Error Minimization of Diffusion Operator

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Outlines

- Derivation of isotropic distribution
- One-dimensional formulation
- Two-dimensional formulation
- Validations

Particle smoothing procedure

- **Original formulation (PS)**

$$\left\langle \nabla^2 \phi \right\rangle_{PS} = \frac{4}{\sum_{j\neq i} \omega(\left|\vec{r}_j - \vec{r}_i\right|, r_e)} \sum_{j\neq i} \frac{(\phi_j - \phi_i)\,\omega(\left|\vec{r}_j - \vec{r}_i\right|, r_e)}{\left|\vec{r}_j - \vec{r}_i\right|^2}$$

or

$$\left\langle \nabla^2 \phi \right\rangle_{PS} = \frac{4\Phi_2^{(0,0)}}{\Omega_0^{(0,0)}}$$

with

$$\Phi_p^{(q,r)} = \sum_{j\neq i} \frac{(\phi_j - \phi_i)(x_j - x_i)^q (y_j - y_i)^r\, \omega(\left|\vec{r}_j - \vec{r}_i\right|, r_e)}{\left|\vec{r}_j - \vec{r}_i\right|^p}$$

$$\Omega_p^{(q,r)} = \sum_{j\neq i} \frac{(x_j - x_i)^q (y_j - y_i)^r\, \omega(\left|\vec{r}_j - \vec{r}_i\right|, r_e)}{\left|\vec{r}_j - \vec{r}_i\right|^p}; \quad O(\Omega_p^{(q,r)}) = O(\delta^{q+r-p})$$

Accuracy analysis

- **Taylor-series expansion**

$$\phi_j - \phi_i = \phi_x(x_j - x_i) + \phi_y(y_j - y_i) + \frac{1}{2}\phi_{xx}(x_j - x_i)^2 + \phi_{xy}(x_j - x_i)(y_j - y_i) + \frac{1}{2}\phi_{yy}(y_j - y_i)^2$$

$$+O(\delta^3)$$

$$\Phi_p^{(q,r)} = \Omega_p^{(q+1,r)}\phi_x + \Omega_p^{(q,r+1)}\phi_y + \frac{1}{2}\Omega_p^{(q+2,r)}\phi_{xx} + \Omega_p^{(q+1,r+1)}\phi_{xy} + \frac{1}{2}\Omega_p^{(q,r+2)}\phi_{yy} + O(\delta^{q+r+3-p})$$

- **Original formulation**

$$\left\langle \nabla^2\phi \right\rangle_{PS} = \frac{4\Omega_2^{(1,0)}}{\Omega_0^{(0,0)}}\phi_x + \frac{4\Omega_2^{(0,1)}}{\Omega_0^{(0,0)}}\phi_y + \frac{2\Omega_2^{(2,0)}}{\Omega_0^{(0,0)}}\phi_{xx} + \frac{4\Omega_2^{(1,1)}}{\Omega_0^{(0,0)}}\phi_{xy} + \frac{2\Omega_2^{(0,2)}}{\Omega_0^{(0,0)}}\phi_{yy} + O(\delta)$$

therefore

$$\left\langle \nabla^2\phi \right\rangle_{PS} - \nabla^2\phi = O(\delta^{-1})$$

Isotropic distribution(1)

- **Invariant after rotation**

$$\sum_{j\neq i}\frac{[(x_j-x_i)+I(y_j-y_i)]^n\,\omega(|\vec{r}_j-\vec{r}_i|,r_e)}{|\vec{r}_j-\vec{r}_i|^p}=0 \qquad n\geq 1;\quad I^2=-1$$

- **Geometrical relation**

$$\Omega_p^{(2+a,b)}+\Omega_p^{(a,2+b)}=\Omega_{p-2}^{(a,b)}$$

n=1

$$\Omega_p^{(1,0)}=\Omega_p^{(0,1)}=0$$

n=2

$$\begin{pmatrix}\Omega_p^{(2,0)}\\ \Omega_p^{(1,1)}\\ \Omega_p^{(0,2)}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix}\Omega_{p-2}^{(0,0)}$$

Isotropic distribution(2)

n=3

$$\Omega_p^{(3,0)} = \Omega_p^{(1,2)} = \Omega_p^{(1,2)} = \Omega_p^{(0,3)} = 0$$

n=4

$$\begin{pmatrix} \Omega_p^{(4,0)} \\ \Omega_p^{(3,1)} \\ \Omega_p^{(2,2)} \\ \Omega_p^{(1,3)} \\ \Omega_p^{(0,4)} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 3 \end{pmatrix} \Omega_{p-4}^{(0,0)}$$

others

$$\begin{pmatrix} \Omega_p^{(6,0)} \\ \Omega_p^{(4,2)} \\ \Omega_p^{(2,4)} \\ \Omega_p^{(0,6)} \end{pmatrix} = \frac{1}{16} \begin{pmatrix} 5 \\ 1 \\ 1 \\ 5 \end{pmatrix} \Omega_{p-6}^{(0,0)}; \quad \begin{pmatrix} \Omega_p^{(8,0)} \\ \Omega_p^{(6,2)} \\ \Omega_p^{(4,4)} \\ \Omega_p^{(2,6)} \\ \Omega_p^{(0,8)} \end{pmatrix} = \frac{1}{128} \begin{pmatrix} 35 \\ 5 \\ 3 \\ 5 \\ 35 \end{pmatrix} \Omega_{p-8}^{(0,0)}$$

Isotropic distribution(3)

- **Consider**

$$\Phi_4^{(2,0)} = \Omega_4^{(3,0)}\phi_x + \Omega_4^{(2,1)}\phi_y + \frac{1}{2}\Omega_4^{(4,0)}\phi_{xx} + \Omega_4^{(3,1)}\phi_{xy} + \frac{1}{2}\Omega_4^{(2,2)}\phi_{yy} = \frac{\Omega_0^{(0,0)}}{16}(3\phi_{xx} + \phi_{yy})$$

$$\Phi_4^{(0,2)} = \Omega_4^{(1,2)}\phi_x + \Omega_4^{(0,3)}\phi_y + \frac{1}{2}\Omega_4^{(2,2)}\phi_{xx} + \Omega_4^{(1,3)}\phi_{xy} + \frac{1}{2}\Omega_4^{(0,4)}\phi_{yy} = \frac{\Omega_0^{(0,0)}}{16}(\phi_{xx} + 3\phi_{yy})$$

We have

$$\phi_{xx} = \frac{2}{\Omega_0^{(0,0)}}(3\Omega_4^{(2,0)} - \Omega_4^{(0,2)});\ \ \phi_{yy} = \frac{2}{\Omega_0^{(0,0)}}(-\Omega_4^{(2,0)} + 3\Omega_4^{(0,2)})$$

and

$$\left\langle \nabla^2\phi \right\rangle_{PS} = \frac{\phi_{xx}}{\Delta x^2} + \frac{\phi_{yy}}{\Delta y^2} = \frac{2}{\Omega_0^{(0,0)}}\left[\left(\frac{3}{\Delta x^2} - \frac{1}{\Delta y^2}\right)\Phi_4^{(2,0)} + \left(\frac{3}{\Delta y^2} - \frac{1}{\Delta x^2}\right)\Phi_4^{(0,2)}\right]$$

One-dimensional formulation

- **First derivative**

$$\Phi_2^{(1)} = \Omega_0^{(0)}\phi_x + O(\delta); \ \langle \phi_x \rangle = \frac{\Phi_2^{(1)}}{\Omega_0^{(0)}}$$

- **Second derivative**

$$\Phi_2^{(0)} = \Omega_2^{(1)}\phi_x + \frac{1}{2}\Omega_2^{(2)}\phi_{xx} + O(\delta);$$

$$\Phi_2^{(0)} - \Omega_2^{(1)}\langle \phi_x \rangle = \Phi_2^{(0)} - \frac{\Omega_2^{(1)}}{\Omega_0^{(0)}}\Phi_2^{(1)} = \frac{1}{2}[\frac{(\Omega_0^{(0)})^2 - \Omega_2^{(1)}\Omega_2^{(3)}}{\Omega_0^{(0)}}]\phi_{xx} + O(\delta)$$

- **Elimination of artificial velocity**

$$\langle \phi_{xx} \rangle = \frac{2\Omega_0^{(0)}}{(\Omega_0^{(0)})^2 - \Omega_2^{(1)}\Omega_2^{(3)}}[\Phi_2^{(0)} - \frac{\Omega_2^{(1)}}{\Omega_0^{(0)}}\Phi_2^{(1)}] = \phi_{xx} + O(\delta)$$

Two-dimensional formulation(1)

- **First derivative**

$$\Phi_2^{(1,0)} = \Omega_2^{(2,0)}\phi_x + \Omega_2^{(1,1)}\phi_y + O(\delta); \quad \Phi_2^{(0,1)} = \Omega_2^{(1,1)}\phi_x + \Omega_2^{(0,2)}\phi_y + O(\delta)$$

$$\left\langle \begin{matrix} \phi_x \\ \phi_y \end{matrix} \right\rangle = \begin{pmatrix} \Omega_2^{(2,0)} & \Omega_2^{(1,1)} \\ \Omega_2^{(1,1)} & \Omega_2^{(0,2)} \end{pmatrix}^{-1} \begin{pmatrix} \Phi_2^{(1,0)} \\ \Phi_2^{(0,1)} \end{pmatrix}$$

- **Second derivative**

$$\Phi_p^{(q,r)} = \begin{pmatrix} \Omega_p^{(q+1,r)} \\ \Omega_p^{(q,r+1)} \end{pmatrix}^T \begin{pmatrix} \phi_x \\ \phi_y \end{pmatrix} + \begin{pmatrix} \Omega_p^{(q+2,r)} \\ \Omega_p^{(q+1,r+1)} \\ \Omega_p^{(q,r+2)} \end{pmatrix}^T \begin{pmatrix} \phi_{xx}/2 \\ \phi_{xy} \\ \phi_{yy}/2 \end{pmatrix} + O(\delta^{q+r+3-p})$$

Two-dimensional formulation(2)

- **Elimination of artificial velocity**

$$\hat{\Phi}_p^{(q,r)} = \Phi_p^{(q,r)} - \begin{pmatrix} \Omega_p^{(q+1,r)} \\ \Omega_p^{(q,r+1)} \end{pmatrix}^T \left\langle \begin{matrix} \phi_x \\ \phi_y \end{matrix} \right\rangle = \Phi_p^{(q,r)} - \begin{pmatrix} \Omega_p^{(q+1,r)} \\ \Omega_p^{(q,r+1)} \end{pmatrix}^T \begin{pmatrix} \Omega_2^{(2,0)} & \Omega_2^{(1,1)} \\ \Omega_2^{(1,1)} & \Omega_2^{(0,2)} \end{pmatrix}^{-1} \begin{pmatrix} \Phi_2^{(1,0)} \\ \Phi_2^{(0,1)} \end{pmatrix}$$

$$= \begin{pmatrix} T_{p,xx}^{(q,r)} \\ T_{p,xy}^{(q,r)} \\ T_{p,yy}^{(q,r)} \end{pmatrix}^T \begin{pmatrix} \phi_{xx}/2 \\ \phi_{xy} \\ \phi_{yy}/2 \end{pmatrix} + O(\delta^{q+r+3-p})$$

- **Effective diffusivity**

$$\begin{pmatrix} T_{p,xx}^{(q,r)} \\ T_{p,xy}^{(q,r)} \\ T_{p,yy}^{(q,r)} \end{pmatrix} = \begin{pmatrix} \Omega_p^{(q+2,r)} \\ \Omega_p^{(q+1,r+1)} \\ \Omega_p^{(q,r+2)} \end{pmatrix} - \begin{pmatrix} \Omega_2^{(3,0)} & \Omega_2^{(2,1)} \\ \Omega_2^{(2,1)} & \Omega_2^{(1,2)} \\ \Omega_2^{(1,2)} & \Omega_2^{(0,3)} \end{pmatrix} \begin{pmatrix} \Omega_2^{(2,0)} & \Omega_2^{(1,1)} \\ \Omega_2^{(1,1)} & \Omega_2^{(0,2)} \end{pmatrix}^{-1} \begin{pmatrix} \Omega_p^{(q+1,r)} \\ \Omega_p^{(q,r+1)} \end{pmatrix}$$

Error measure

- **Difference between numerical and physical diffusivity tensor**

$$E = \left\| \begin{matrix} T_{p,xx}^{(q,r)} - 2 & T_{p,xy}^{(q,r)} \\ T_{p,xy}^{(q,r)} & T_{p,yy}^{(q,r)} - 2 \end{matrix} \right\|$$

