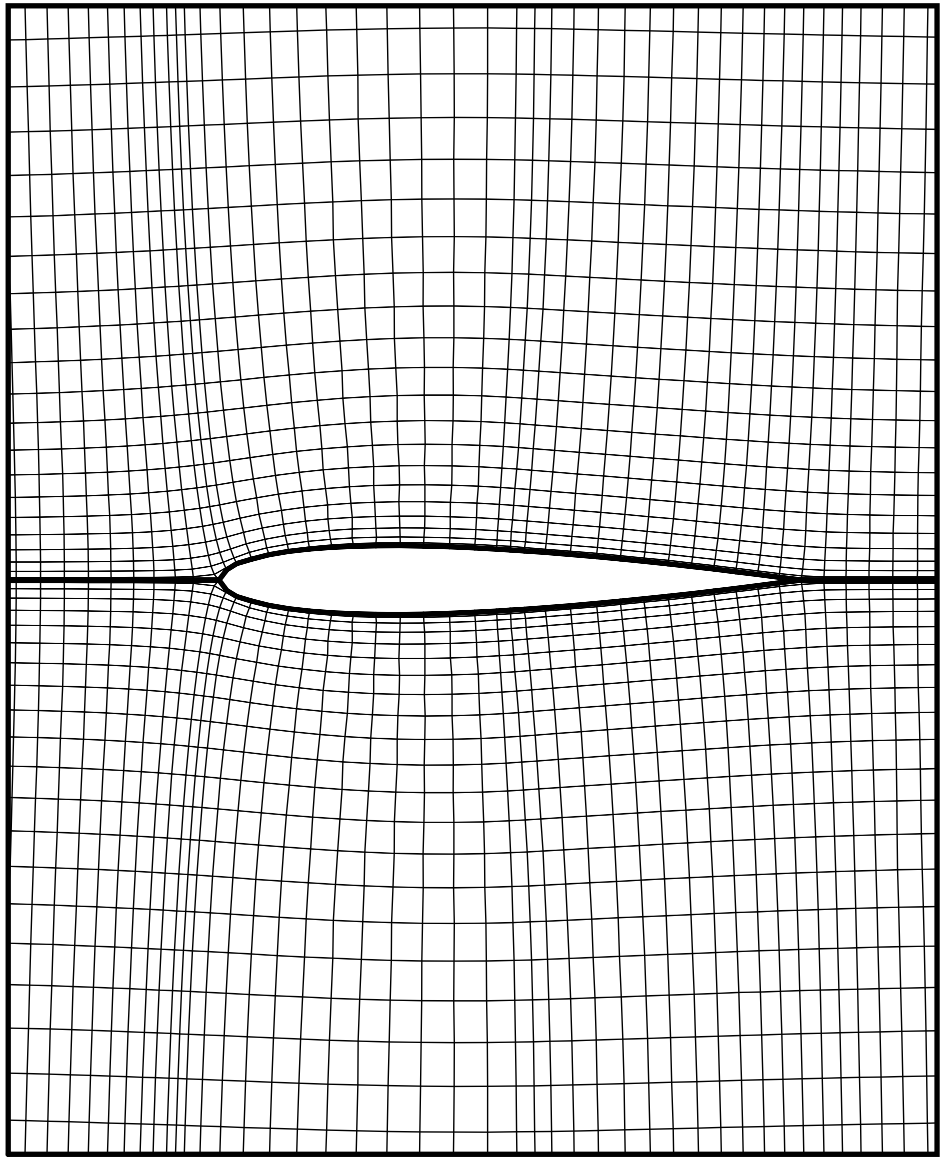


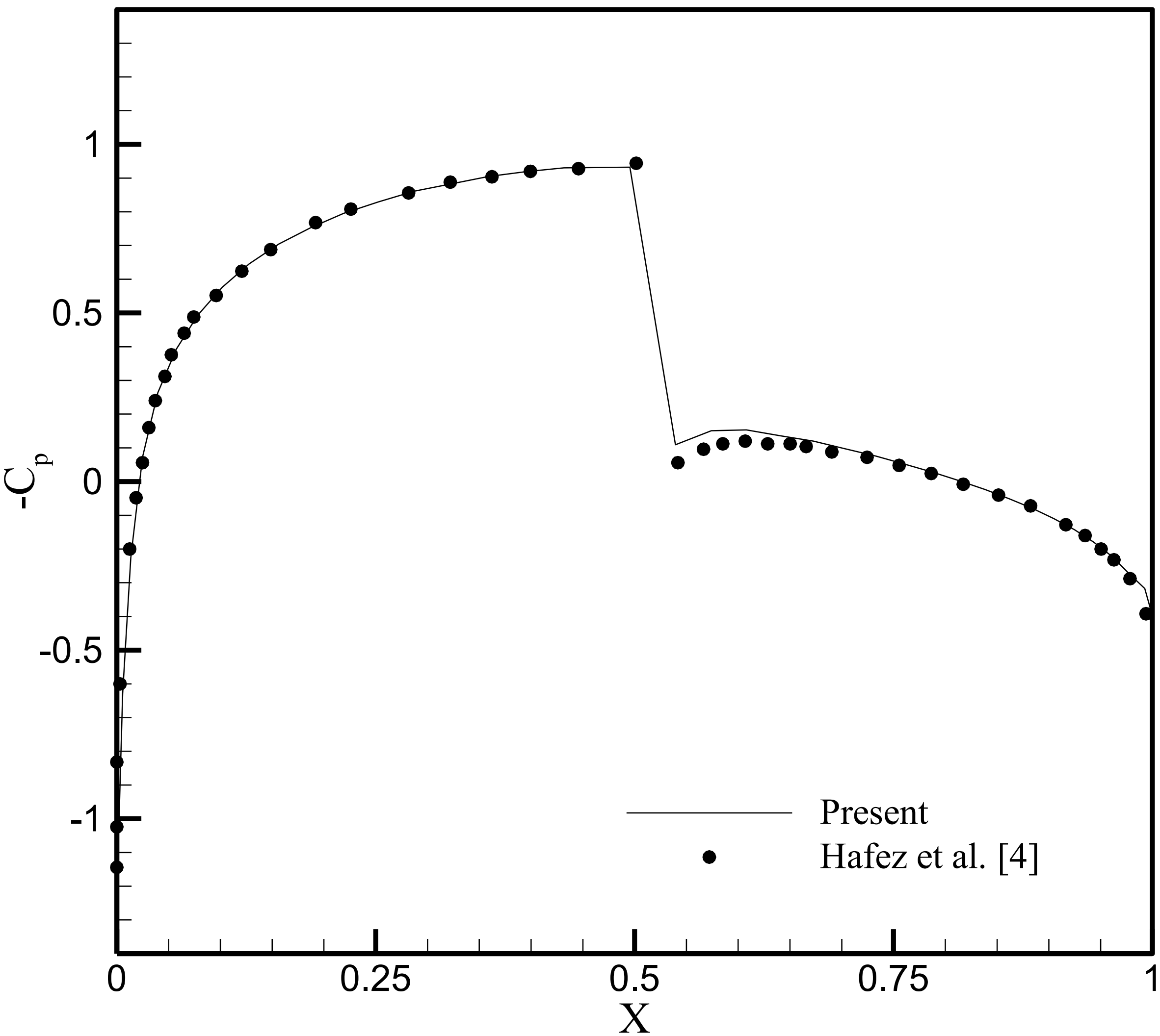
This resembles the flow in the wind tunnel—numerical wind tunnel



-C_P
1
0.5
0
-0.5
-1
0
0.25
0.5
0.75
1
X
Pressure at t=10.1
Hafez et al. [4]





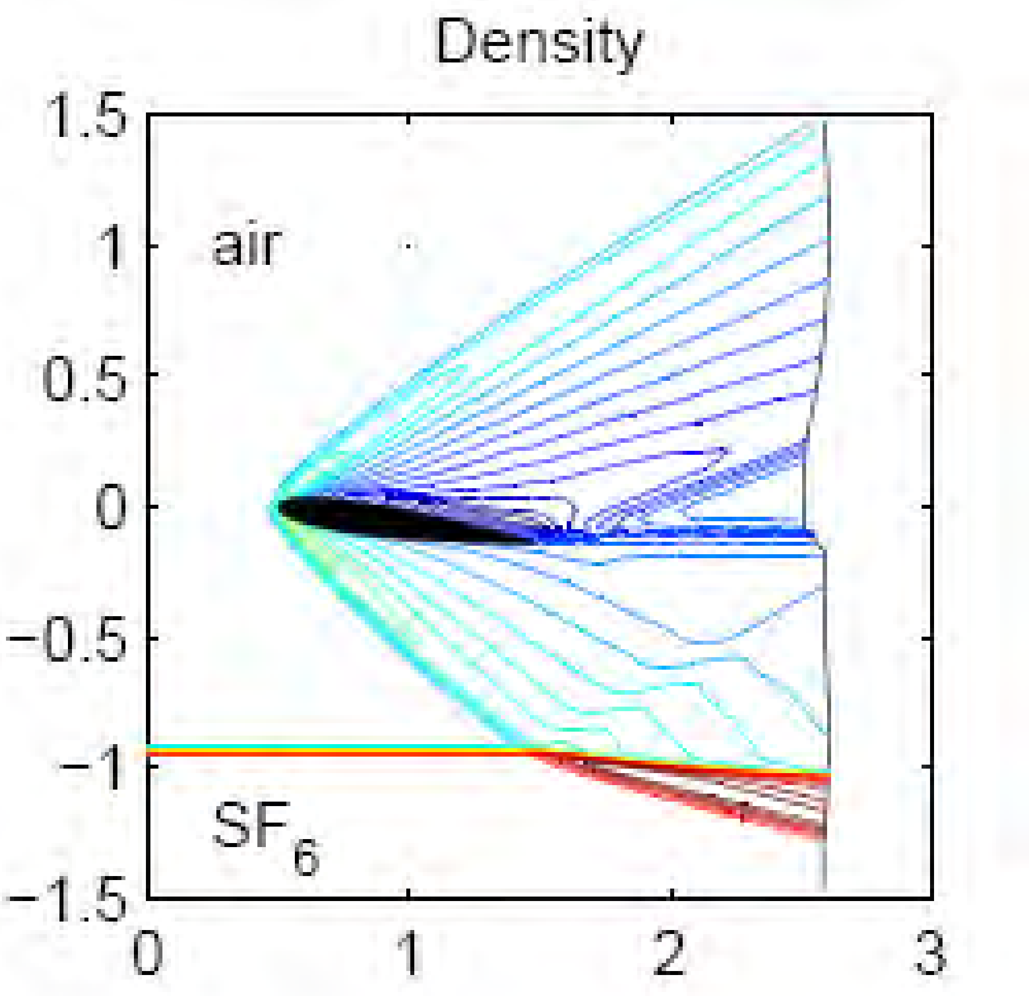


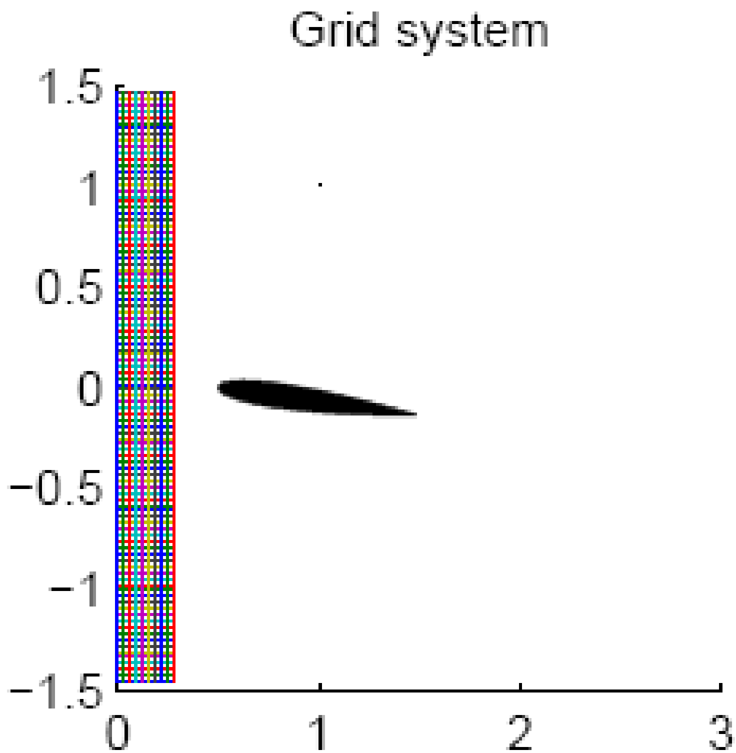
-C_p
1
0.5
0
-0.5
-1
0
0.25
0.5
0.75
1
X
Present
Hafez et al. [4]

