Essential Features of Moving Particle with Pressure Mesh

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Merits of particle method

- Proceeds without topological connection among computational nodes (meshless)
- Lagrangean treatment on convection term
- Easy particle manipulations (addition and/or deletion)
- Versatility in engineering problems

MPPM

Moving particle method with embedded pressure mesh (MPPM) – Incompressible Flow Computations

Features of existing schemes

- SPH, MPS, FVPM….
- Material particles (mass point, global mass conservation)
- Operators realized with randomly particle cloud
- Artificial compressibility or projection method for continuity constraint

Disadvantages of existing schemes

- Inaccurate (inconsistent) operator realization (particle smoothing)
- Incommensurate particle distribution (invariable length scale)
- Incapable particle management
- Assignment of boundary condition
- Computationally inefficient (Pressure equation)

Motivation: role of pressure

- **Continuity constraints – Langragian or Eulerian?**
- **Governing equation without convection term**
- **Two types of computational particles: Langragian velocity/Eulerian pressure**
- **Velocity particle liberated from mass constraint**

Essentials of present proposition (1)

- Inserted pressure mesh to realize pressure-related operators
- Background mesh and length scale
- Projection method for continuity constraint
- Local mass conservation
- Particle as observation point

Essentials of present proposition (2)

- No particle management constraint
- Feasible non-uniform distribution
- No special boundary treatment
- Constant and diagonally dominant coefficient matrix of pressure equation

Governing equations

- **Continuity**

$$\nabla \cdot \vec{\mathbf{u}} = 0$$

- **Momentum**

$$\frac{D\vec{\mathbf{u}}}{Dt} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \vec{\mathbf{u}}$$

Projection method

1. **Intermediate step**

$$\vec{\mathbf{u}}_p^* = \vec{\mathbf{u}}_p^n + \Delta t \nu \nabla^2 \vec{\mathbf{u}}_p^n, \quad \vec{\mathbf{r}}_p^* = \vec{\mathbf{r}}_p^n + \Delta t \vec{\mathbf{u}}_p^*$$

2. **Pressure step**

$$\nabla^2 p^{n+1} = \frac{\rho}{\Delta t} \nabla \cdot \vec{\mathbf{u}}^*$$

3. **Final step**

$$\vec{\mathbf{u}}_p^{n+1} = \vec{\mathbf{u}}_p^* - \Delta t \frac{\nabla p_p^{n+1}}{\rho}, \qquad \vec{\mathbf{r}}_p^{n+1} = \vec{\mathbf{r}}_p^* - \frac{\Delta t^2}{\rho} \nabla p_p^{n+1}$$

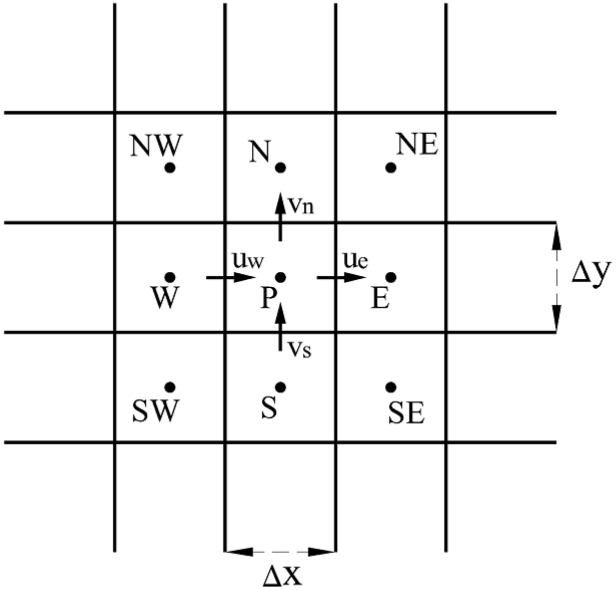
Pressure mesh

- **Continuity**

$$(u_e^{n+1} - u_w^{n+1})\Delta y + (v_n^{n+1} - v_s^{n+1})\Delta x = 0$$

- **Facial velocity splitting**

$$u_e^{n+1} = u_e^{*} - \frac{\Delta t}{\rho}\frac{p_E^{n+1} - p_P^{n+1}}{\Delta x}$$



Pressure equation

$$2\left(\frac{\Delta x}{\Delta y}+\frac{\Delta y}{\Delta x}\right)p_P^{n+1}=\frac{\Delta y}{\Delta x}p_E^{n+1}+\frac{\Delta y}{\Delta x}p_N^{n+1}+\frac{\Delta x}{\Delta y}p_W^{n+1}$$

$$+\frac{\Delta x}{\Delta y}p_S^{n+1}-\frac{\rho}{\Delta t}[(u_e^*-u_w^*)\Delta y+(v_n^*-v_s^*)\Delta x]$$