One-parameter

- **Based on isotropic formulation**

$$\left\langle \nabla^2 \phi \right\rangle = c[(\frac{3}{\Delta x^2} - \frac{1}{\Delta y^2})\hat{\Phi}_4^{(2,0)} + (\frac{3}{\Delta y^2} - \frac{1}{\Delta x^2})\hat{\Phi}_4^{(0,2)}]$$

- **Minimization of error**

$$\frac{\partial E}{\partial c} = 0$$

which leads to

$$c = \frac{2(m_{xx} + m_{xy})}{m_{xx}^2 + 2m_{xy}^2 + m_{yy}^2}$$

and

$$m_{xx} = [3 - (\frac{\Delta x}{\Delta y})^2]T_{4,xx}^{(2,0)} + [3(\frac{\Delta x}{\Delta y})^2 - 1)]T_{4,xx}^{(0,2)}; \quad m_{xy} = (3\frac{\Delta y}{\Delta x} - \frac{\Delta x}{\Delta y})T_{4,xy}^{(2,0)} + (3\frac{\Delta x}{\Delta y} - \frac{\Delta y}{\Delta x})T_{4,xy}^{(0,2)}$$

$$m_{yy} = [3(\frac{\Delta y}{\Delta x})^2 - 1)]T_{4,yy}^{(2,0)} + [3 - (\frac{\Delta y}{\Delta x})^2]T_{4,yy}^{(0,2)}$$

Two-parameter

- **Add cross derivative term**

$$\left\langle \nabla^2 \phi \right\rangle = c[(\frac{3}{\Delta x^2} - \frac{1}{\Delta y^2})\hat{\Phi}_4^{(2,0)} + (\frac{3}{\Delta y^2} - \frac{1}{\Delta x^2})\hat{\Phi}_4^{(0,2)}] + \frac{c_{xy}}{\Delta x \Delta y}\hat{\Phi}_4^{(1,1)}$$

- **Minimization of error**

$$\frac{\partial E}{\partial c} = \frac{\partial E}{\partial c_{xy}} = 0$$

which leads to

$$[m_{xx}^2 + 2m_{xy}^2 + m_{yy}^2]c + [\frac{\Delta x}{\Delta y} m_{xx} T_{4,xx}^{(1,1)} + 2m_{xy} T_{4,xy}^{(1,1)} + \frac{\Delta y}{\Delta x} m_{yy} T_{4,yy}^{(1,1)}]c_{xy} = 2[m_{xx} + m_{yy}]$$

$$[\frac{\Delta x}{\Delta y} m_{xx} T_{4,xx}^{(1,1)} + 2m_{xy} T_{4,xy}^{(1,1)} + \frac{\Delta y}{\Delta x} m_{yy} T_{4,yy}^{(1,1)}]c + [(\frac{\Delta x}{\Delta y})^2 (T_{4,xx}^{(1,1)})^2 + 2(T_{4,xy}^{(1,1)})^2 + (\frac{\Delta y}{\Delta x})^2 (T_{4,yy}^{(1,1)})^2]c_{xy}$$

$$= 2[\frac{\Delta x}{\Delta y} T_{4,xx}^{(1,1)} + \frac{\Delta y}{\Delta x} T_{4,yy}^{(1,1)}]$$

Three-parameter

- **Complete second derivative terms**

$$\left\langle \nabla^2 \phi \right\rangle = c_{xx}\hat{\Phi}_4^{(2,0)} + c_{xy}\hat{\Phi}_4^{(1,1)} + c_{yy}\hat{\Phi}_4^{(0,2)}$$

- **Consistent expression**

$$E = 0$$

leads to

$$\begin{pmatrix} T_{4,xx}^{(2,0)} & T_{4,xx}^{(1,1)} & T_{4,xx}^{(0,2)} \\ T_{4,xy}^{(2,0)} & T_{4,xy}^{(1,1)} & T_{4,xy}^{(0,2)} \\ T_{4,yy}^{(2,0)} & T_{4,yy}^{(1,1)} & T_{4,yy}^{(0,2)} \end{pmatrix} \begin{pmatrix} c_{xx} \\ c_{xy} \\ c_{yy} \end{pmatrix} = \begin{pmatrix} 2/\Delta x^2 \\ 0 \\ 2/\Delta y^2 \end{pmatrix}$$

1a-smooth field

- **Solution field**

$$\phi(x) = ax^2 + bx + c$$

- **Resulting terms**

$$\Phi_2^{(0)} = a\Omega_2^{(2)} + b\Omega_2^{(1)}; \quad \Phi_2^{(1)} = a\Omega_2^{(3)} + b\Omega_2^{(2)}$$

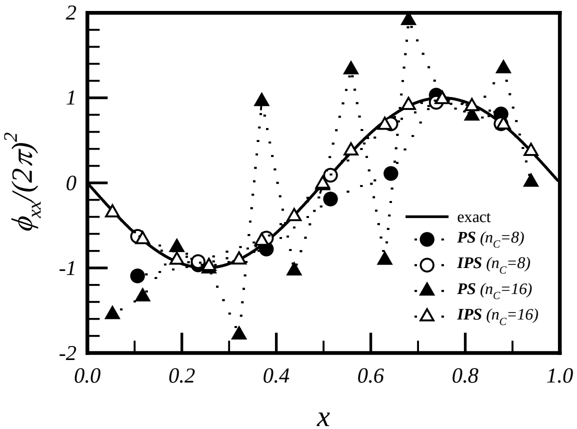
leads to exact solution: $\langle \phi_{xx} \rangle = 2a = \phi_{xx}$

1b-operator test

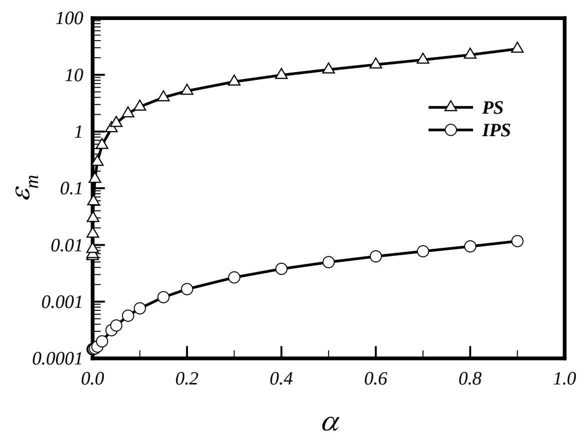
- **Solution field and particle distribution**

$$\phi(x) = \sin(2\pi x); \qquad x_i = (i-1)\Delta x + \alpha(\chi - 0.5)\Delta x, \quad \Delta x = 1/n_C$$

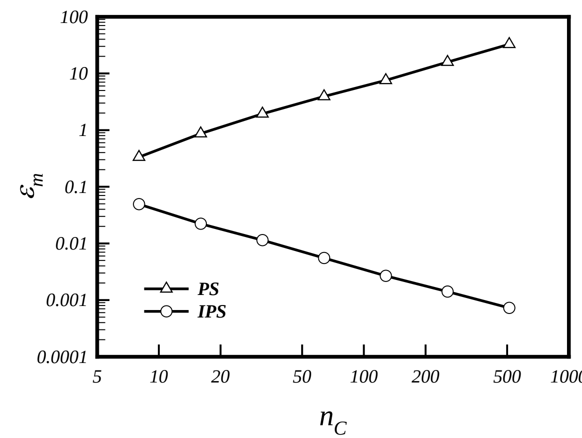
- **Results**



$\alpha = 0.3$



$n_C = 128$



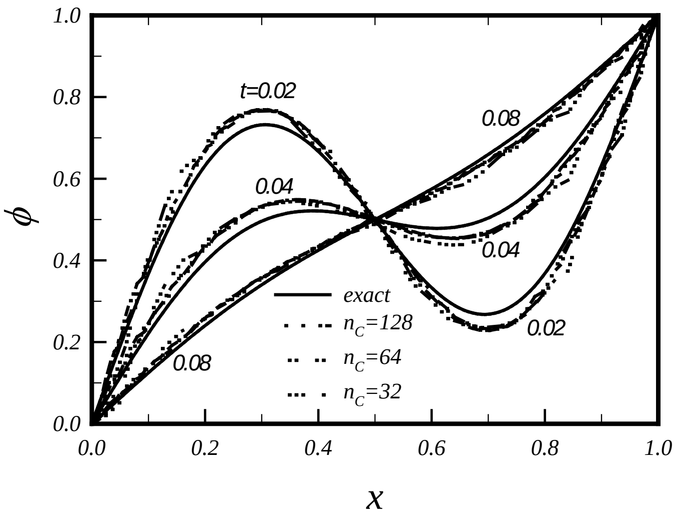
$\alpha = 0.3$

1c-conduction

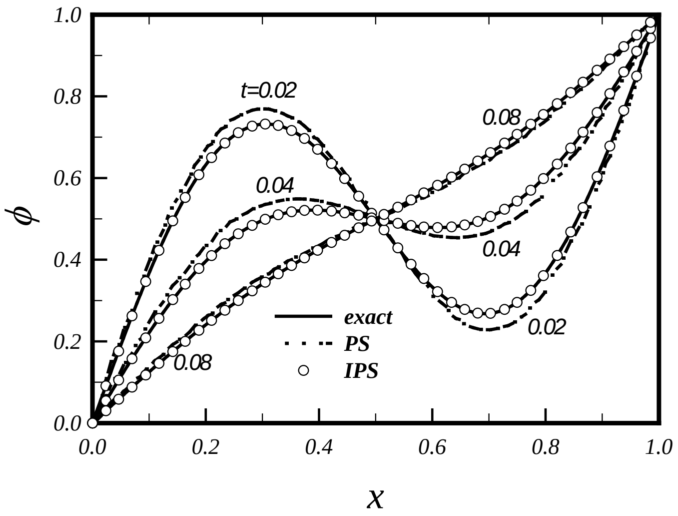
- **Governing equation and exact solution**

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}, \quad \phi_{ex}(x,t) = e^{-(2\pi)^2 t} \sin(2\pi x) + x$$