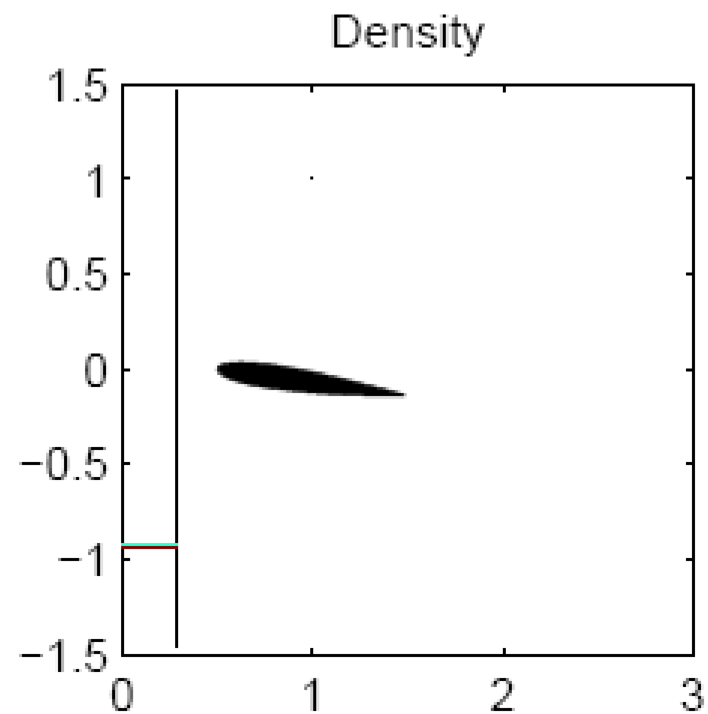
Example 3

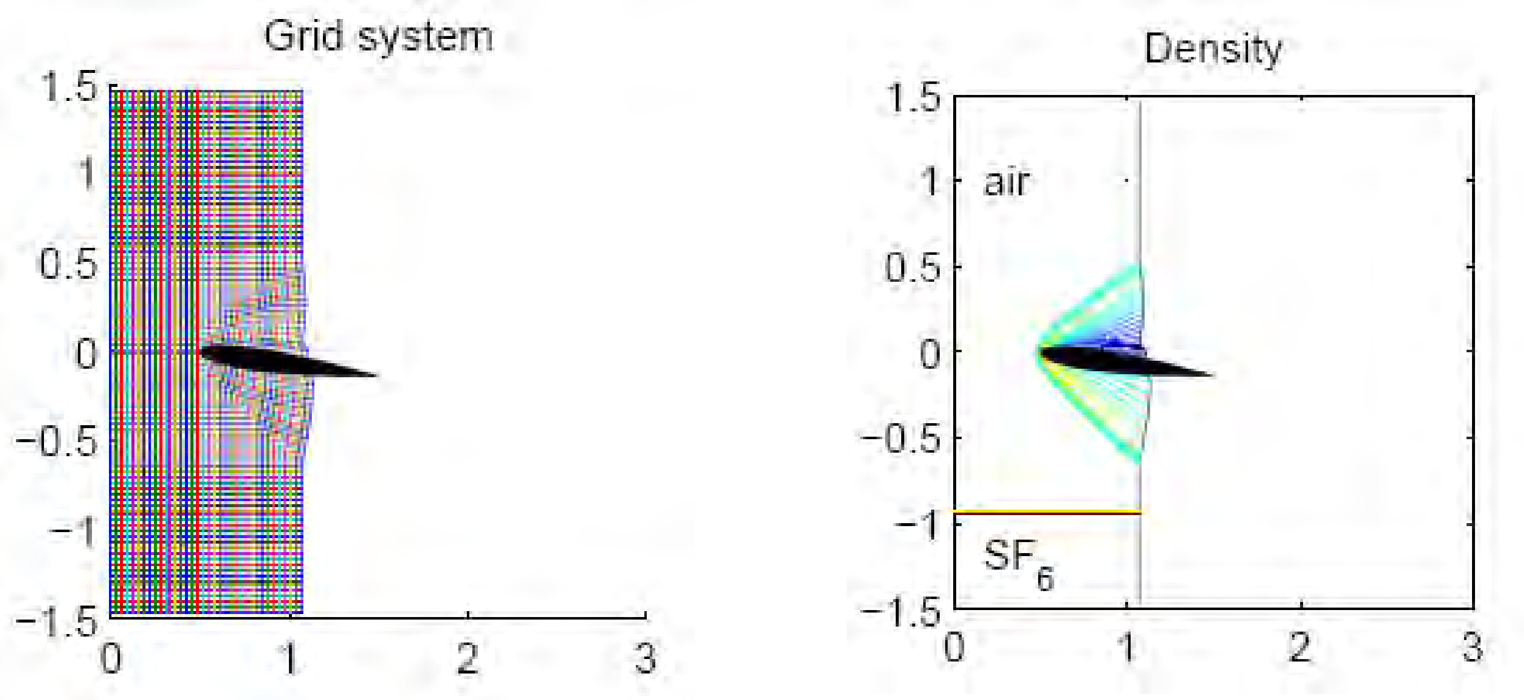
Two-Fluids Flow past a NACA 0012 airfoil

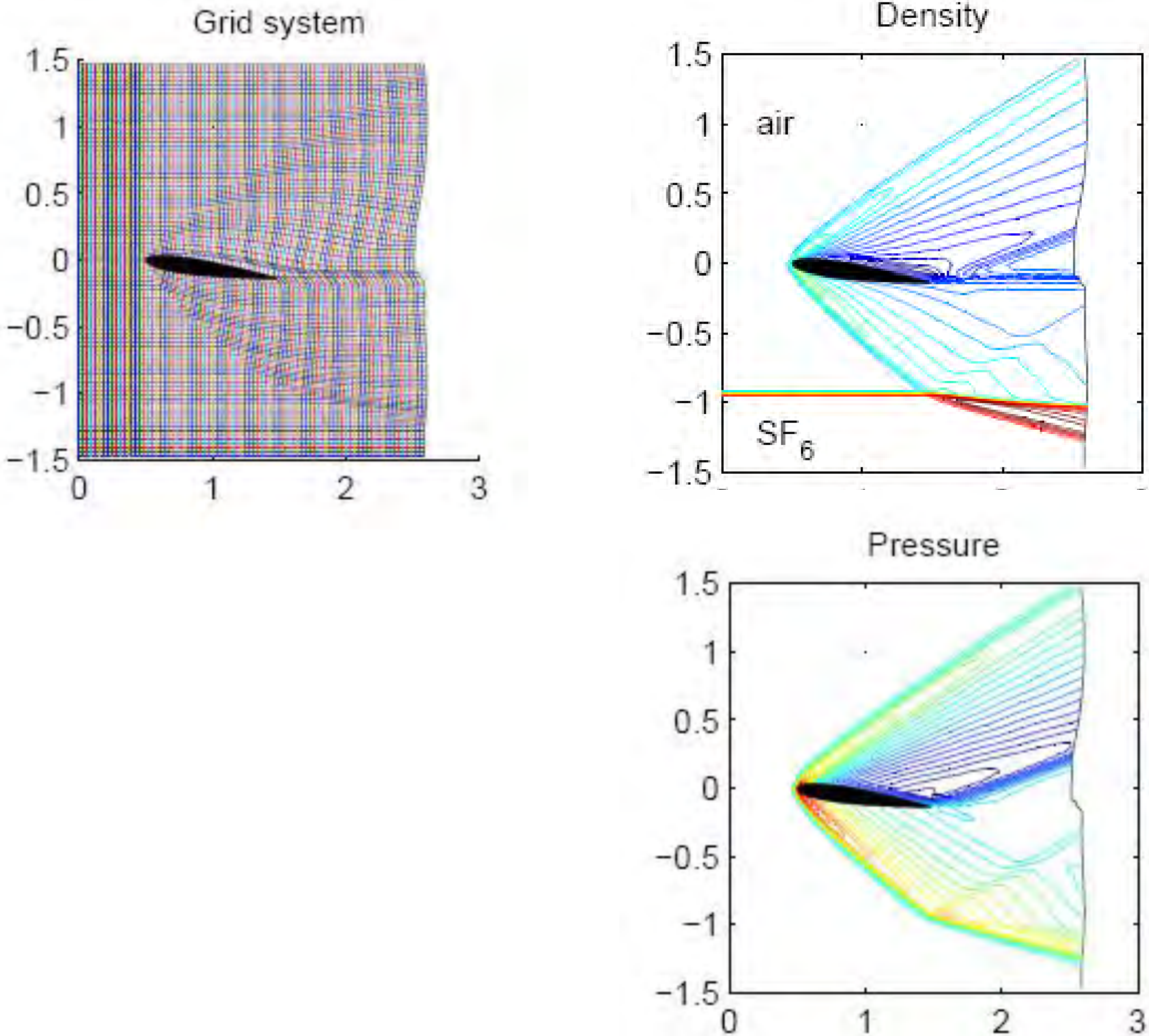
M = 2.2 *AoA = 8 degrees* (Hui, Hu, Shyue)




Grid system
1.5
1
0.5
0
−0.5
−1
−1.5
0
1
2
3

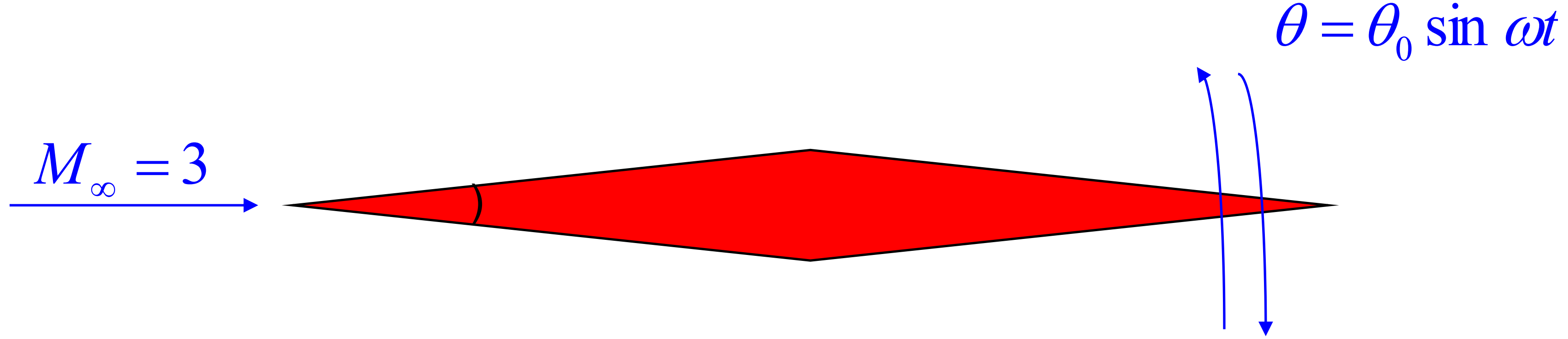

Density
1.5
1
0.5
0
−0.5
−1
−1.5
0
1
2
3


Grid system
1.5
1
0.5
0
−0.5
−1
−1.5
0
1
2
3
Density
1.5
1
0.5
0
−0.5
−1
−1.5
0
1
2
3
air
SF6

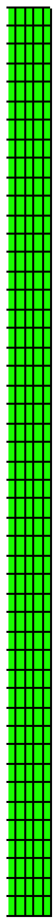

Grid system
Density
air
SF6
Pressure
1.5
1
0.5
0
−0.5
−1
−1.5
0
1
2
3

Example 4. Supersonic flow past a pitching oscillating diamond-shape airfoil

(GP Zhao & WH Hui)