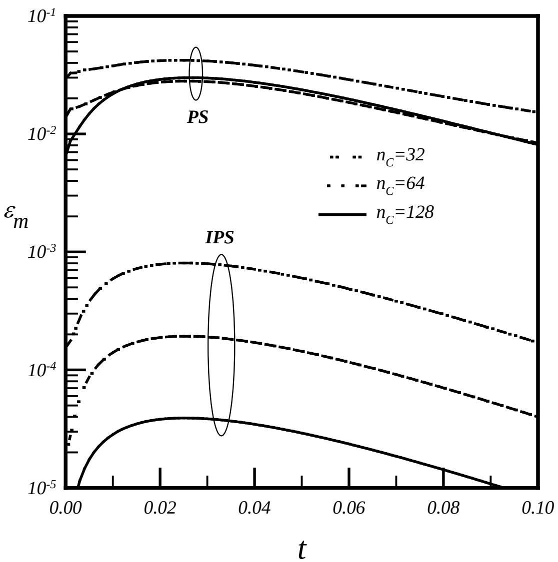
- **Results**



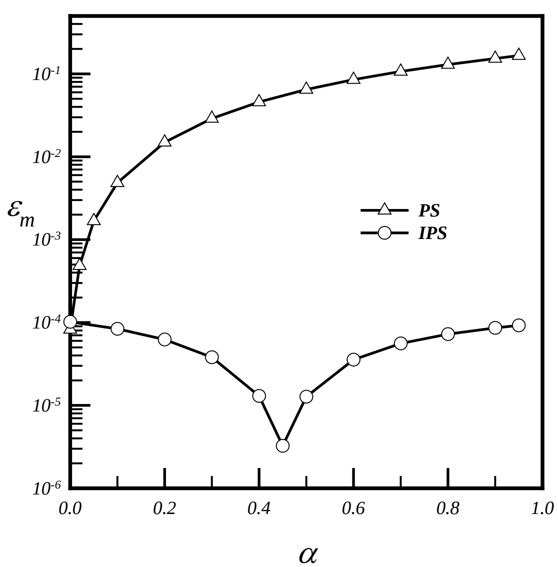
$PS\ (\alpha = 0.3, C_D = 1.0)$



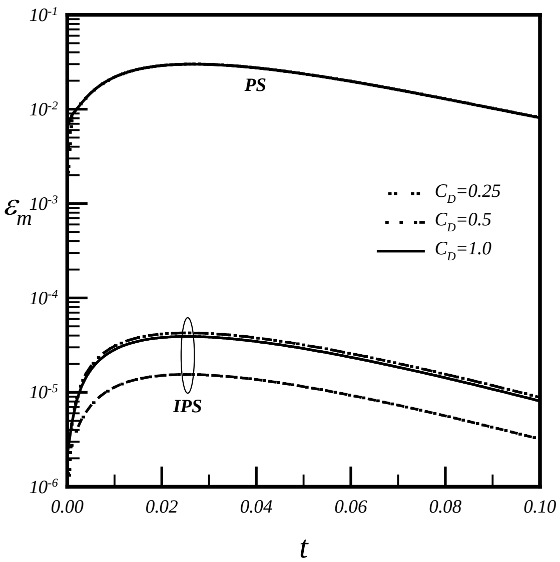
$n_C = 128, \alpha = 0.3, C_D = 1.0$



$\alpha = 0.3, C_D = 1.0$



$n_C = 128, C_D = 1.0$

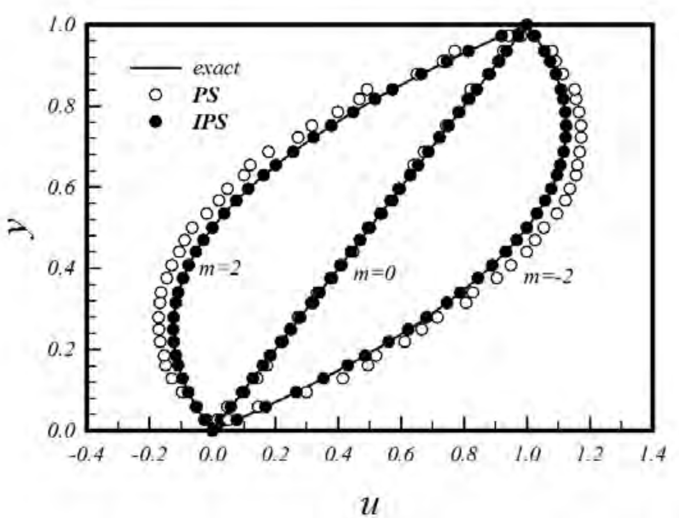


$n_C = 128, \alpha = 0.3$

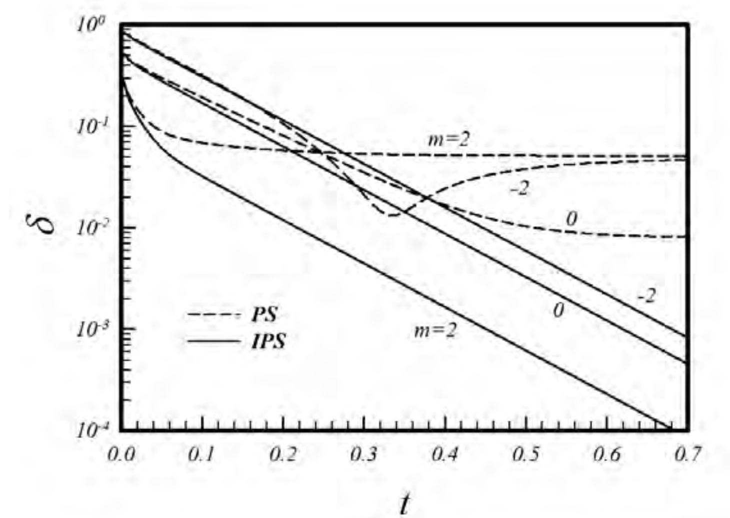
1d-flow

- **Governing equation and exact solution**

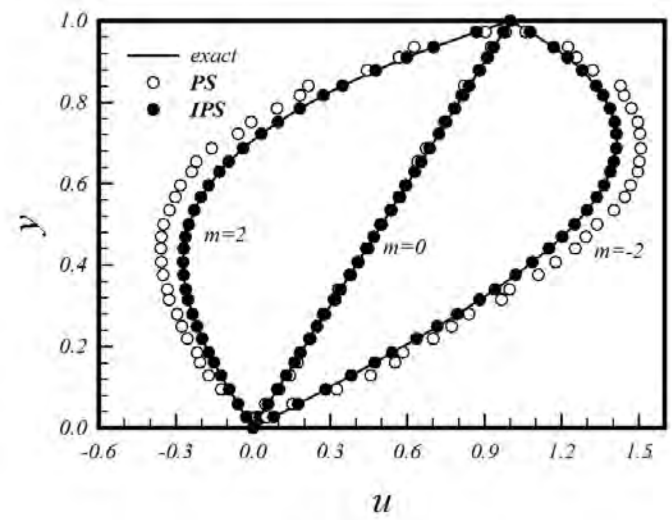
$$\rho \frac{\partial u}{\partial t} = -\frac{dp}{dz} y^{\kappa} + \mu \frac{\partial^2 u}{\partial y^2}, \quad u_{ex}(y,t) = m(y^{\kappa+2} - y) + u_T y, m = \frac{1}{(\kappa+1)(\kappa+2)\mu} \frac{dp}{dz}$$



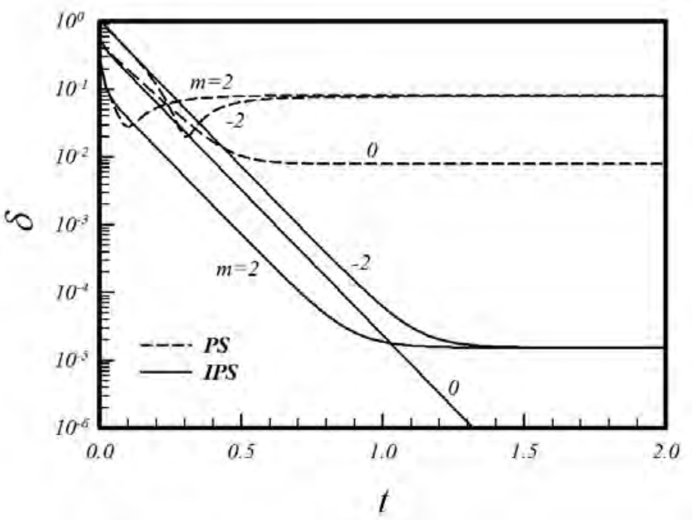
$\kappa = 0, \quad t = 1.0\,(n_C = 32, \alpha = 0.3, C_D = 1.0)$



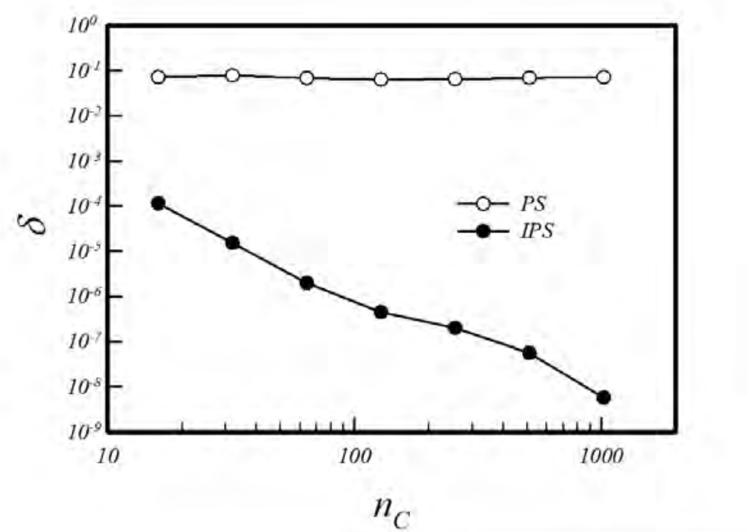
$\kappa = 0\,(n_C = 32, \alpha = 0.3, C_D = 1.0)$



$\kappa = 1, \quad t = 1.0\,(n_C = 32, \alpha = 0.3, C_D = 1.0)$



$\kappa = 1\,(n_C = 32, \alpha = 0.3, C_D = 1.0)$



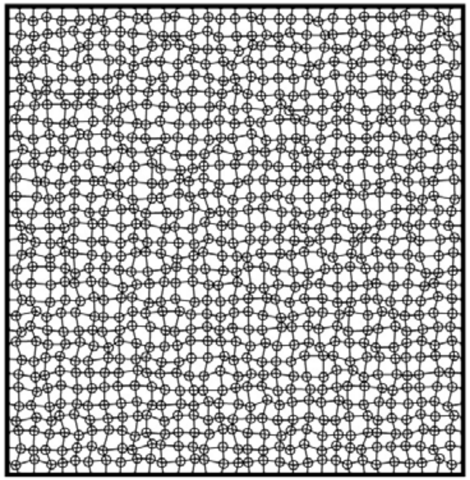
$\kappa = 1, \quad m = 2, \quad t = 2.0\,(\alpha = 0.3, C_D = 1.0)$

2a-operator test

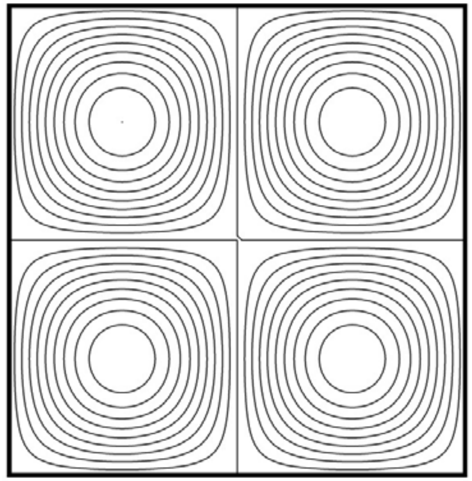
- **Solution field and particle distribution**

$$\phi(x,y) = \sin(2\pi x)\sin(2\pi y)$$

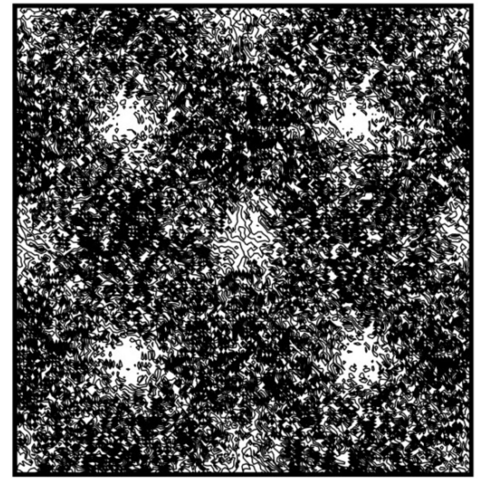
$$x_{ij} = (i-1)\Delta x + \alpha(\chi_{ij}^{(x)} - 0.5)\Delta x, \quad y_{ij} = (j-1)\Delta y + \alpha(\chi_{ij}^{(y)} - 0.5)\Delta y$$



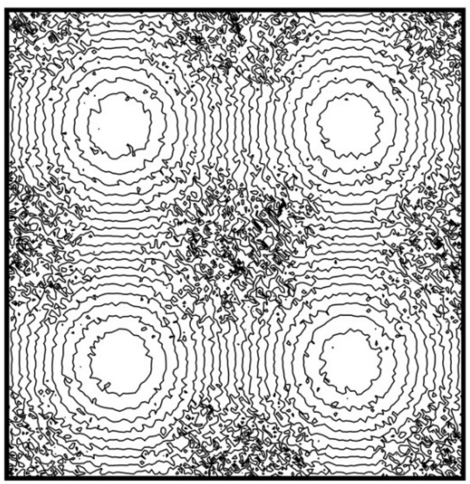
$n_C = 32, \quad \alpha = 0.5$



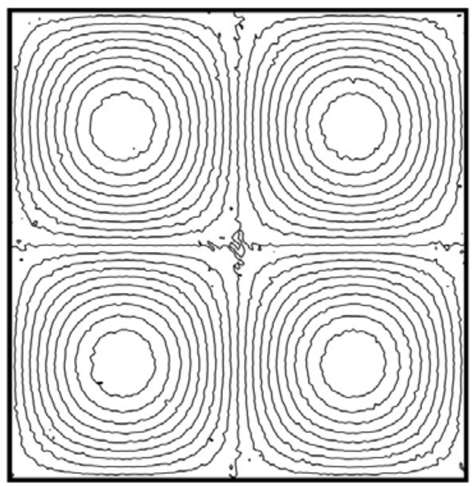
exact



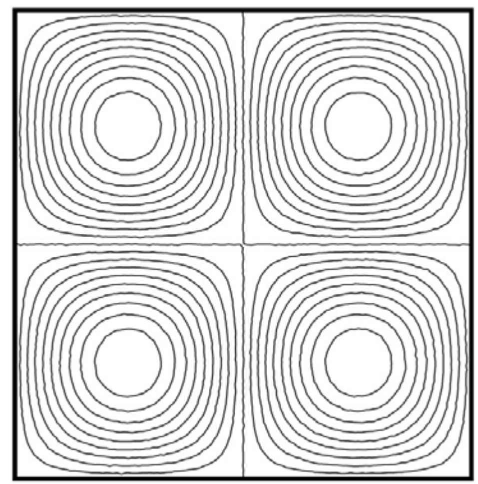
$PS\,(n_C = 128, \quad \alpha = 0.5)$



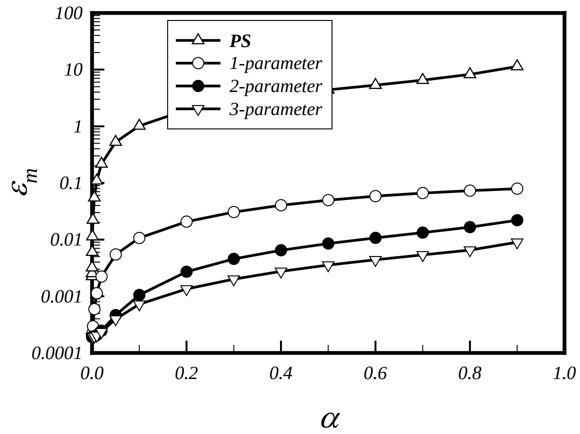
$1-parameter\,(n_C = 128, \quad \alpha = 0.5)$

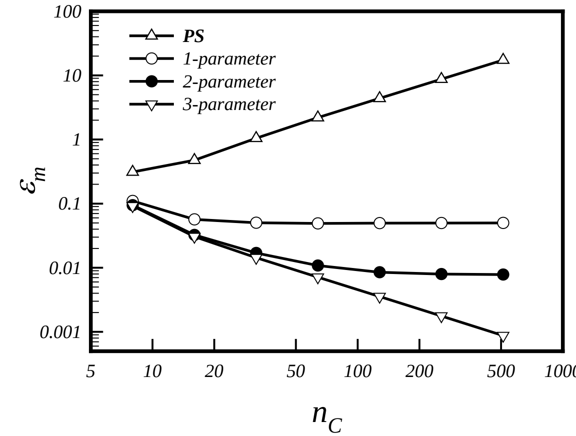


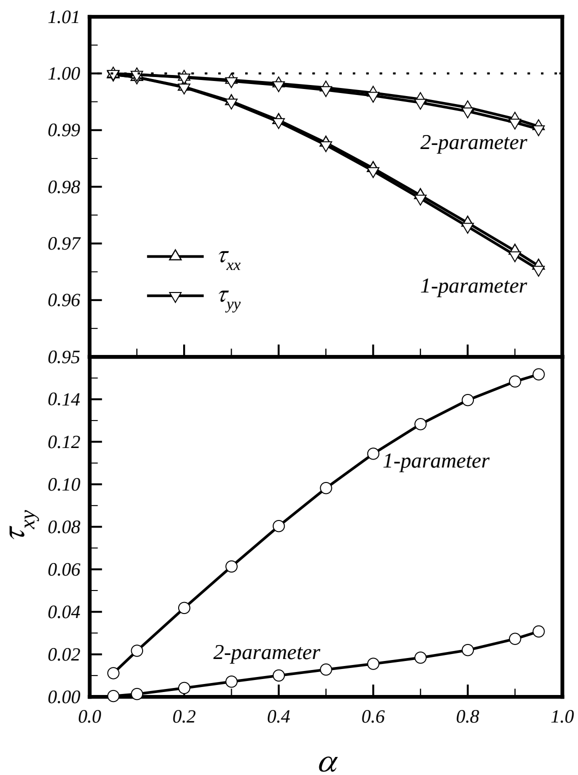
$2-parameter\,(n_C = 128, \quad \alpha = 0.5)$



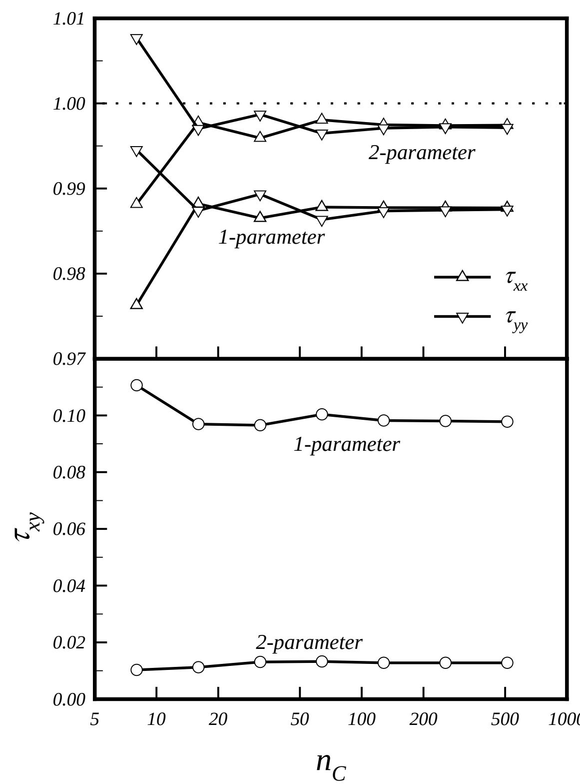
$3-parameter\,(n_C = 128, \quad \alpha = 0.5)$







$n_C = 128$



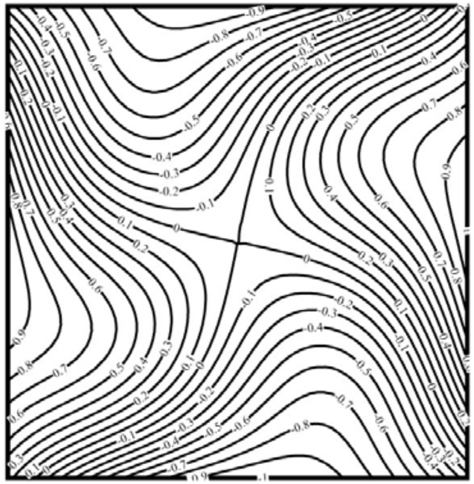
$\alpha = 0.5$

2b-conduction

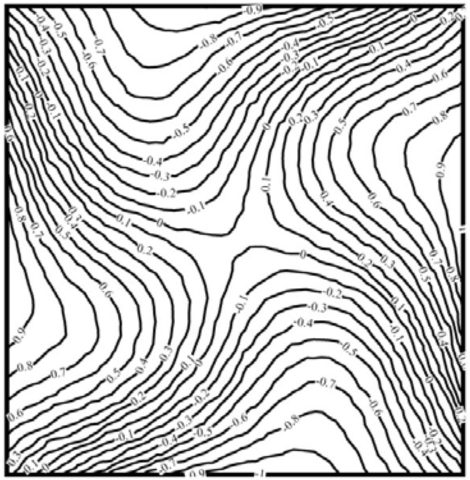
- **Governing equation and exact solution**

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

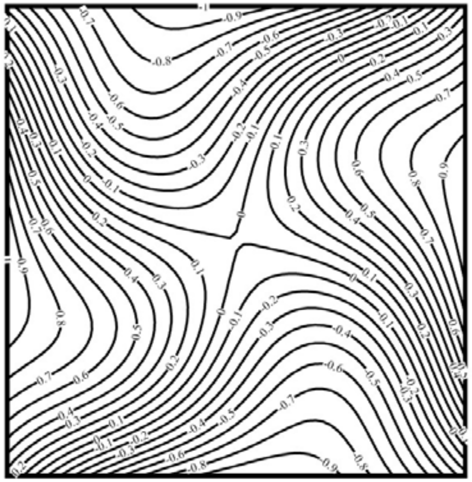
$$\phi_{ex}(x, y, t) = e^{-2(2\pi)^2 t} \sin(2\pi x)\sin(2\pi y) + 4x(x-1) - 4y(y-1) + (x-0.5)(y-0.5)$$



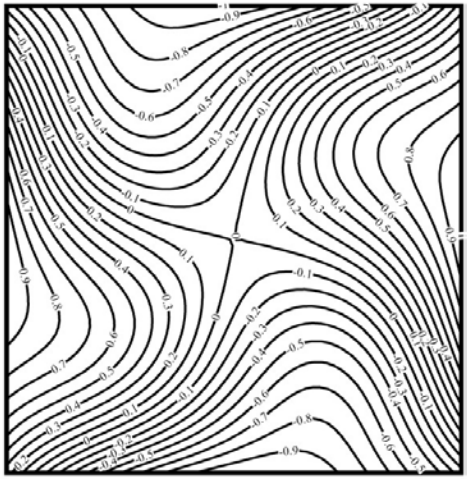
exact



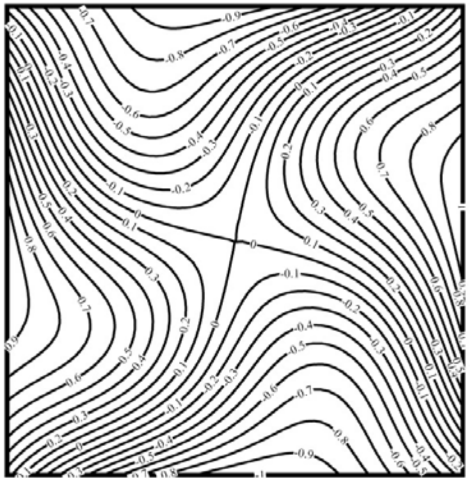
PS



$1-parameter$

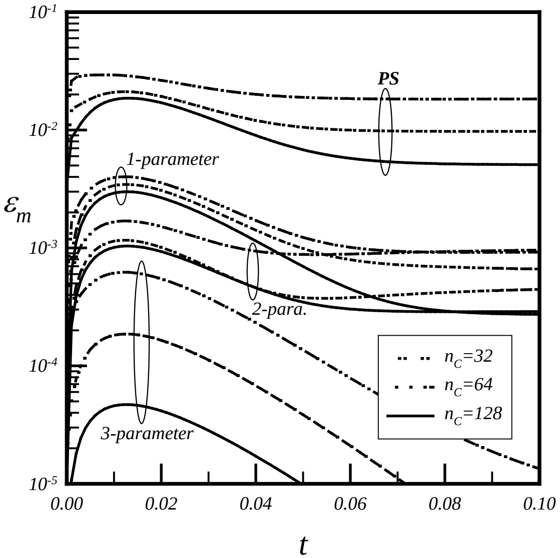


$2-parameter$

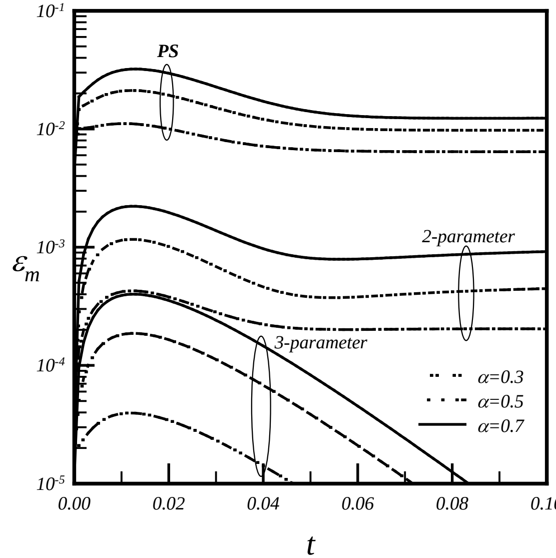


$3-parameter$

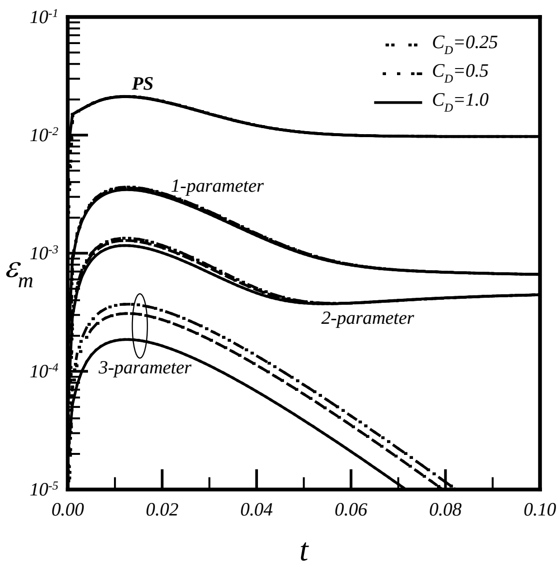
$$t = 0.01\,(\,n_C = 64, \alpha = 0.5, C_D = 1.0\,)$$