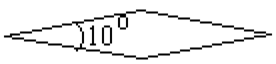


Unsteady pressure distribution



$M_\infty = 3$

$\theta = 2^\circ \sin 30t$

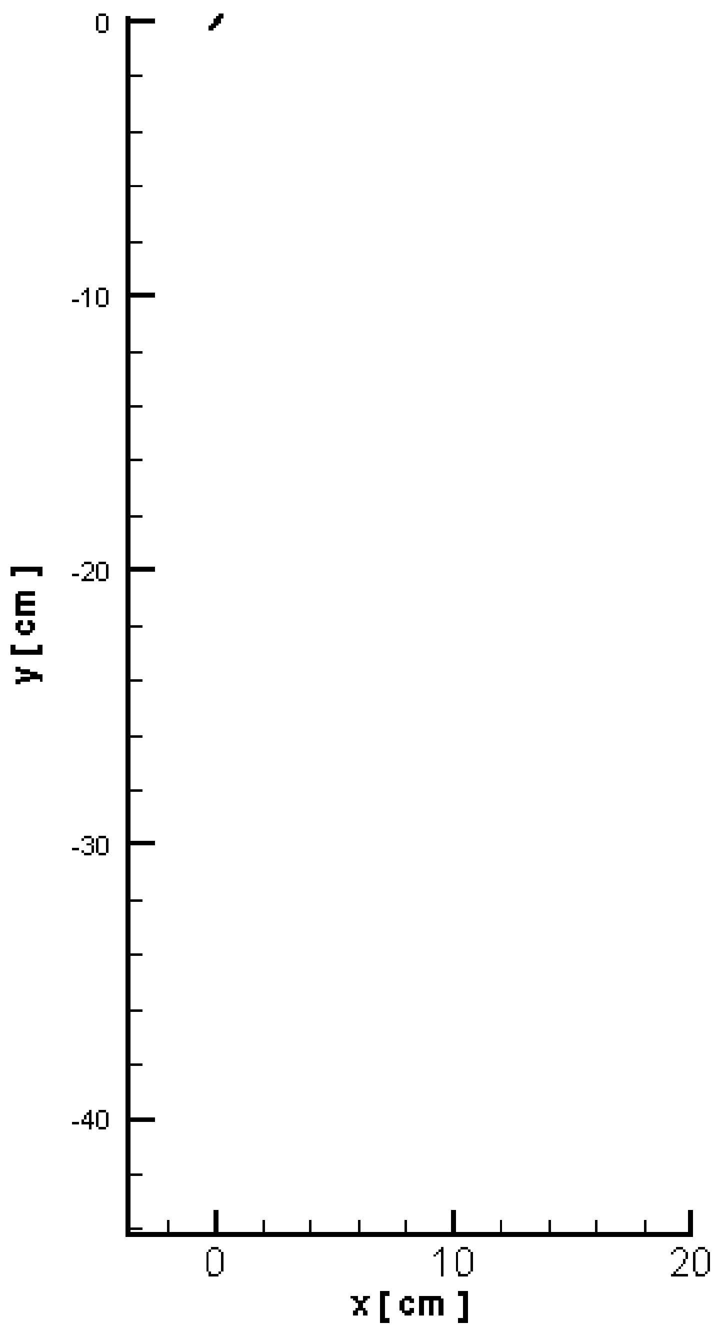


7. Aerodynamics of Freely Falling Leaves

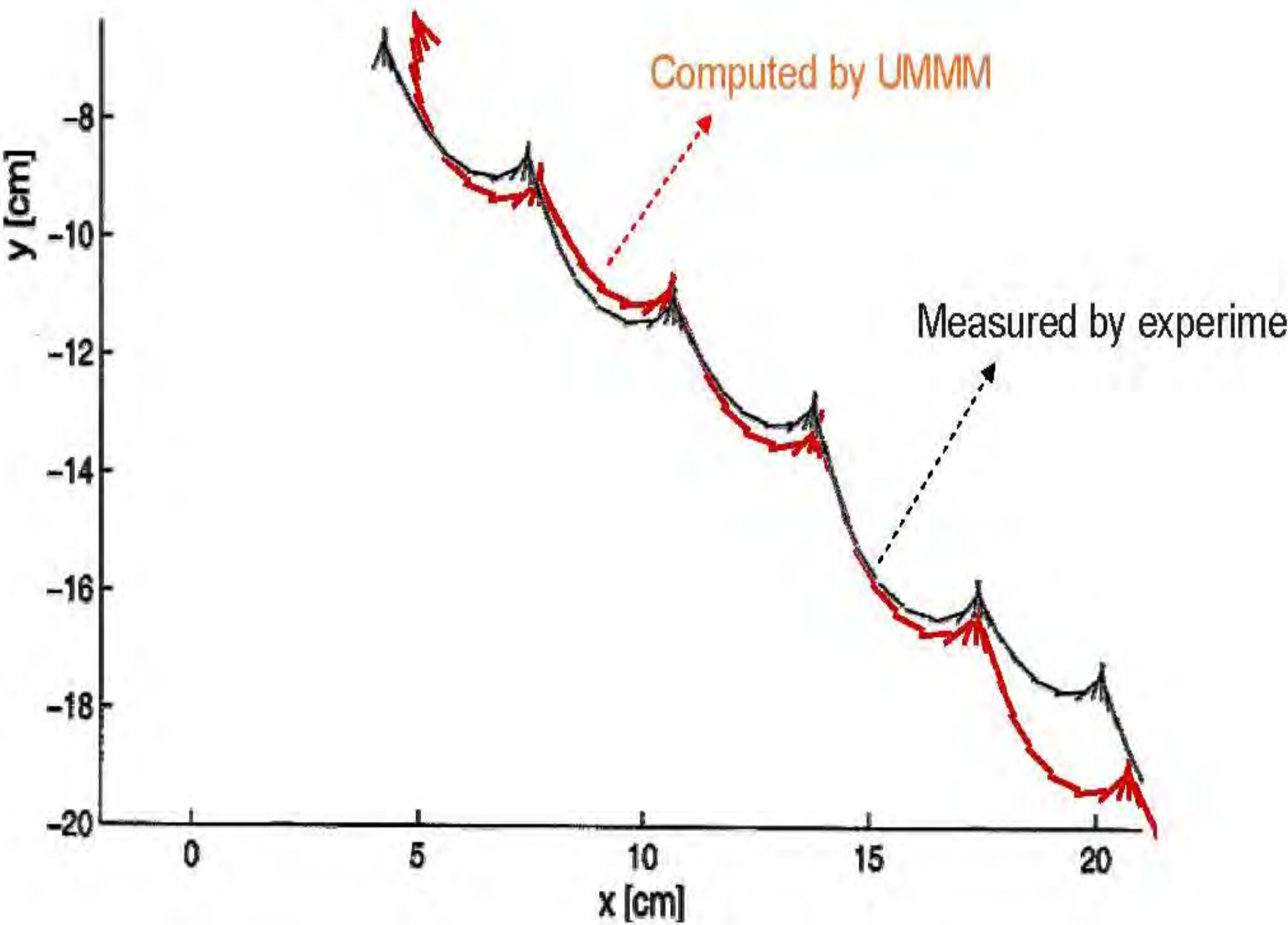
(C Jin and K Xu)

(another way to choose the mesh velocity)

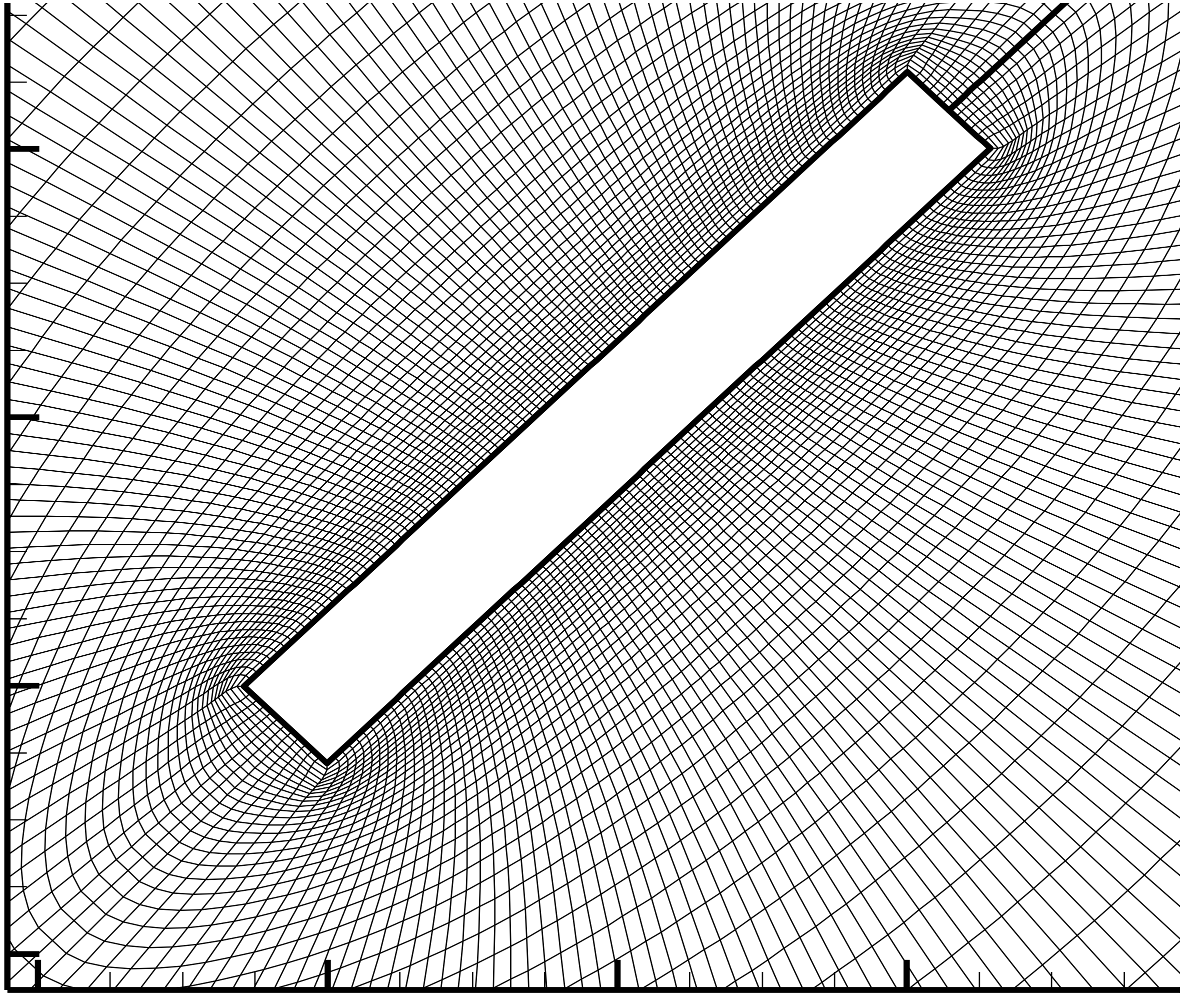
falling leaves with fluttering and tumbling motion



0
-10
-20
-30
-40
y [cm]
0
10
20
x [cm]

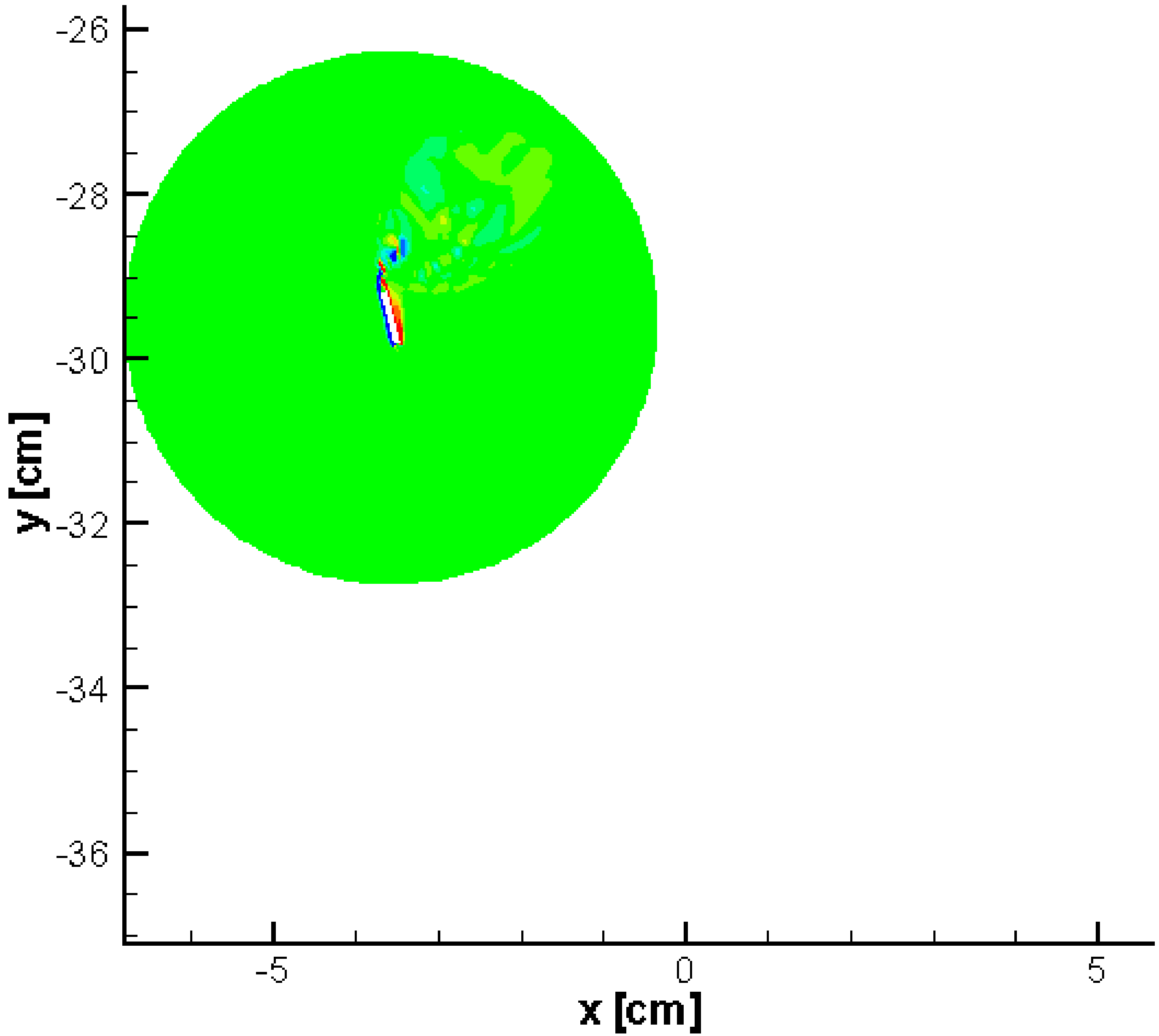


Computed by UMMM
Measured by experime
y [cm]
-8
-10
-12
-14
-16
-18
-20
0
5
10
15
20
x [cm]



Mesh moves with the body; this specifies the mesh velocity (U, V) at all time.

All fluid-solid interaction problems can be tackled this way.



-26
-28
-30
-32
-34
-36
y [cm]
-5
0
5
x [cm]

- In this example you can easily choose the mesh to be fixed with the plate, but it is **impossible to write down the finite transformation functions**: they are part of the flow solution which is unknown.

- On the other hand, if you choose the transformation functions F, G and H first, then at the same time you have also completely determined the mesh velocity, which may not be what you want.

8. Future Research Directions

The Unified Coordinate Method has provided new theoretical foundation and methodology for the following major research directions:

(1) Automatic Mesh Generation

(2) General Moving Mesh Method

(3) Lagrangian CFD

Thank You

Secret of Perfection

"As I am not pressed and work more for my pleasure than from duty,……….