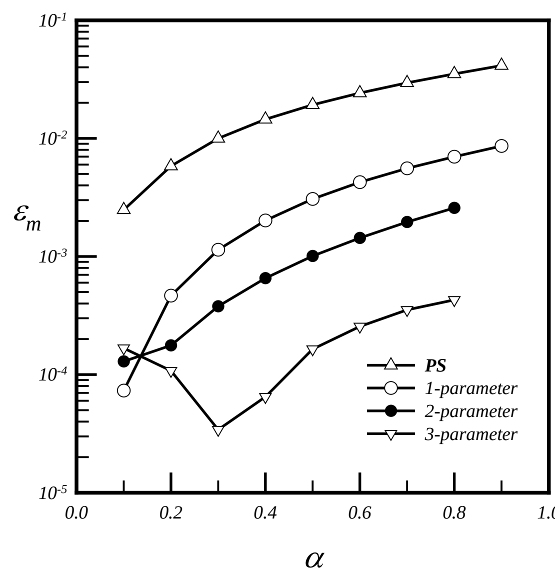
$\alpha = 0.5, C_D = 1.0$



$n_C = 64, C_D = 1.0$



$n_C = 64, \alpha = 0.5$



$t = 0.02\,(n_C = 64, C_D = 1.0)$

2c-Elliptic duct flow

- **Governing equation and elliptical duct**

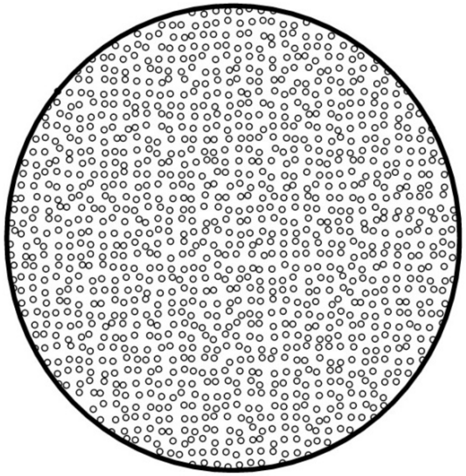
$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{\partial u}{\partial x^2} + \frac{\partial u}{\partial y^2}\right), \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- **Steady-state solution**

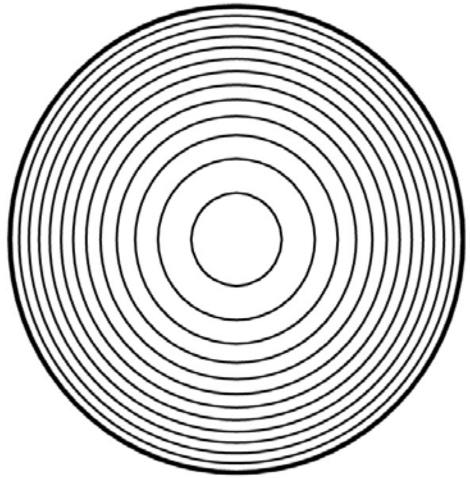
$$u = \frac{1}{2\mu}\left(-\frac{\partial p}{\partial z}\right)\frac{a^2b^2}{a^2+b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

with

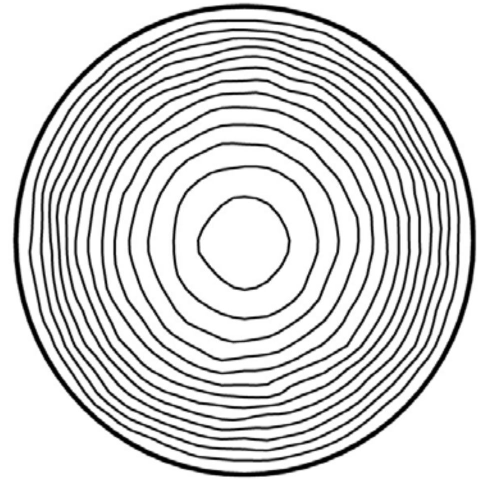
$$\rho = \mu = -\frac{\partial p}{\partial z} = a = 1$$



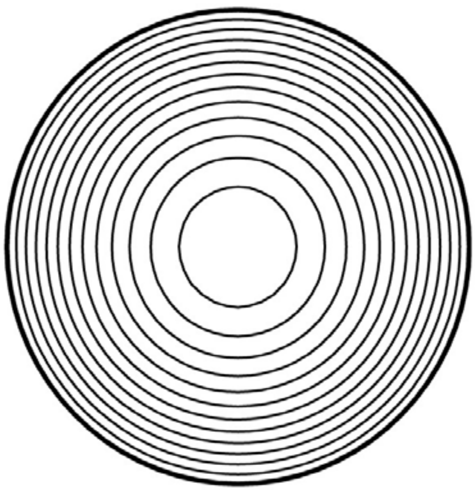
$\Delta x = 1/40, \alpha_x = \alpha_y = 0.5$



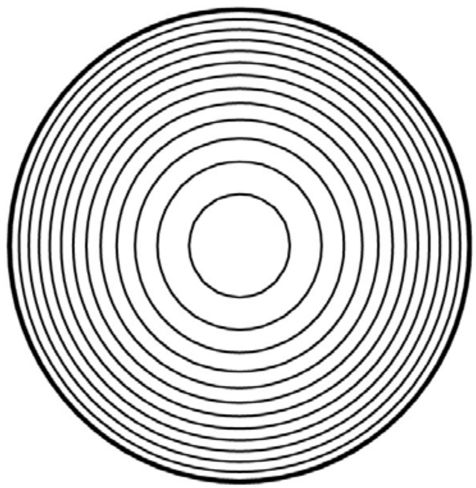
$exact\ (\Delta u = 0.02)$



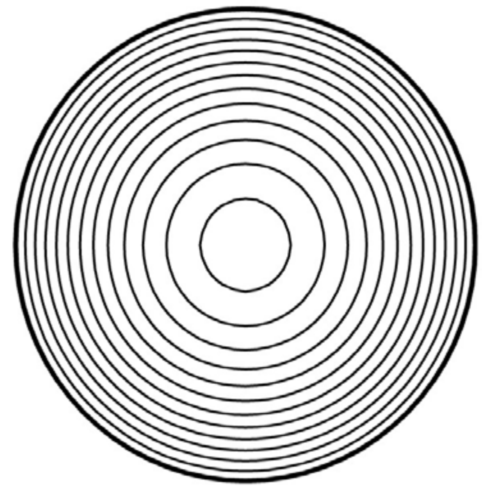
PS



$1-parameter$

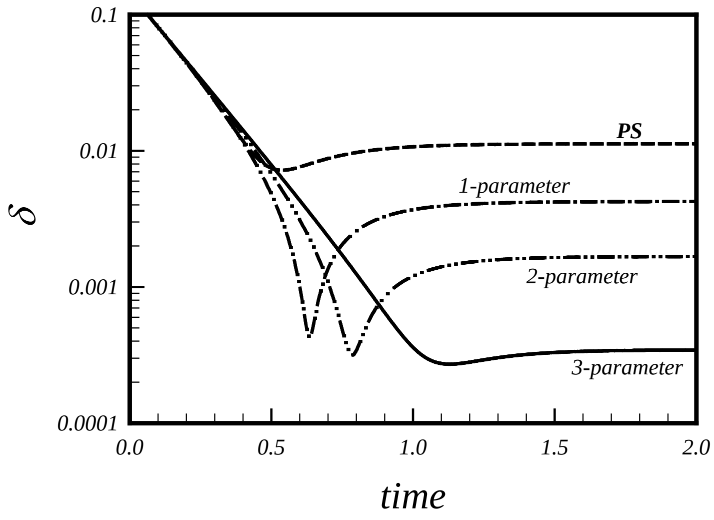


$2-parameter$

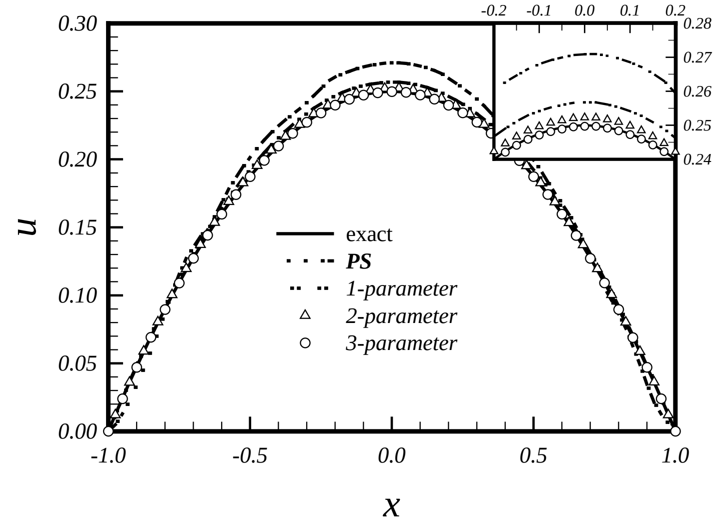


$3-parameter$

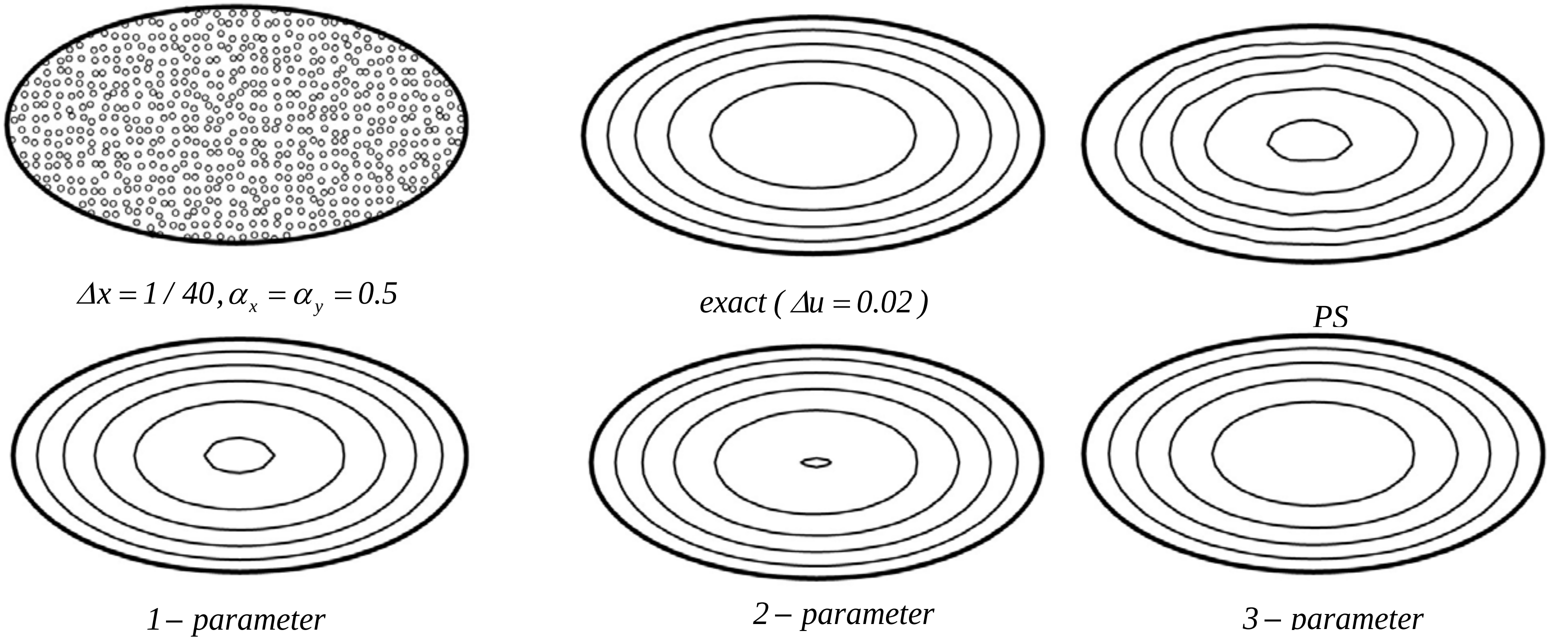
$$t = 10\ (b = 1, C_D = 1.0)$$

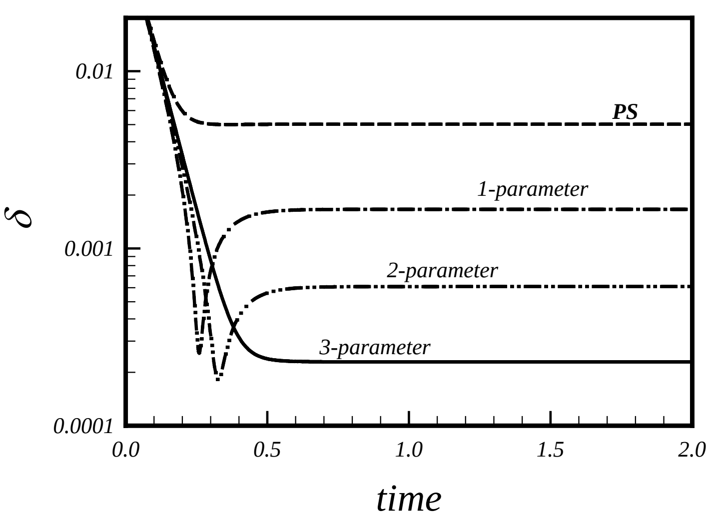


Derivation from steady – state solution

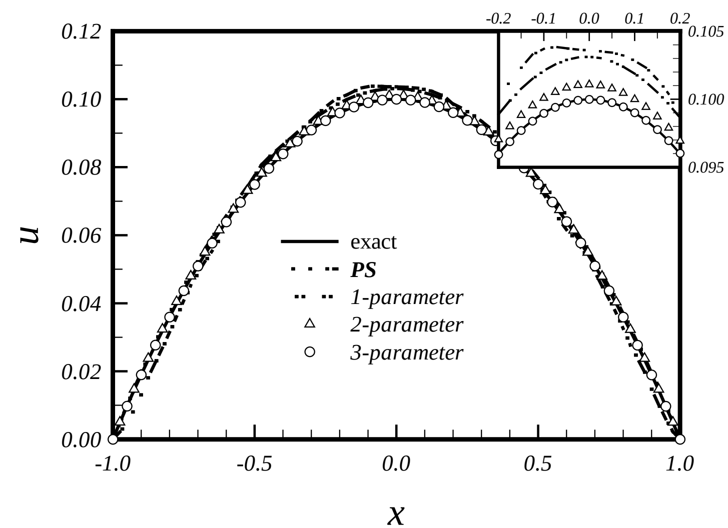


along $y = 0$



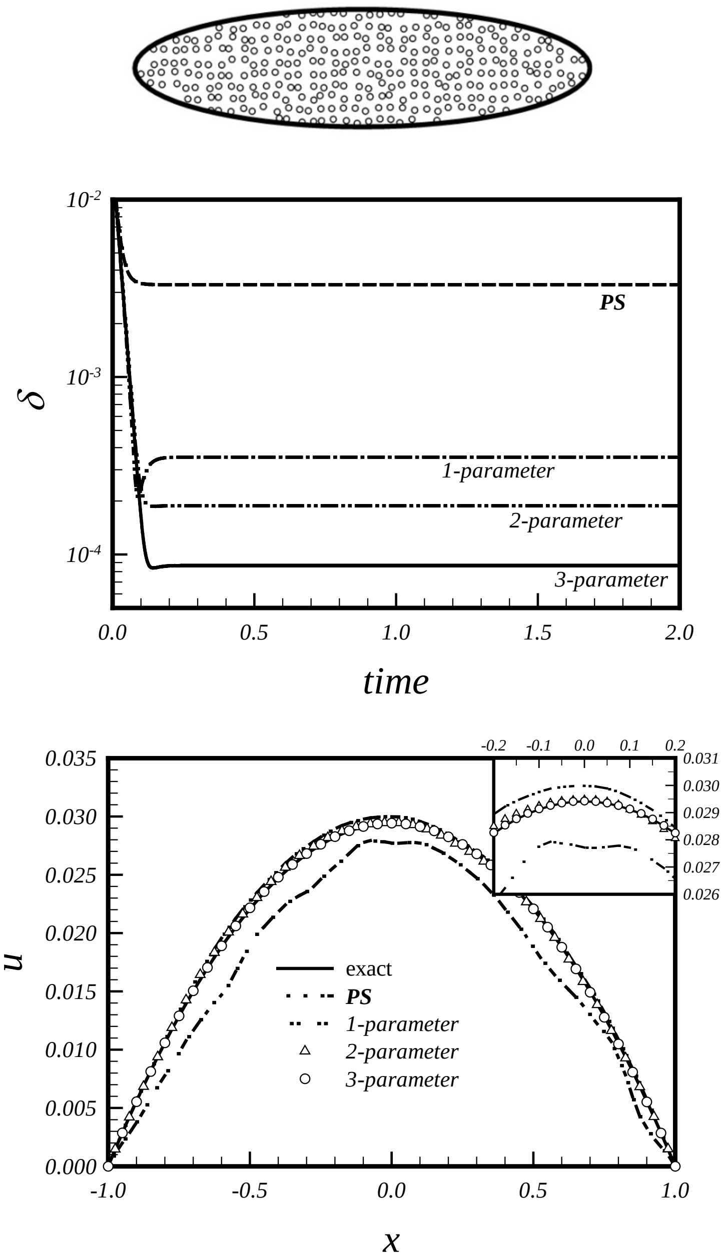


Derivation from steady – state solution



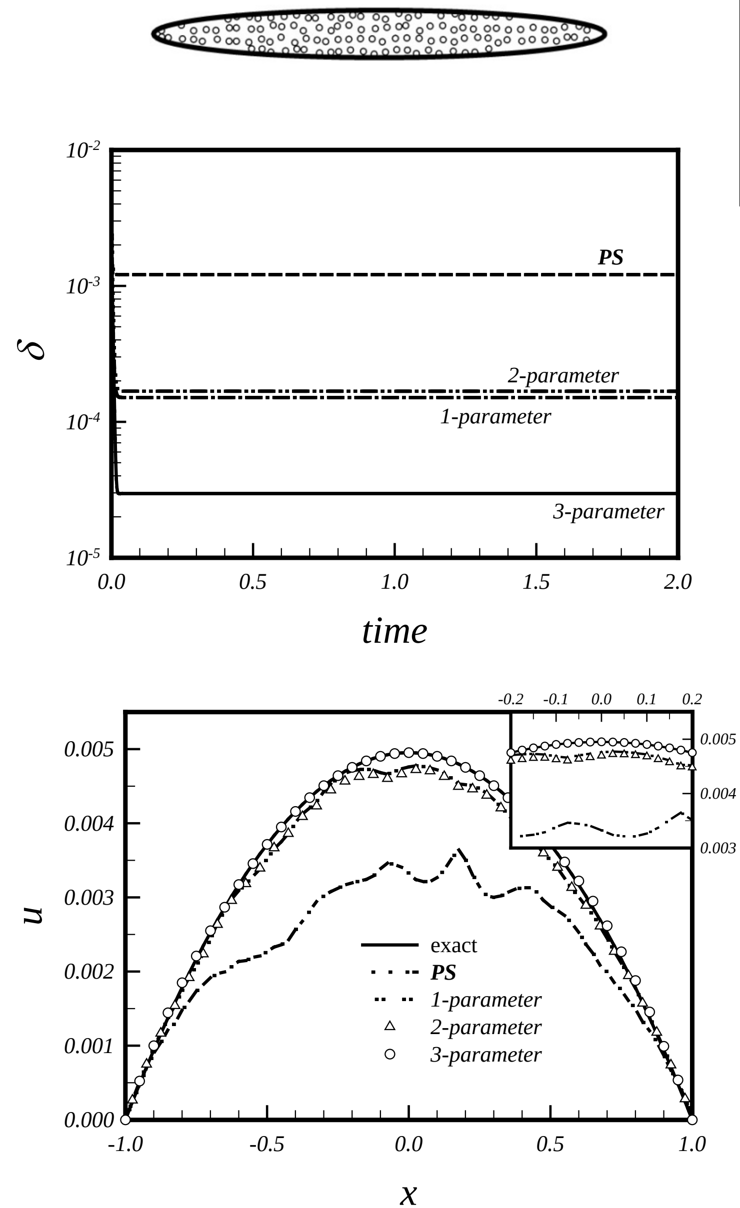
along $y = 0$

$b = 0.5$



10-2
10-3
10-4
δ
PS
1-parameter
2-parameter
3-parameter
0.0 0.5 1.0 1.5 2.0
time
0.035 0.030 0.025 0.020 0.015 0.010 0.005 0.000
u
-0.2 -0.1 0.0 0.1 0.2
0.031 0.030 0.029 0.028 0.027 0.026
exact
PS
1-parameter
2-parameter
3-parameter
-1.0 -0.5 0.0 0.5 1.0
x

$b = 0.5$



10-2
10-3
10-4
10-5
δ
PS
2-parameter
1-parameter
3-parameter
0.0 0.5 1.0 1.5 2.0
time
0.005 0.004 0.003 0.002 0.001 0.000
u
-0.2 -0.1 0.0 0.1 0.2
0.005 0.004 0.003
exact
PS
1-parameter
2-parameter
3-parameter
-1.0 -0.5 0.0 0.5 1.0
x

$b = 0.25$

2-d Fully developed temperature

- **Governing equation and elliptical duct**

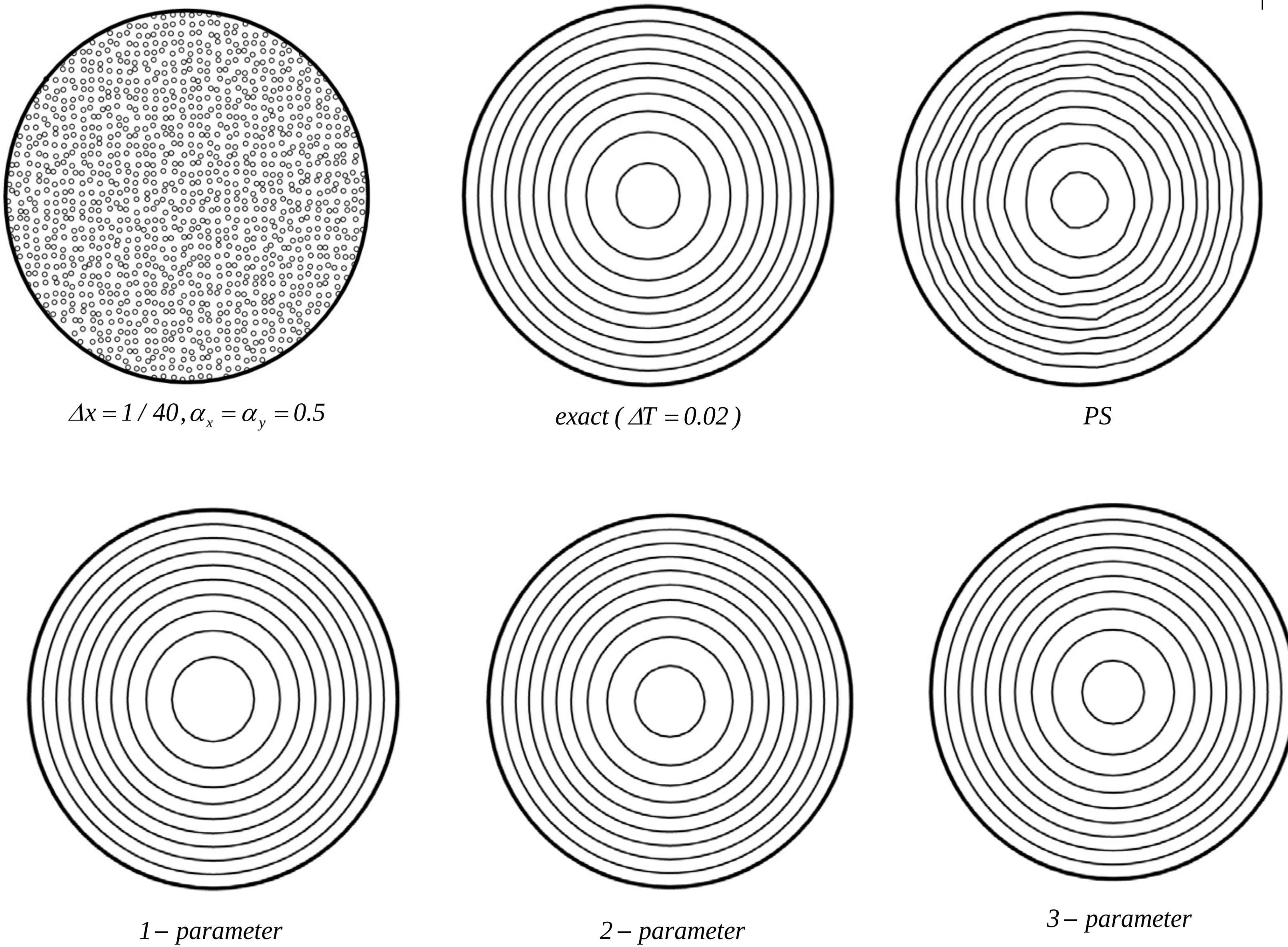
$$\frac{\partial T}{\partial t} + u\frac{dT_m}{dz} = \alpha\left(\frac{\partial T}{\partial x^2} + \frac{\partial T}{\partial y^2}\right)$$