I make, unmake, and remake, until I am passably satisfied with my results, which happens only rarely"

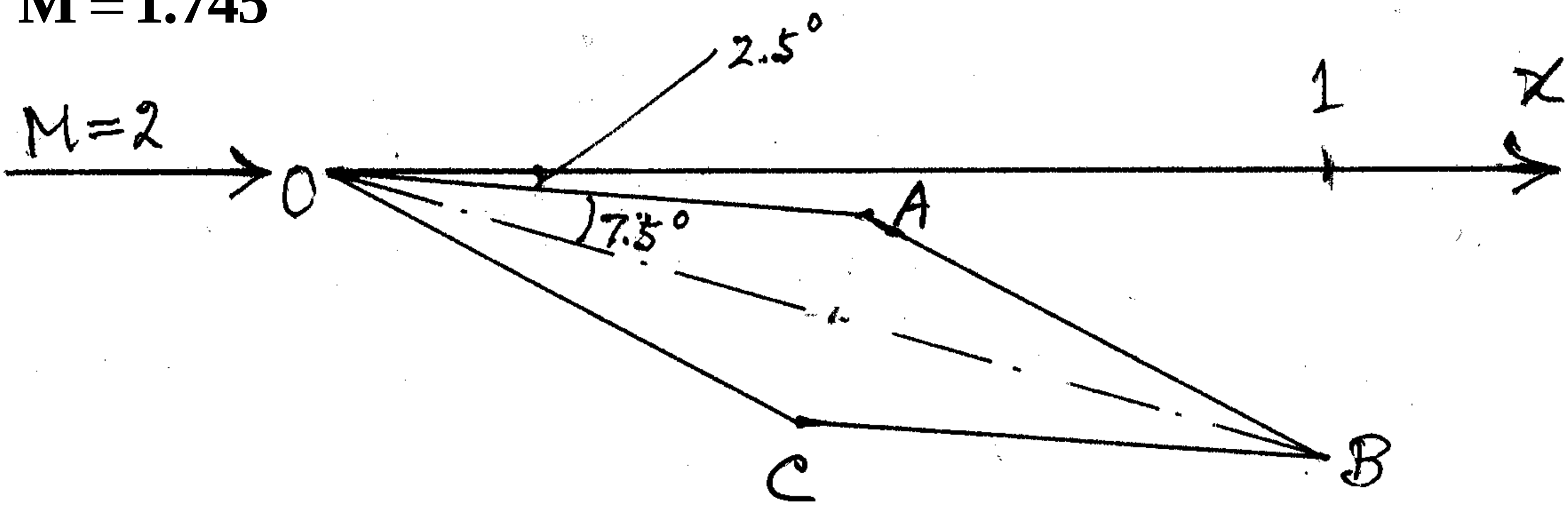
Joseph-Louis Lagrange (1736-1813)

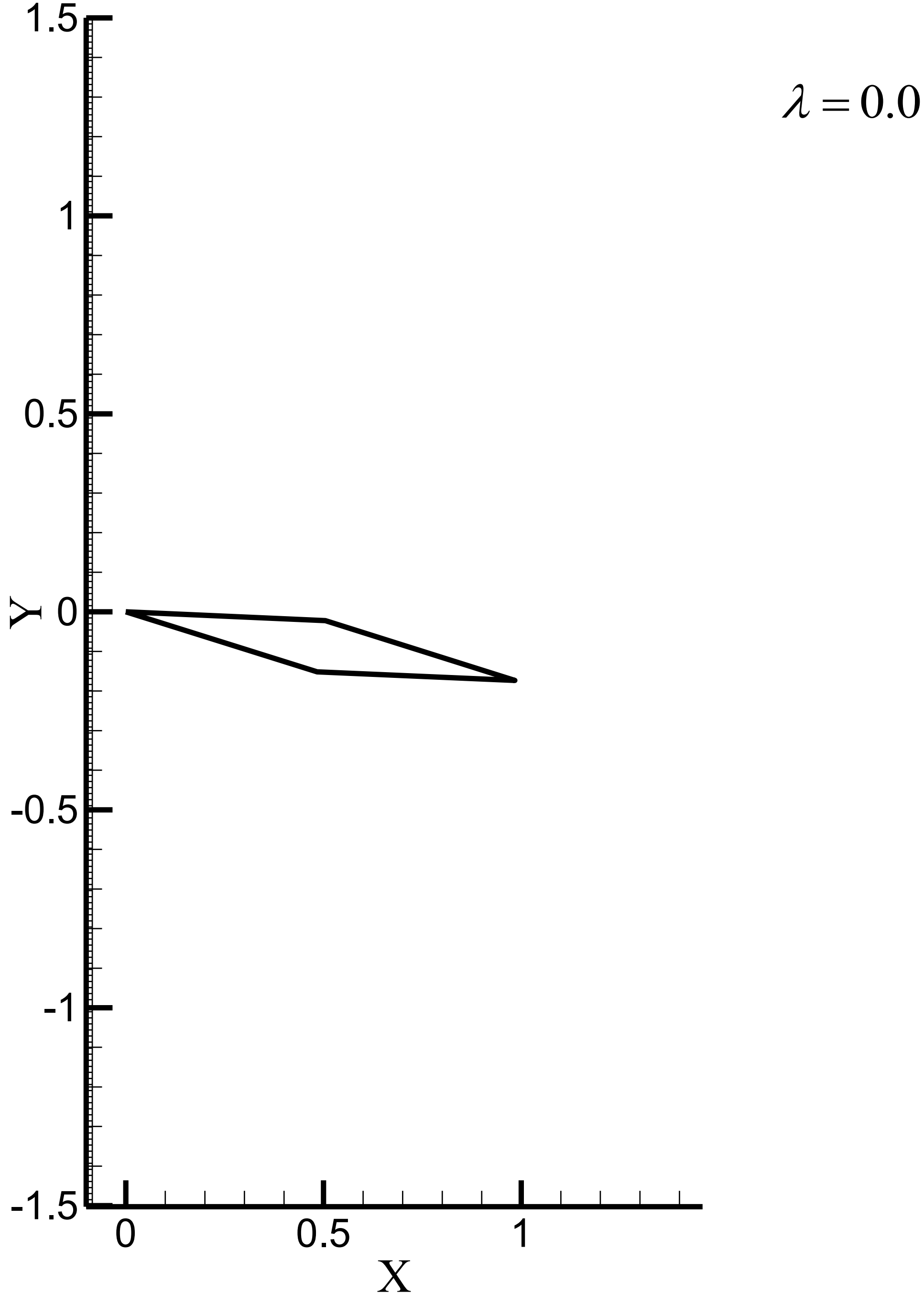
6. Automatic Mesh Generation

Example 2 . Steady supersonic flow

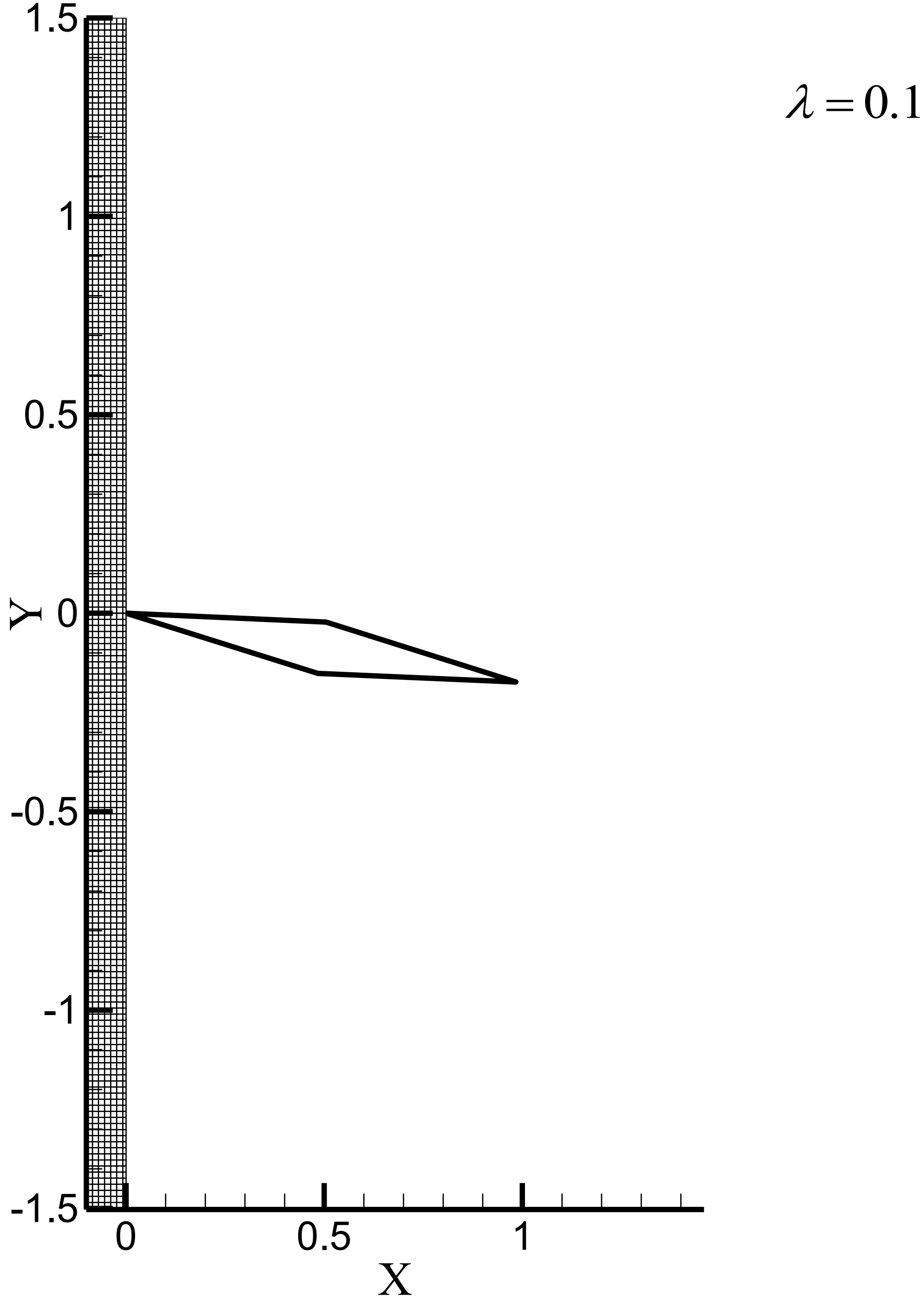
(Space-marching, WH Hui & JJ Hu)

$\mathbf{M} = \mathbf{1.745}$

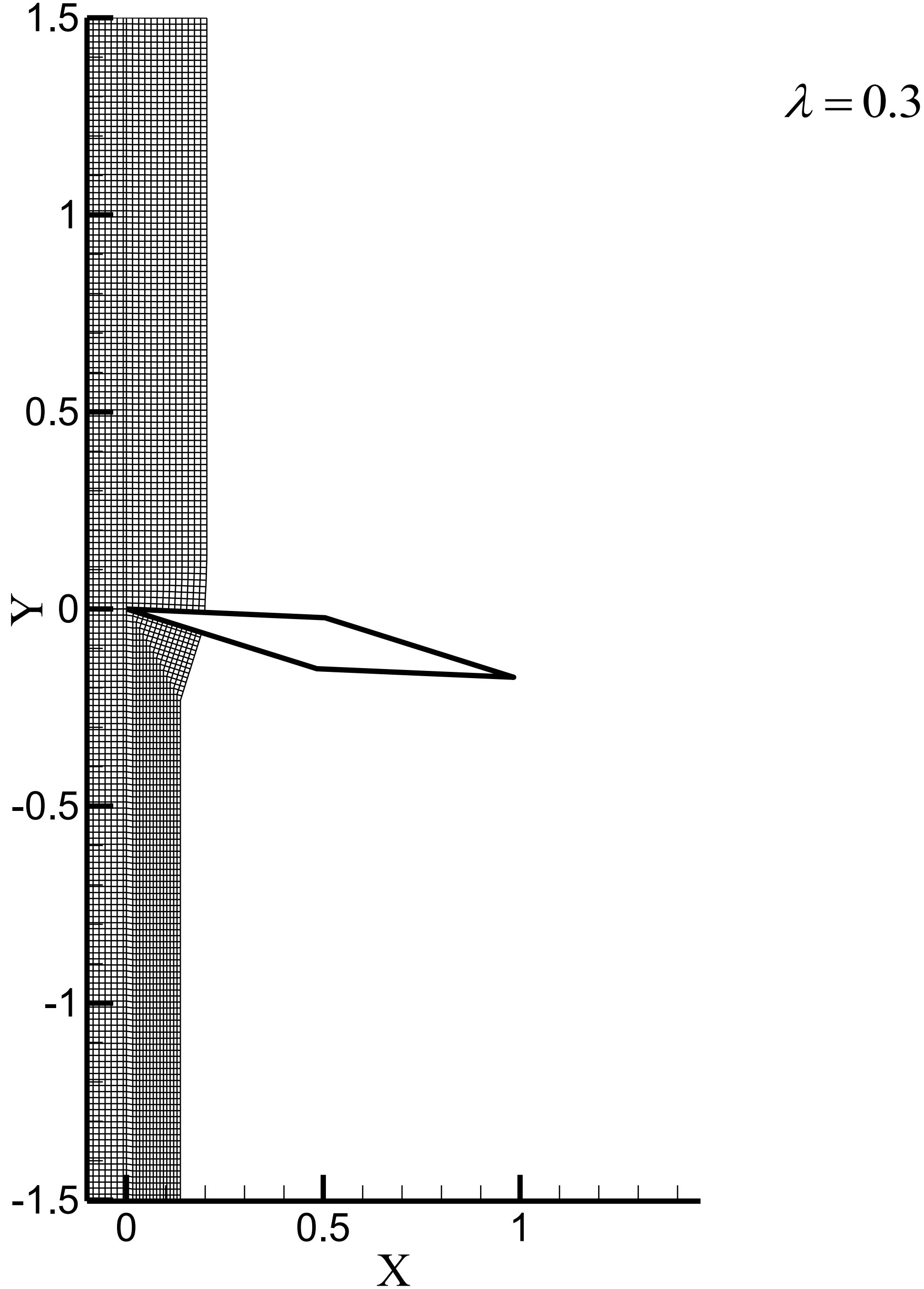




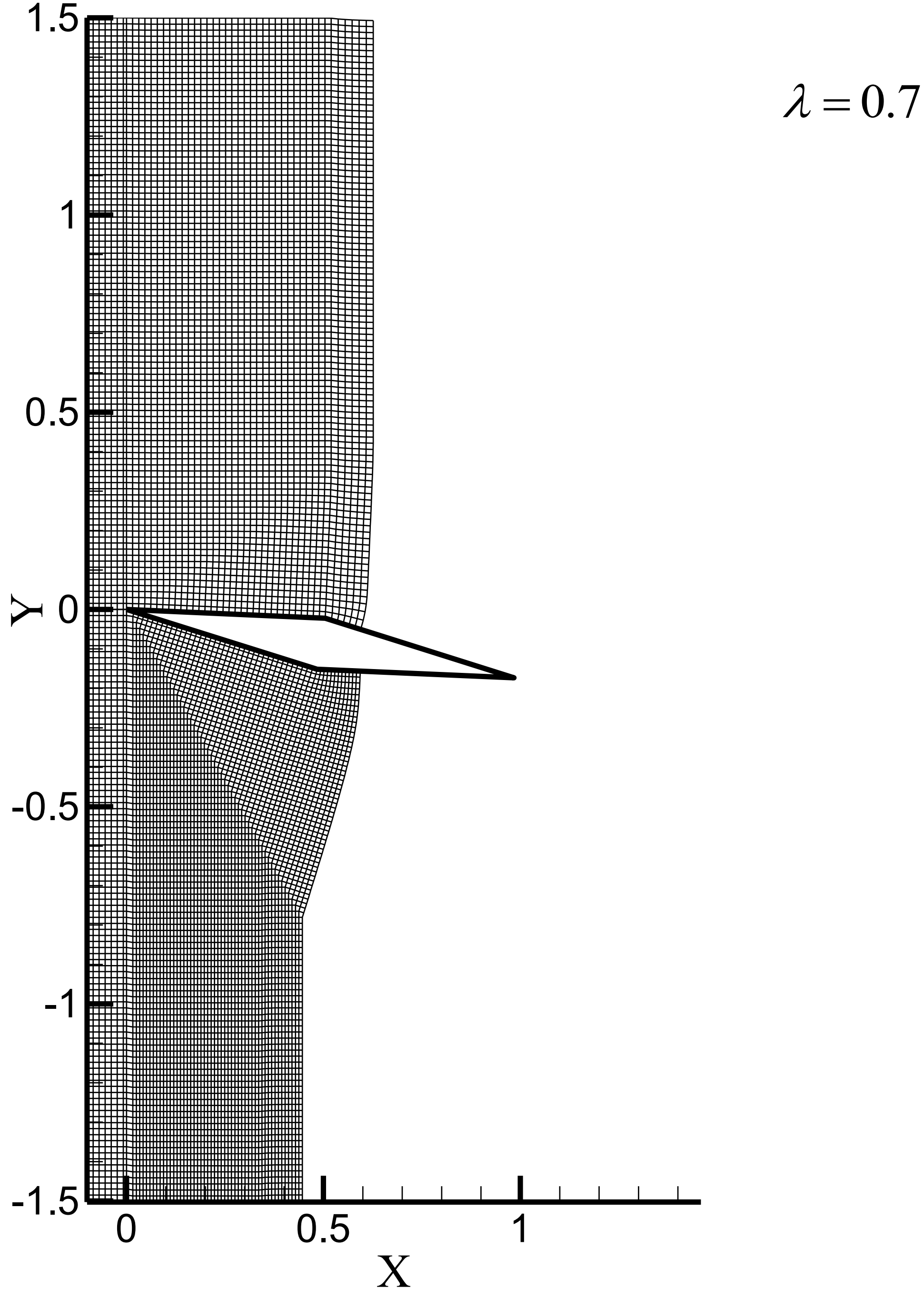
$\lambda = 0.0$
1.5
1
0.5
0
-0.5
-1
-1.5
Y
0
0.5
1
X



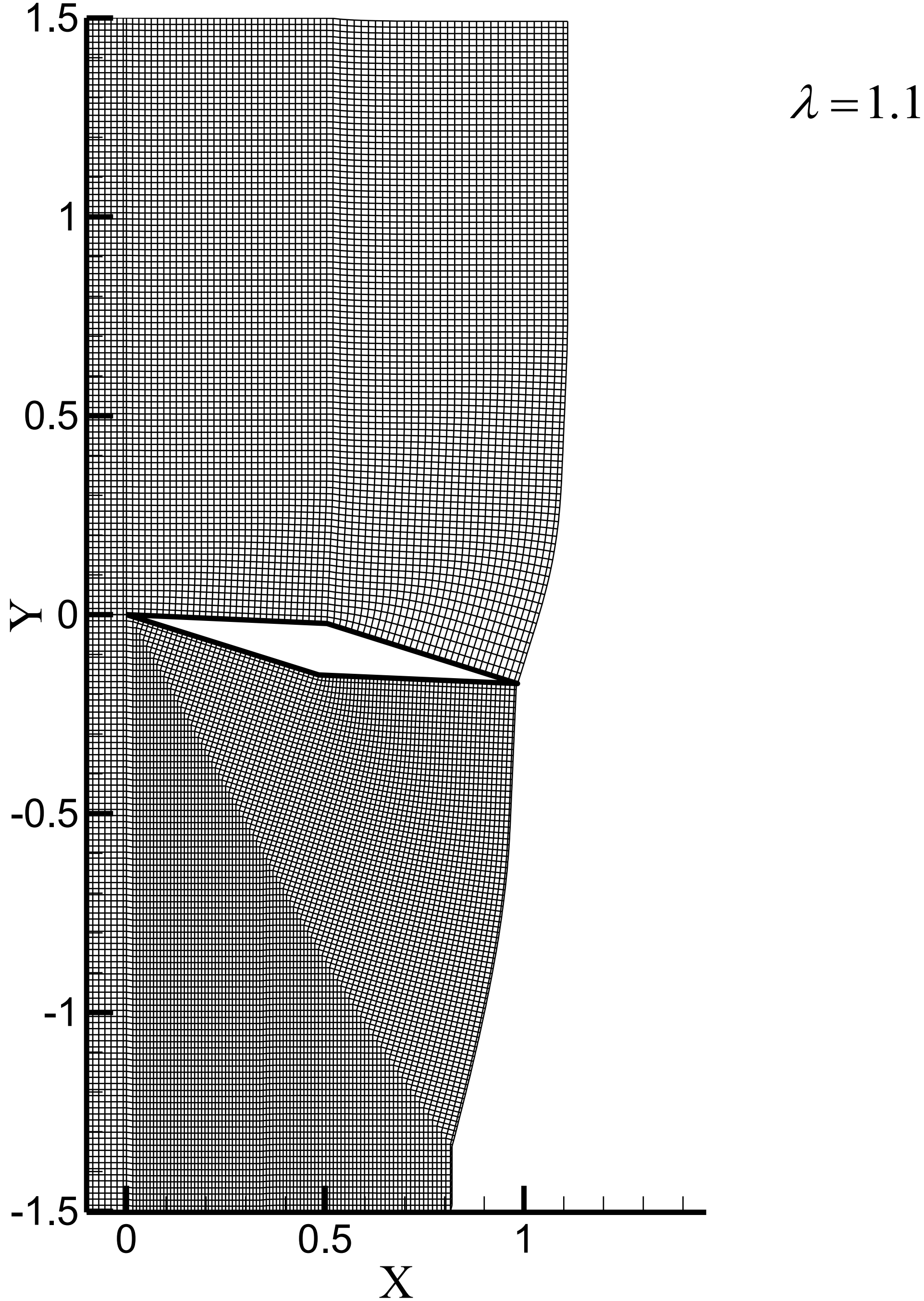
$\lambda = 0.1$
1.5
1
0.5
0
-0.5
-1
-1.5
Y
0
0.5
1
X