- **Fully developed solution**

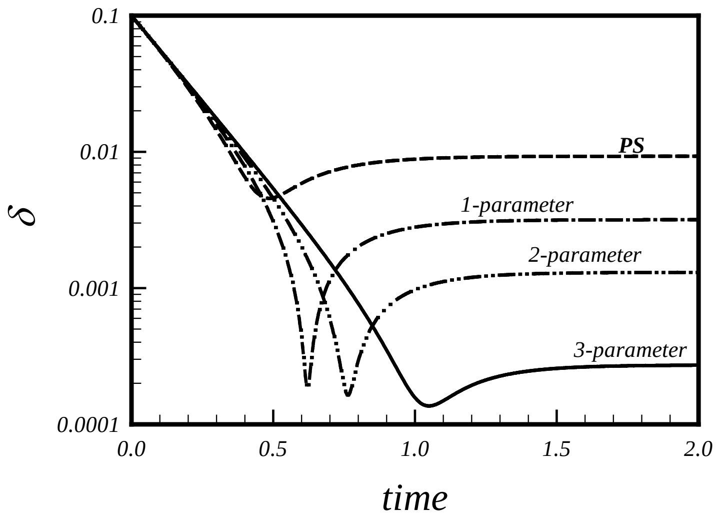
$$T - T_W = -AR^2\left[\frac{3}{16} + \frac{1}{16}\left(\frac{r}{R}\right)^4 - \frac{1}{4}\left(\frac{r}{R}\right)^2\right]$$

with

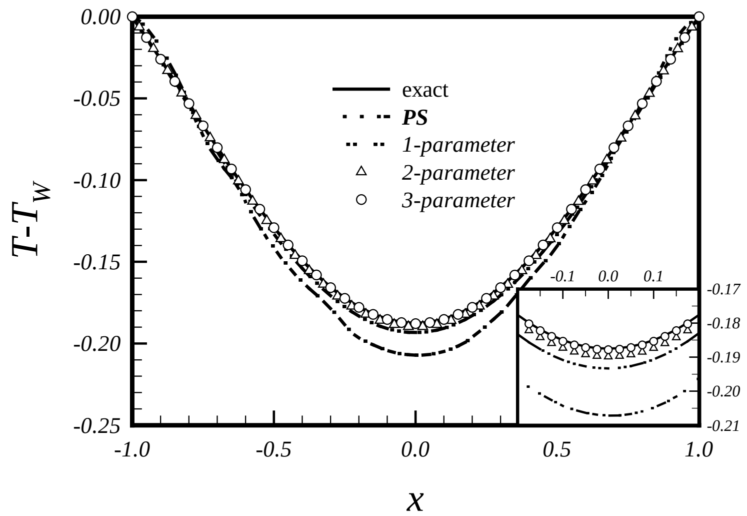
$$A = \frac{2u_m}{\alpha}\frac{dT_m}{dz}, u = 2u_m\left[1 - \left(\frac{r}{R}\right)^2\right], R = 1, \alpha = 1, u_m = 1/2, \frac{dT_m}{dz} = 1$$


$\Delta x = 1/40, \alpha_x = \alpha_y = 0.5$
$exact\ (\Delta T = 0.02)$
PS
$1-parameter$
$2-parameter$
$3-parameter$

$$t = 10\ (C_D = 1.0)$$



Derivation from steady – state solution



along $y = 0$

2-e Flow over cylinder

- **Governing equation**

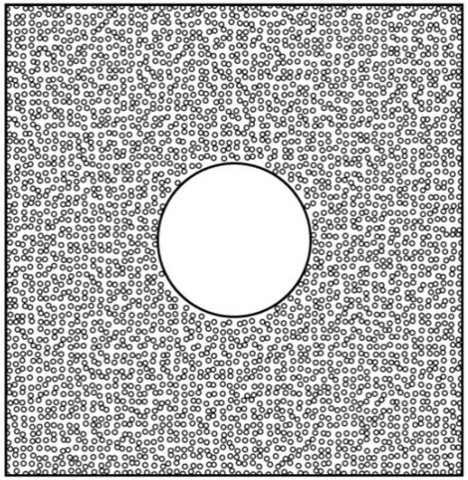
$$\frac{\partial \psi}{\partial t} = (\frac{\partial \psi}{\partial x^2} + \frac{\partial \psi}{\partial y^2})$$

- **Steady-state solution**

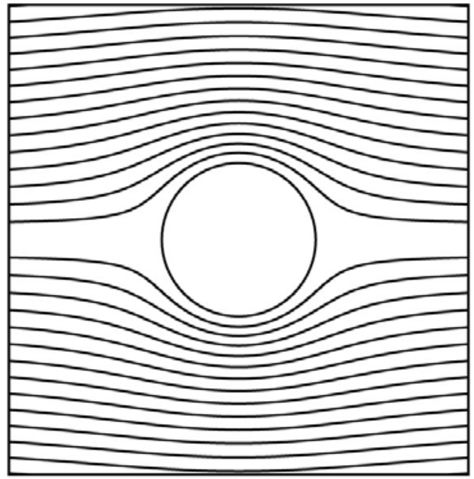
$$\psi = V_0 (r - \frac{r_0^2}{r}) \sin\theta + \frac{\Gamma}{2\pi} \ell n(r)$$

with

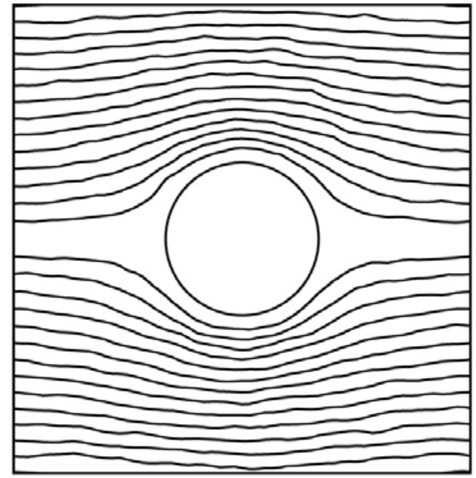
$$A = \frac{2u_m}{\alpha} \frac{dT_m}{dz}, u = 2u_m [1 - (\frac{r}{R})^2], R = 1, \alpha = 1, u_m = 1/2, \frac{dT_m}{dz} = 1$$



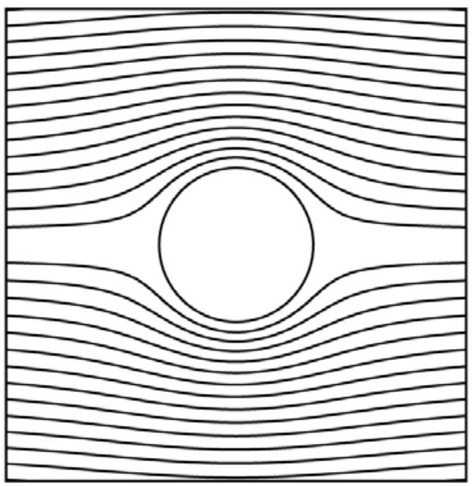
$\Delta x = 1/40, \alpha_x = \alpha_y = 0.5$



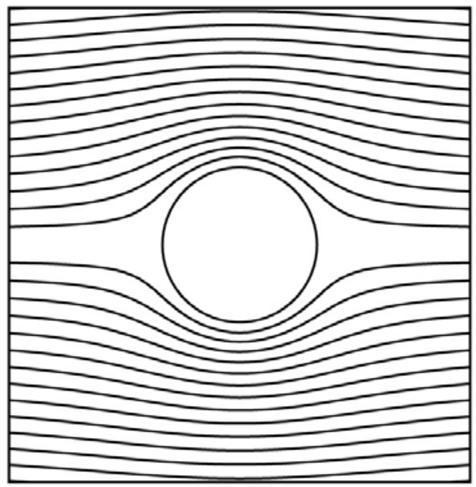
$exact\ (\Delta\psi = 0.2)$



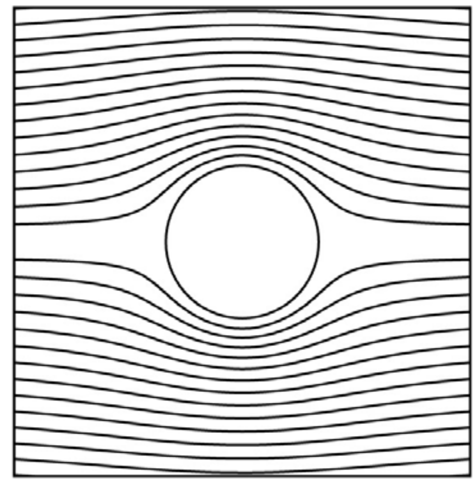
PS



$1-parameter$

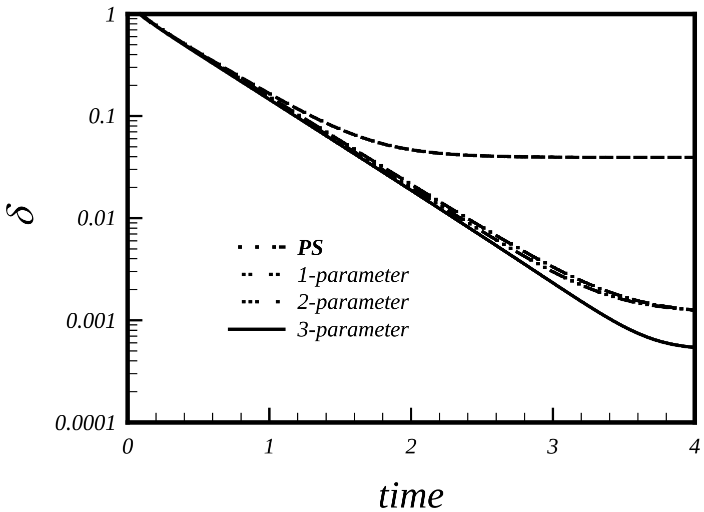


$2-parameter$

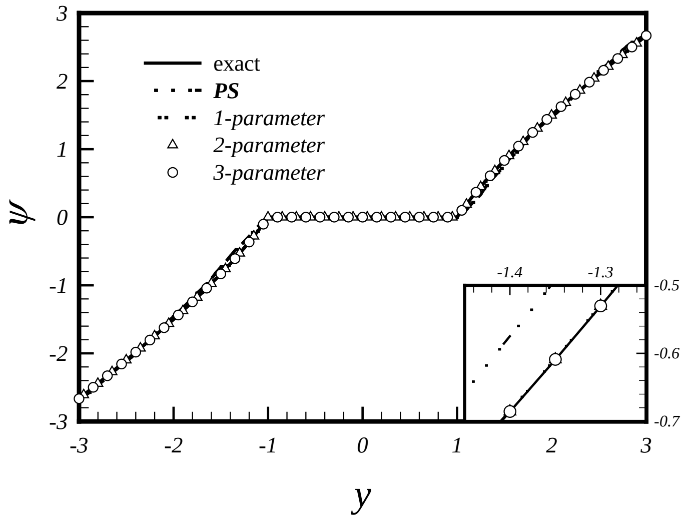


$3-parameter$

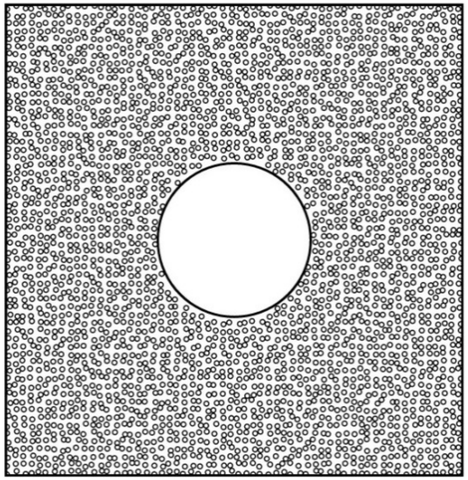
$$\Gamma = 0$$



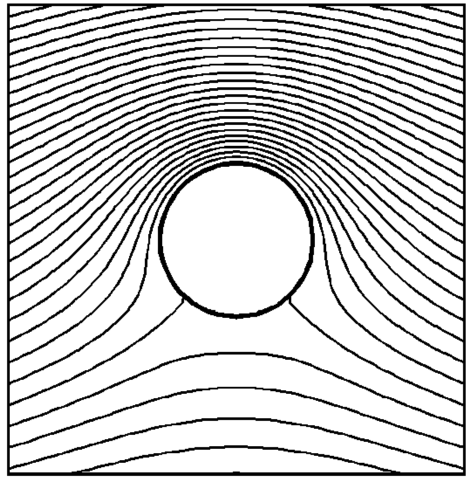
Derivation from steady – state solution



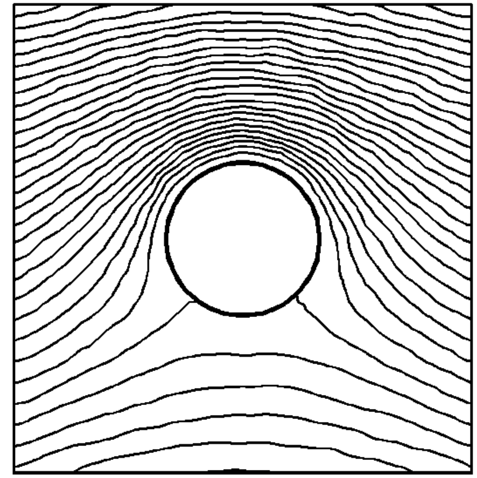
along $y = 0$



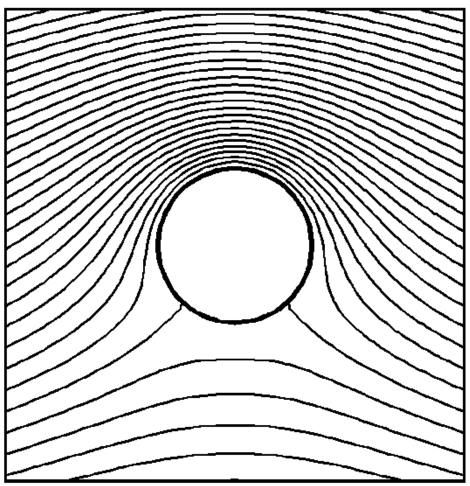
$\Delta x = 1/40, \alpha_x = \alpha_y = 0.5$