$\lambda = 0.3$
1.5
1
0.5
Y 0
-0.5
-1
-1.5
0
0.5
1
X

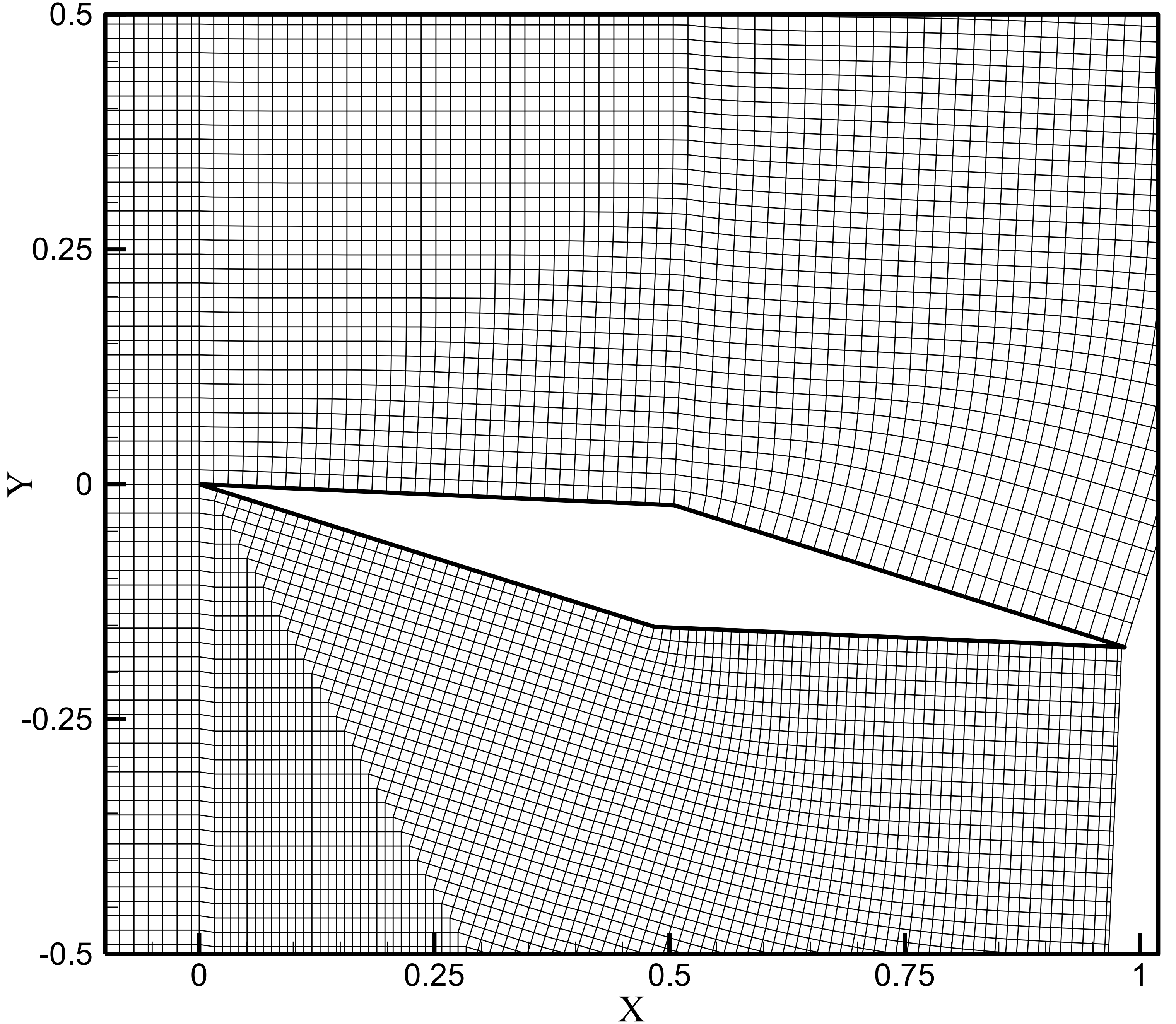


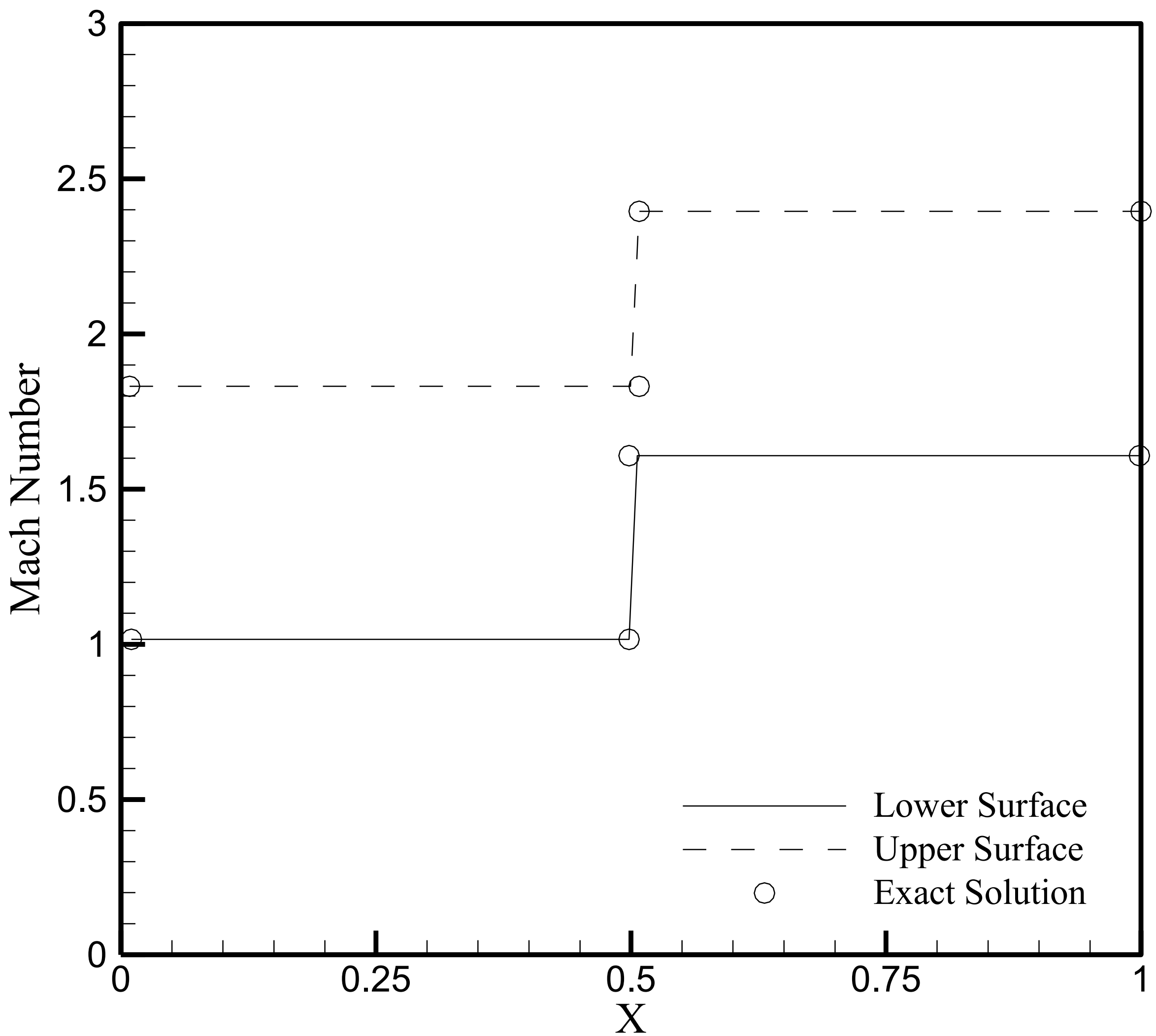
1.5
1
0.5
0
-0.5
-1
-1.5
Y
0
0.5
1
X
$\lambda = 0.7$



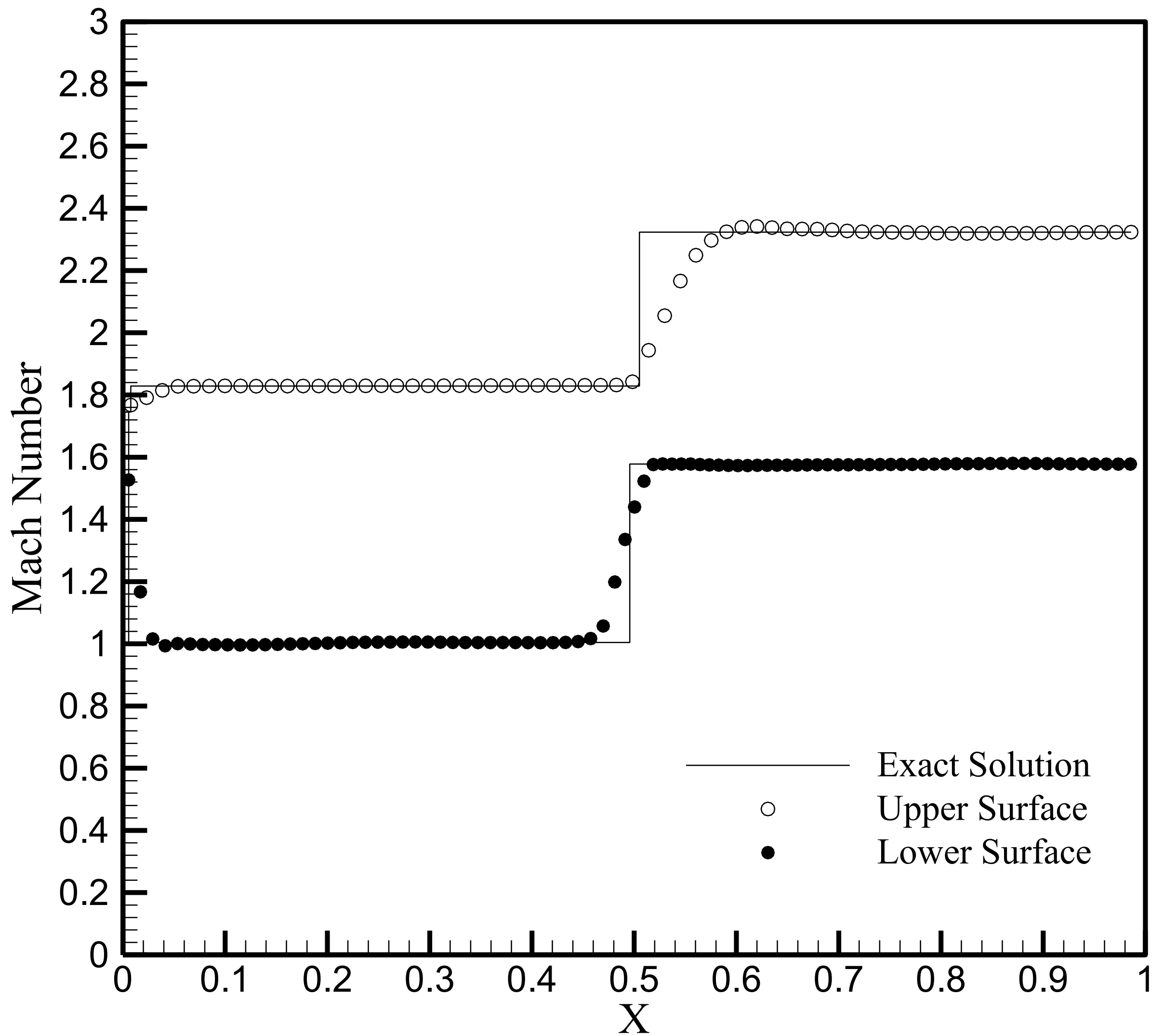
λ = 1.1
Y
1.5
1
0.5
0
-0.5
-1
-1.5
X
0
0.5
1

Flow-Generated Mesh (Close View)





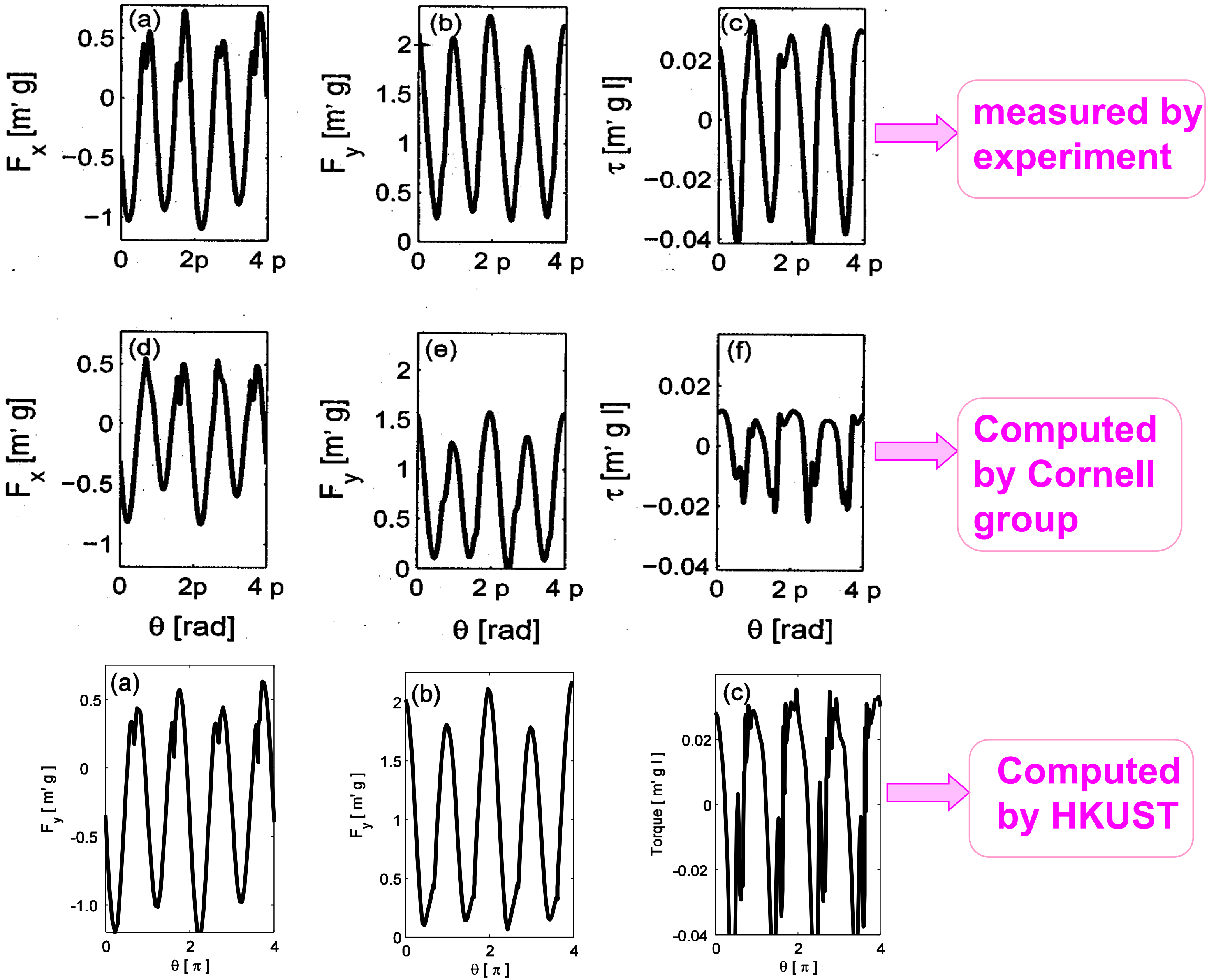
Surface Mach number distribution, 120 cells
Computing time: 1.8s (P4, 2.8GHz)



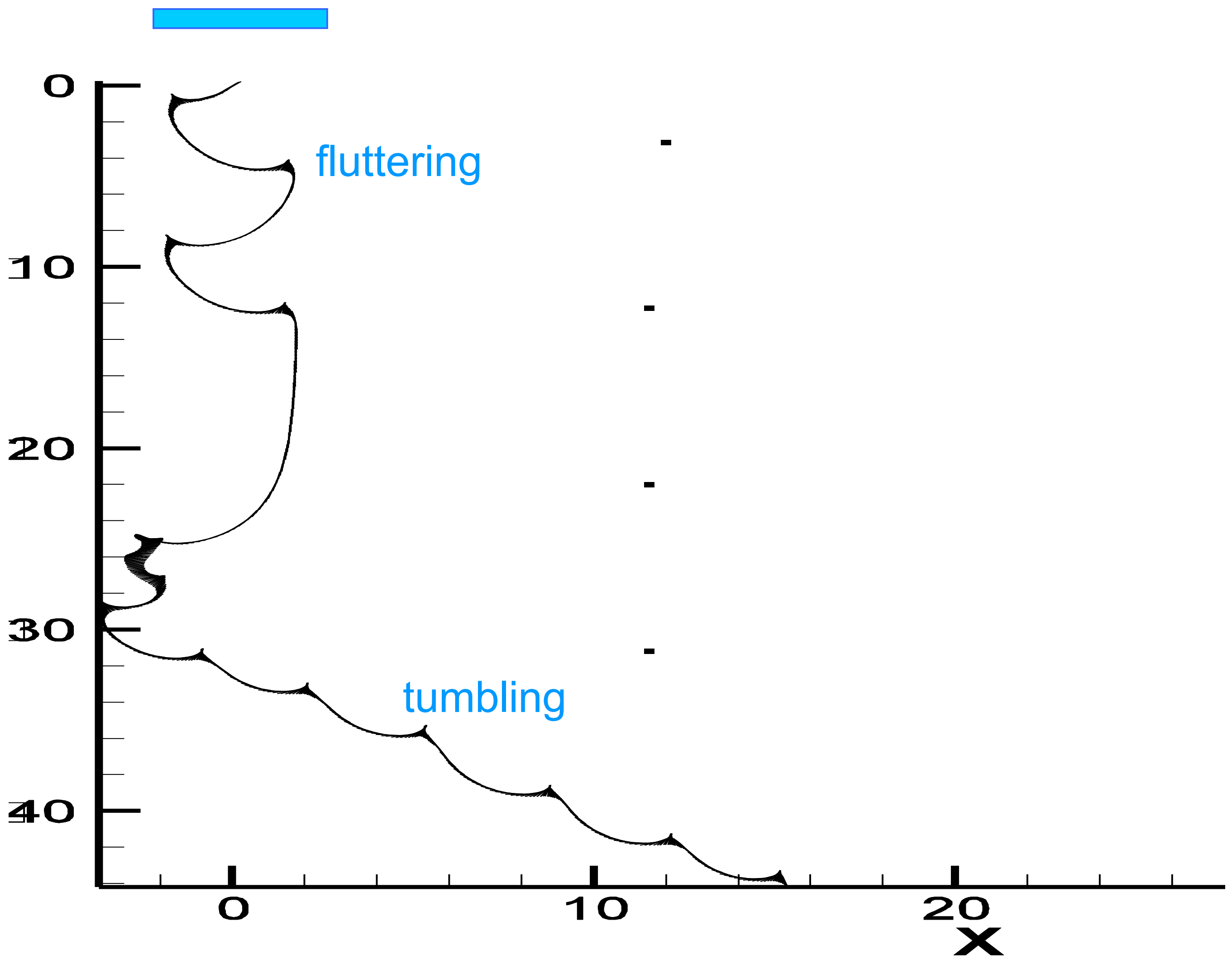
Surface Mach Number Distributions on Diamond-Shaped Airfoil.

Eulerian Computation (5th Order WENO Scheme), 100 x 200 cells.

Computing Time to 20,000 Steps: 2,393s on P4, 2.8 GHz.


(a)
(b)
(c)
F_x [m' g]
F_y [m' g]
τ [m' g l]
0 2p 4 p
measured by experiment
(d)
(e)
(f)
θ [rad]
Computed by Cornell group
(a)
(b)
(c)
F_y [m' g]
Torque [m' g l]
θ [π]
Computed by HKUST

Computed paths (10 full rotations)



- Fluttering and then tumbling

- tumbling

"Optimal" Coordinates

For compressible flow, we want a coordinate system to possess the following properties:

(a) Conservation PDEs exist, as in Eulerian;

(b) Contacts are sharply resolved, as in Lagrangian;

(c) Body-fitted mesh can be generated automatically;

(d) Mesh to be orthogonal;

(e) ……

The unified coordinate system satisfies these requirements

Computational procedure:

(1) At time t, the body position and mesh velocity are given;

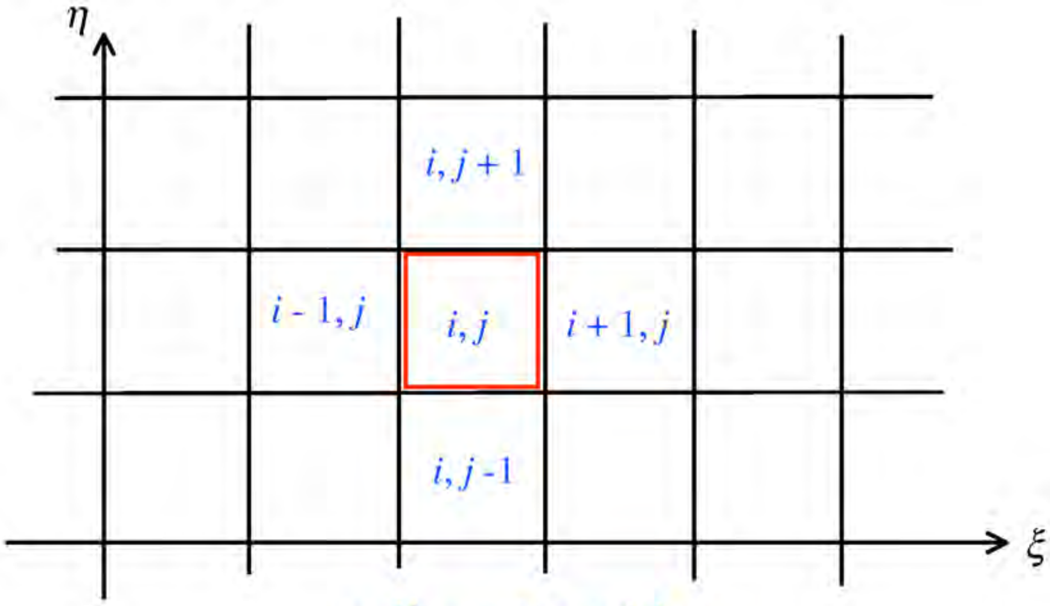
(2) Use Xu's gas kinetic-BGK solver to find the flow and hence the aerodynamic forces on the body *(Note: the effects of mesh movement on the flow are correctly and fully accounted for through the geometric conservation laws);*

(3) Use Newton's 2nd law to compute the motion of the body under the aerodynamic and gravitational forces, giving the body position and the mesh movement at new time;

Repeat (1) – (3).

UC Computation

As we have conservation form, computations are done like Eulerian in λ-ξ-η space by marching in time λ, (eg, Godunov-MUSCL scheme plus splitting). At each time step *the mesh velocity (U, V) is either given or computed.*



cell-center grid

(3) The system of **Lagrangian gas dynamics equations is written in conservation PDE** form, thus providing a foundation for developing Lagrangian schemes as moving mesh schemes.

(4) The **Lagrangian system of gas dynamics equations in 2-D and 3-D are shown to be only *weakly* hyperbolic,** in contrast to the Eulerian system which is fully hyperbolic; hence the **two systems are not equivalent** to each other.

(5) The *UC* possesses the advantages of Lagrangian system: **contact discontinuities are resolved sharply**.

(6) In using the *UC*, there is no need to generate a body-fitted mesh prior to computing flow past a body; the **mesh is automatically generated by the flow.**

- **CFD as Numerical Solution to Nonlinear Hyperbolic PDEs**

- G F B Riemann (1860) gave the theoretical foundation:
- ***Riemann invariants*** and ***Riemann problem***

- L F Richardson (1922) proposed weather prediction by numerical computation.

- (wanting to find numerical solutions to fluid flow immediately raises many interesting theoretical and practical questions, and progresses are made in answering them).

- **1**. The discovery of the ***CFL condition*** (Courant, R.,   Friedrichs, K.O. and Lewy, 1928). It simply says that in a time-marching process to find a numerical solution, marching too fast causes numerical instability and destroys the solution.

- **2**. Practical methods for computing solution with shocks are developed:
-
- ***Artificial viscosity method*** of von Neumann and Richtmyer (1950)
- ***Godunov method*** (1959)
- ***Glimm random choice method*** (1965)
- ***Shock-fitting method*** (1972)

- **3.** P D Lax & B Wendroff (1960) pointed out: in order to numerically capture shock discontinuities correctly, the governing PDE should be written in ***conservation form***.

- in Eulerian cordinates: ***easy***
- in Lagrangian coordinates
- for1-D flow: ***easy***
- for 2-D d 3-D flow: Hui, Li & Li (1999)

- 4. To extend Godunov’s method to higher order accuracy: ***limiters*** and ***TVD*** were introduced which avoid non-physical oscillations in high resolution schemes. (J P Boris & D L Book,1973; B van Leer, 1973)

- 5. To search for the optimum coordinate system:

  Particle-in-Cell method (F H Harlow, 1955);
- ***Arbitrary- Lagrangian-Eulerian method*** (C W Hirt, A A Amsden & J L Cook, 1974);
- ***Moving mesh methods*** (J U Brackbill & J S Saltzman, 1982);
- ***Unified coordinate method*** (W H Hui, 2007).

- **6.** For computing flow past a given body:
- ***Automatic mesh generation***
- (W H Hui, G P Zhao & J J J Hu, 2005)

A typical Lagrangian grid

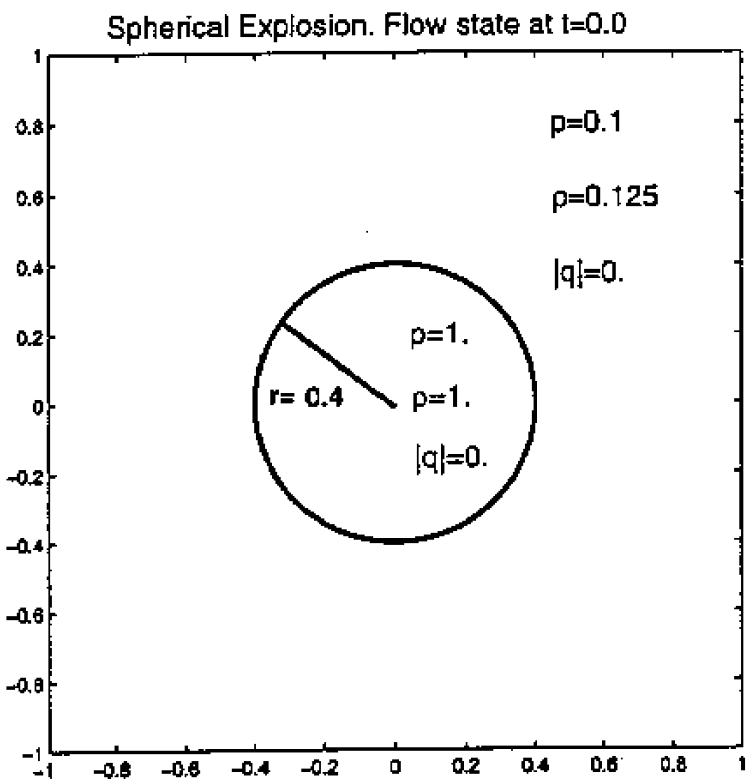
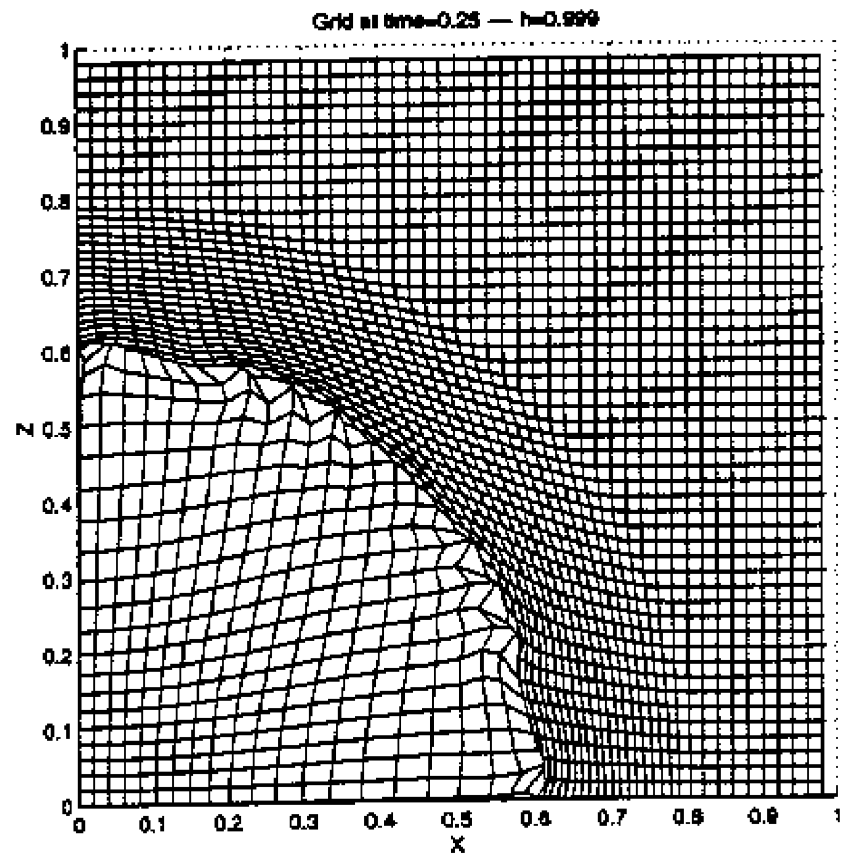
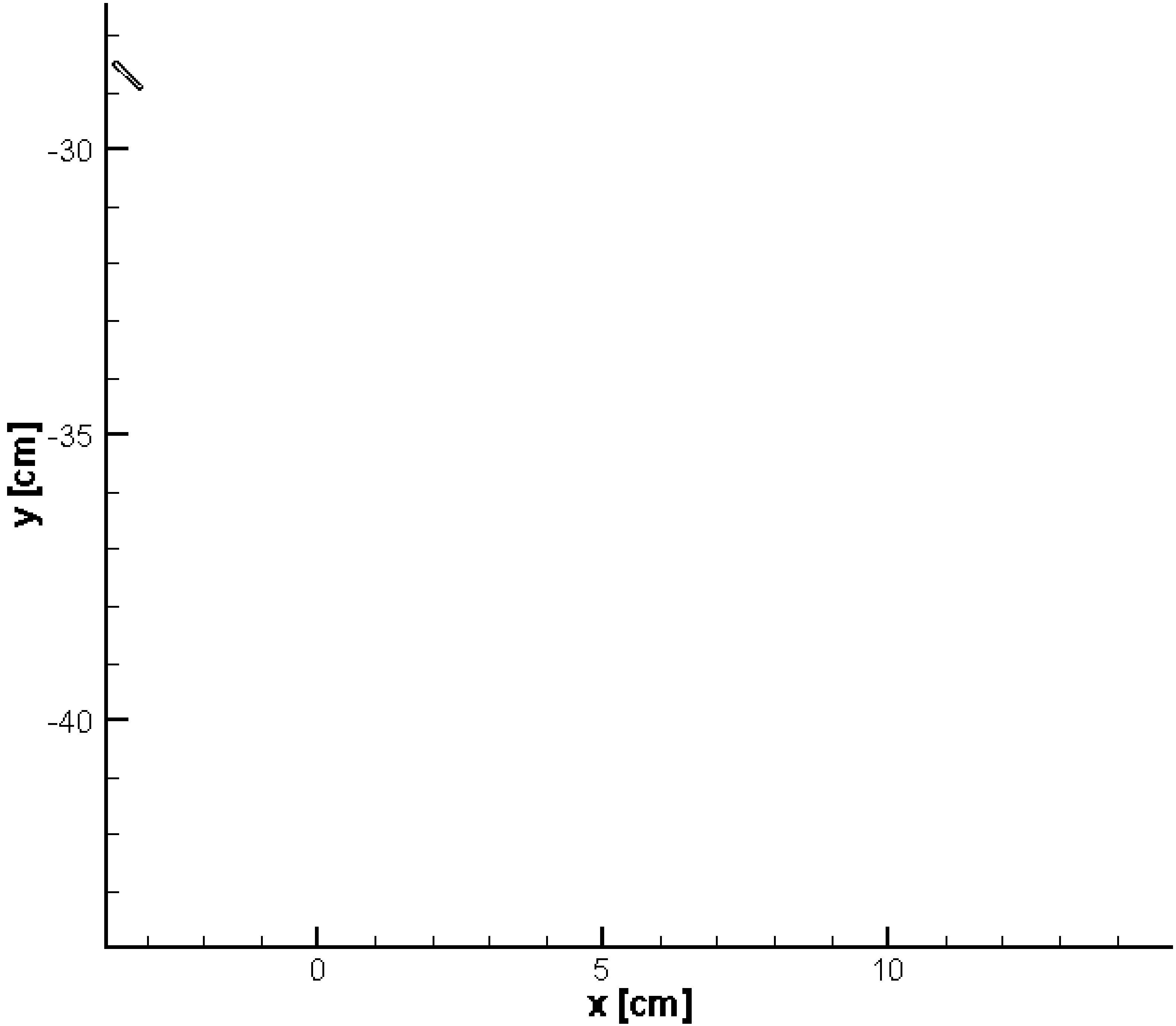