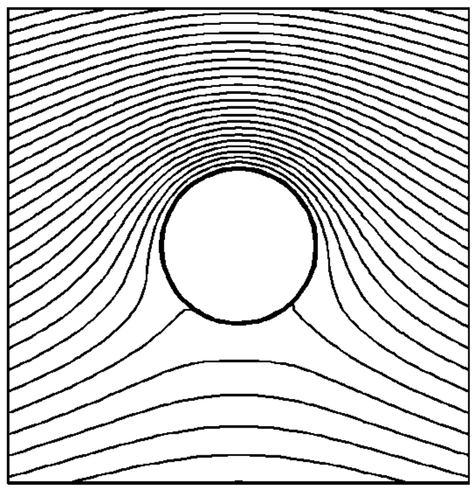
$exact\ (\Delta\psi = 0.2)$



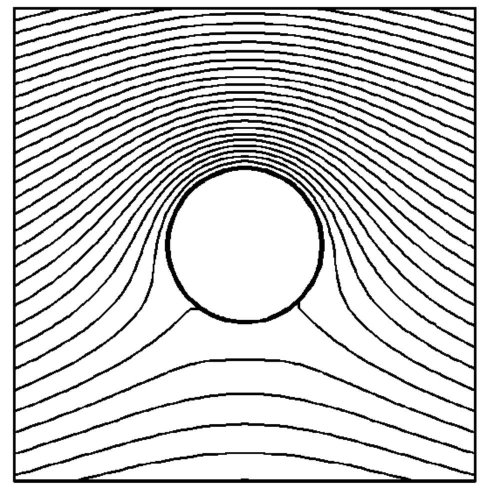
PS



$1-parameter$

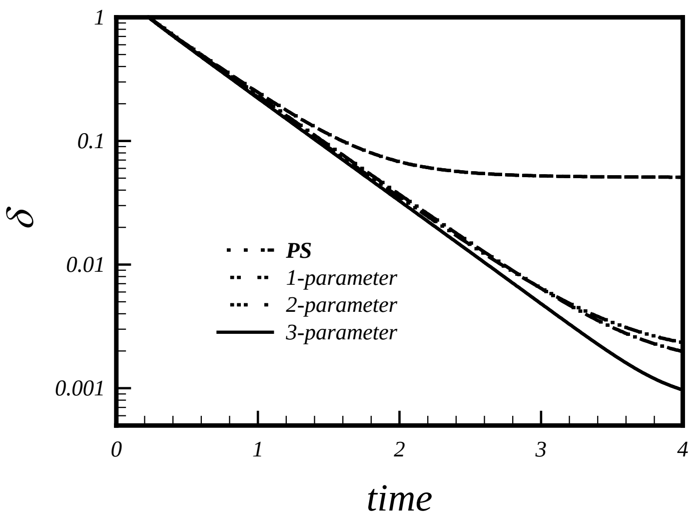


$2-parameter$

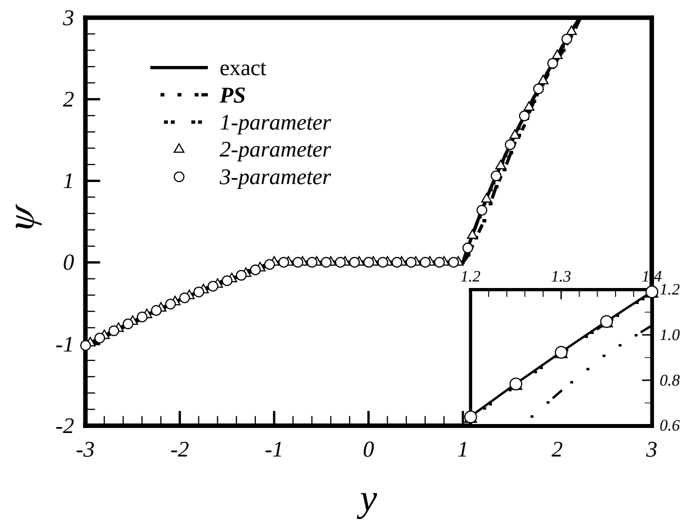


$3-parameter$

$$\Gamma / 2\pi = 1.5$$



Derivation from steady – state solution



along $y = 0$

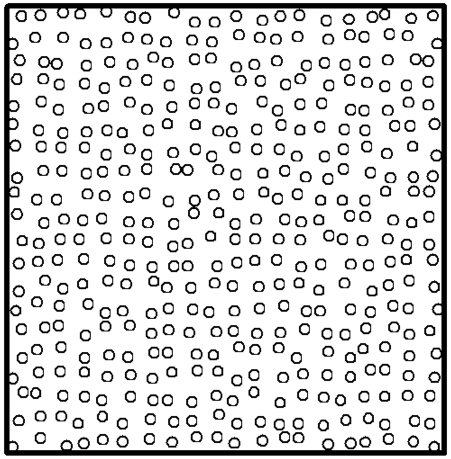
2-e Flow across corner

- **Governing equation**

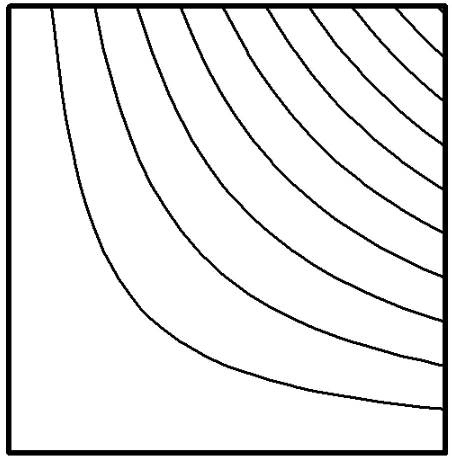
$$\frac{\partial \psi}{\partial t} = (\frac{\partial \psi}{\partial x^2} + \frac{\partial \psi}{\partial y^2})$$

- **Steady-state solution**

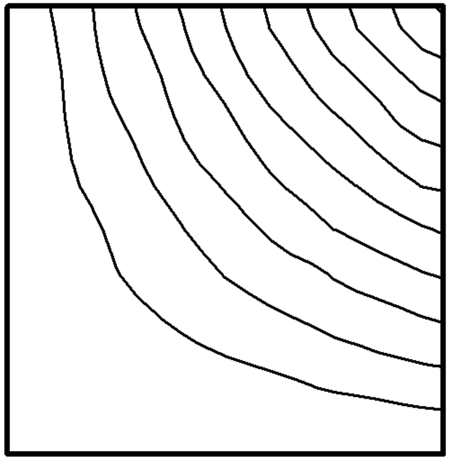
$$\psi = r^2 \sin(2\theta)$$



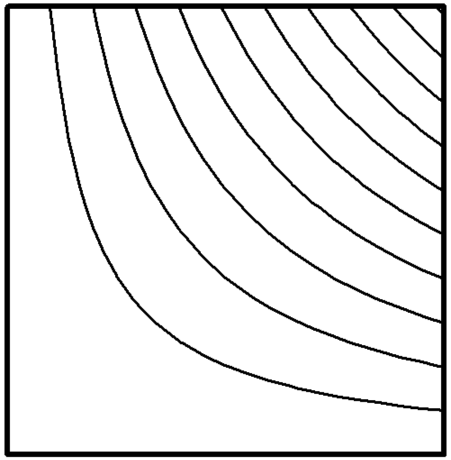
$\Delta x = 1/20, \alpha_x = \alpha_y = 0.5$



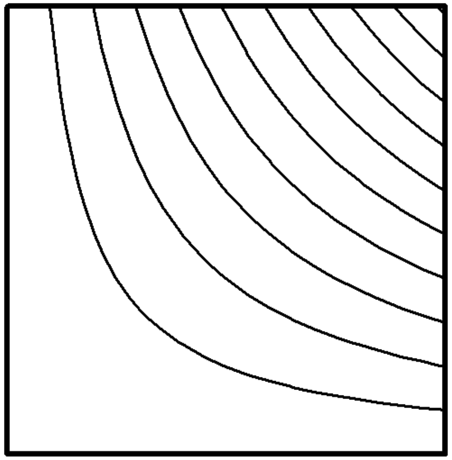
exact ($\Delta \psi = 0.2$)



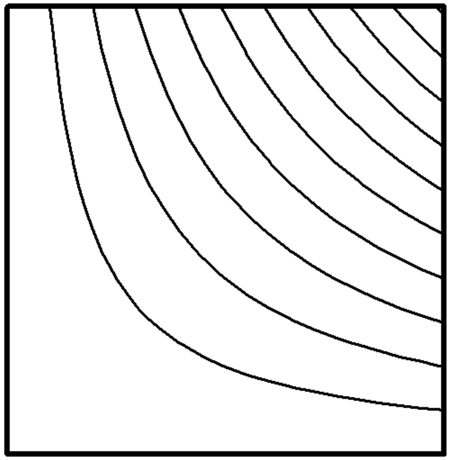
PS



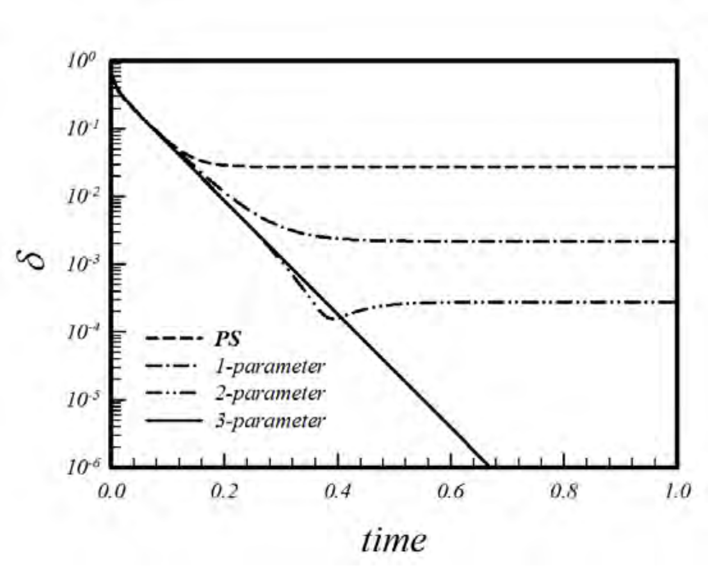
1 – parameter



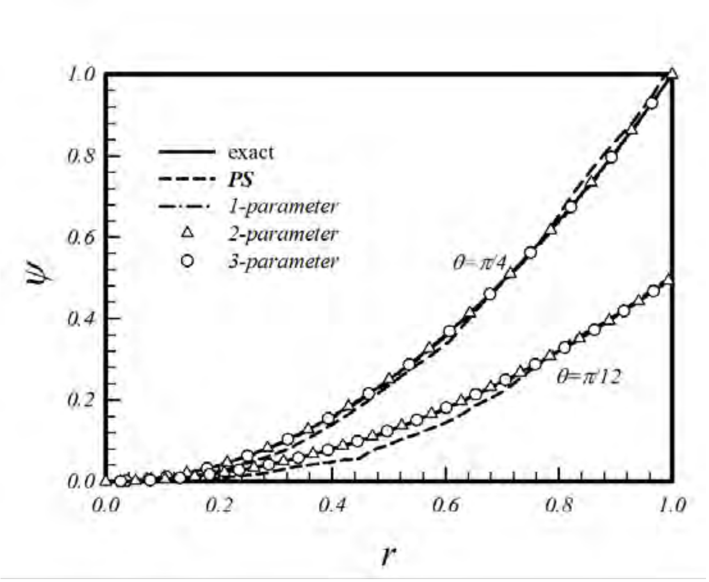
2 – parameter



3 – parameter



Derivation from steady – state solution



stream function