Figure 21: Spherical explosion: Flow state at $t = 0.0$ in the plane $z =$



Computation breaks down soon afterwards


-30
-35
-40
y [cm]
0
5
10
x [cm]

D Serre in "Systems of Conservation Laws" (1999) states "Writing the equations of gas dynamics in Lagrangian coordinates is very complicated if $D \geq 2$ ".

The difficulty lies in the momentum equation

$$\rho \frac{\partial^2 \boldsymbol{x}}{\partial t^2} = -\nabla_{\boldsymbol{x}}\, p$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho u\\ \rho e\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho u\\ \rho u^2+p\\ u(\rho e+p)\end{pmatrix}=0$$

(Eulerian, conservation form and hyperbolic)

$$e=\frac{1}{2}u^2+\frac{1}{\gamma-1}\frac{p}{\rho}$$

By transformation

$$\begin{cases} dt=d\lambda \\ dx=ud\lambda+\frac{1}{\rho}d\xi \end{cases}$$

we get

$$\frac{\partial}{\partial\lambda}\begin{pmatrix}1/\rho\\ u\\ e\end{pmatrix}+\frac{\partial}{\partial\xi}\begin{pmatrix}-u\\ p\\ up\end{pmatrix}=\mathbf{0}$$

(Classical Lagrangian, conservation form and hyperbolic)

(The seminal papers by Von Neumann(1946) and Godunov (1959)

use Lagrangian system. Equivalency established by D H Wagner (1987))

In ***UC***, we use **adaptive Godunov scheme** (Hui & Lapage, *JCP,* ***122,*** *291(1995))* : the shock is fitted (using the Riemann solution), hence we can use a different conservative system

$$\frac{\partial}{\partial\lambda}\begin{pmatrix}1/\rho\\ u\\ p/\rho^{\gamma}\end{pmatrix}+\frac{\partial}{\partial\xi}\begin{pmatrix}-u\\ p\\ 0\end{pmatrix}=\mathbf{0}$$

Determination of Mesh Velocity (*U, V*)

We place two requirements:

(A) coordinate lines η = const. shall be material lines of fluid particles, meaning $\frac{D_q \eta}{Dt} = 0$. With $\frac{D_Q \eta}{Dt} = 0$ this gives

$$(V - v)\,A - (U - u)\,B = 0 \qquad \text{(a)}$$

Observations:

(1) Contact lines, being material lines, must coincide with coordinate lines η = const and, therefore, can be resolved sharply.

(2) As the body surface is a material line, condition (a) guarantees that the unified mesh is automatically a body-fitted mesh.

(3) η (x, y, t) is a level set function.

(B) Mesh angles and orthogonality shall be preserved, yielding an ODE for *U*

$$\frac{\partial U}{\partial \eta} + P(\eta;\lambda,\xi)\,U = Q(\eta;\lambda,\xi) \qquad \text{(b)}$$

U (η) prescribed at η= const.

$$P(\eta;\lambda,\xi) = \frac{S^2}{T^2 J}\left(A\frac{\partial B}{\partial \xi} - B\frac{\partial A}{\partial \xi}\right) - \frac{L}{AJ}\left(A\frac{\partial B}{\partial \eta} - B\frac{\partial A}{\partial \eta}\right)$$

$$Q(\eta;\lambda,\xi) = \frac{S^2 A}{T^2 J}\left(B\frac{\partial u}{\partial \xi} - A\frac{\partial v}{\partial \xi}\right) + \frac{L}{J}\left(A\frac{\partial v}{\partial \eta} - B\frac{\partial u}{\partial \eta}\right) + uP(\eta;\lambda,\xi)$$

$$S^2 = L^2 + M^2, \qquad T^2 = A^2 + B^2$$

Alternatively, we may require area (Jacobian) be preserved.

Hyperbolicity of Lagrangian Gas Dynamic Equations

All eigenvalues are real.

For eigenvalue 0 (multiplicity 6), there exist 4, or 3, or 5 linearly independent eigenvectors, depending on how the differential constraints

$$\frac{\partial A}{\partial \eta} = \frac{\partial L}{\partial \xi}, \quad \frac{\partial B}{\partial \eta} = \frac{\partial M}{\partial \xi}$$

are used.

In all cases, the system is weakly hyperbolic.

Lagrangian re-formulation of Eulerian hyperbolic system will generically leads to a weak hyperbolic system.

Consider inviscid *hyperbolic* Burgers equation

$$u_t + uu_x = 0$$

Transformation

$$\begin{cases} dt = d\lambda \\ dx = ud\lambda + Ad\xi \end{cases} \quad \text{or} \quad \begin{cases} d\lambda = dt \\ d\xi = \frac{1}{A}(dx - udt) \end{cases} \Rightarrow \frac{D\xi}{Dt} = 0$$

leads to

$$\begin{pmatrix} u \\ A \end{pmatrix}_\lambda + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ A \end{pmatrix}_\xi = 0$$

Eigenvalues are: 0, 0. But there is only one eigenvector: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

. Therefore, the system is *weakly hyperbolic*

Derivation of Governing Equations in Conservation PDE Form

in Unified Coordinates

Consider conservation of mass

$$0 = \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j} \qquad (j = 1, 2, 3)$$

$$= \frac{\partial(\rho u_\alpha)}{\partial x_\alpha} \qquad (\alpha = 0, 1, 2, 3), \quad (x_0 = t, \quad u_0 = 1)$$

$$= \int_{\partial\Omega} \rho u_\alpha \hat{d} x_\alpha \qquad (\text{by Gauss devergencetheorem})$$

where Ω is a control volume in $\vec{x}$ space, and

$$\begin{cases} \hat{d}x_0 = dx_1 dx_2 dx_3 \\ \hat{d}x_1 = -dx_2 dx_3 dx_0 \\ \hat{d}x_2 = dx_3 dx_0 dx_1 \\ \hat{d}x_3 = -dx_0 dx_1 dx_2 \end{cases}$$

$\hat{d}\xi_\beta$ are defined similarly.

Transform Cartesian coordinates (t, x, y, z) to the unified coordinates (λ, ξ, η, ζ) via

$$dt = d\lambda$$

$$dx = U d\lambda + A d\xi + L d\eta + P d\varsigma$$

$$dy = V d\lambda + B d\xi + M d\eta + Q d\varsigma$$

$$dz = W d\lambda + C d\xi + N d\eta + R d\varsigma$$

Since from the transformation, we have

$$\hat{d}x_\alpha = U_{\alpha\beta}\hat{d}\xi_\beta$$

where

$$U_{\alpha\beta} = \frac{\partial \hat{x}_\alpha}{\partial \hat{\xi}_\beta}, \qquad \begin{cases} \hat{x}_0 = (x_1, x_2, x_3) \\ \hat{x}_1 = (x_2, x_3, x_0) \\ \hat{x}_2 = (x_3, x_0, x_1) \\ \hat{x}_3 = (x_0, x_1, x_2) \end{cases}$$

we get (by Gauss divergence theorem)

$$0 = \int_{\partial\Omega} \rho u_\alpha \hat{d}x_\alpha = \oint_{\partial\Omega} K_\beta \hat{d}\xi_\beta \;\Rightarrow\; \frac{\partial K_\beta}{\partial \xi_\beta} = 0 \qquad (\beta = 0,1,2,3)$$

This is the mass equation in conservation form, where

$$K_\beta = \rho u_\alpha U_{\alpha\beta}$$

Similarly for the momentum and energy equations.

Summary

in Cartesian coordinates

$$\begin{cases} \dfrac{\partial(\rho u_\alpha)}{\partial x_\alpha} = 0 & (\alpha = 0,1,2,3) \\ \dfrac{\partial(\rho u_j u_\alpha)}{\partial x_\alpha} + \dfrac{\partial p}{\partial x_j} = 0 & (j = 1,2,3) \\ \dfrac{\partial(\rho u_\alpha H)}{\partial x_\alpha} - \dfrac{\partial p}{\partial x_0} = 0 & \end{cases}$$

$$\left(H = e + {}^{p}\!/_{\rho} \right)$$

in unified coordinates

$$\Rightarrow \begin{cases} \dfrac{\partial K_\beta}{\partial \xi_\beta} = 0 & (\beta = 0,1,2,3) \\ \dfrac{\partial\left(K_\beta u_j + pU_{j\beta}\right)}{\partial \xi_\beta} = 0 & \\ \dfrac{\partial\left(K_\beta H - pU_{0\beta}\right)}{\partial \xi_\beta} = 0 & \end{cases}$$

$$(K_\beta = \rho u_\alpha U_{\alpha\beta})$$

The physical laws are now written in conservation PDE form in the unified coordinates, including the Lagrangian.

Summary

$$\begin{cases} \dfrac{\partial(\rho u_\alpha)}{\partial x_d} = 0 & (\alpha = 0,1,2,3) \\ \dfrac{\partial(\rho u_i u_\alpha)}{\partial x_d} + \dfrac{\partial p}{\partial x_i} = 0 & (i = 1,2,3) \\ \dfrac{\partial(\rho u_\alpha H)}{\partial x_d} - \dfrac{\partial p}{\partial x_0} = 0 & \end{cases} \Rightarrow \begin{cases} \dfrac{\partial K_\beta}{\partial \xi_\beta} = 0 & (\beta = 0,1,2,3) \\ \dfrac{\partial\left(K_\beta u_i + pU_{i\beta}\right)}{\partial \xi_\beta} = 0 & \\ \dfrac{\partial\left(K_\beta H - pU_{0\beta}\right)}{\partial \xi_\beta} = 0 & \end{cases}$$

$$\left(H = e + {}^{p}\!/_{\rho}\right)$$

$$\begin{cases} \oint\limits_{\partial\Omega} \rho u_\alpha \hat{d}x_\alpha = 0 \\ \oint\limits_{\partial\Omega} \left(\rho u_i u_\alpha \hat{d}x_\alpha + p\hat{d}x_i\right) = 0 \\ \oint\limits_{\partial\Omega} \left(\rho u_\alpha H \hat{d}x_\alpha + p\hat{d}x_0\right) = 0 \end{cases} \Rightarrow \begin{cases} \oint\limits_{\partial\Omega} K_\beta \hat{d}\xi_\beta = 0 \\ \oint\limits_{\partial\Omega} \left(K_\beta u_i + pU_{i\beta}\right)\hat{d}\xi_\beta = 0 \\ \oint\limits_{\partial\Omega} \left(K_\beta H - pU_{0\beta}\right)\hat{d}\xi_\beta = 0 \end{cases}$$

Potential Applications

(1) Multi-fluid flow: material interfaces

(2) Fluid-solid Interactions: moving interface and small cut cells

(3) Blood flow in arteries: deforming boundary

(4) Debris flow: terrain-following coordinates and moving boundary

(5) Typhoon: bottom topography, slip surface resolution and grid uniformity