- **Facial velocity interpolation**

$$u_e^*=\sum_p u_p^*\omega(r_{pe},r_e)/\sum_p \omega(r_{pe},r_e)$$

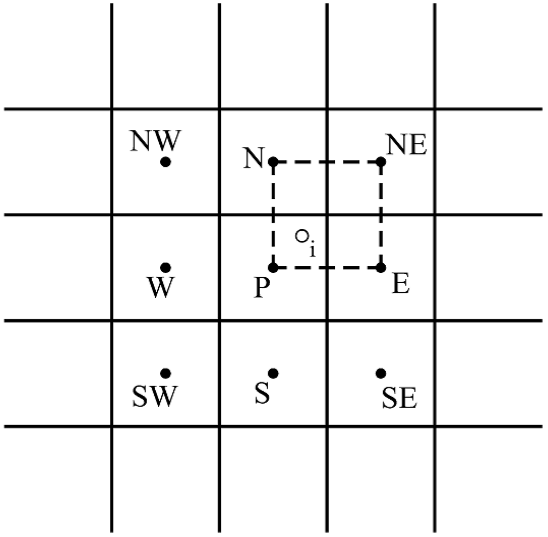
Pressure gradient

- **Shape function**

$$p(\xi,\eta) = p_P(1-\xi)(1-\eta) + p_E\xi(1-\eta)$$
$$+p_N(1-\xi)\eta + p_{NE}\xi\eta$$

$$\xi = (x - x_P)/(x_E - x_P)$$

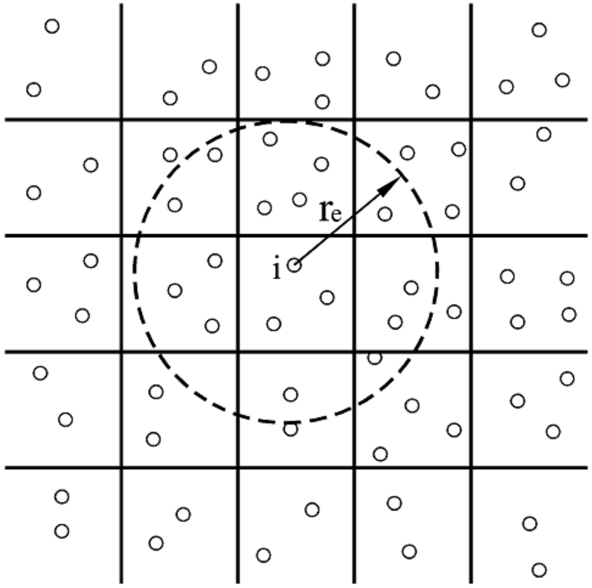
$$\eta = (y - y_P)/(y_N - y_P)$$



Velocity Diffusion

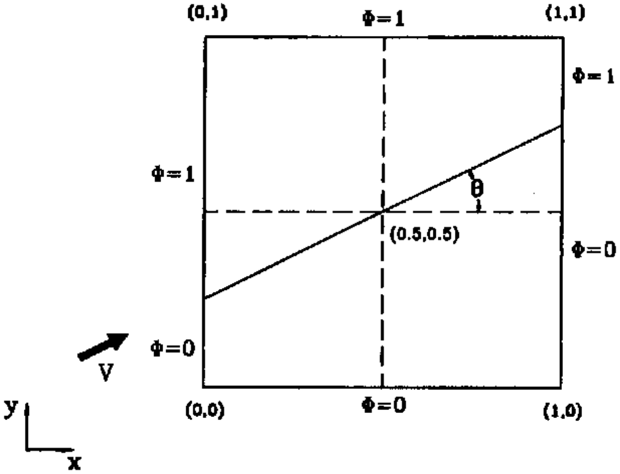
- **Particle smoothing**

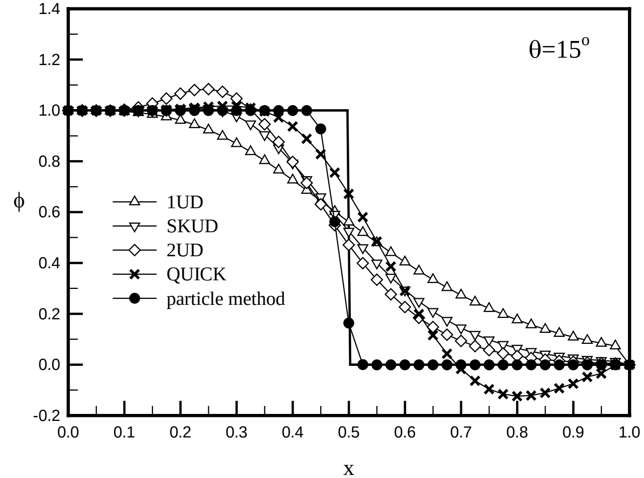
$$\left\langle \nabla^2 \phi \right\rangle_i = \frac{4}{\sum_{j \neq i} \omega_{ji}} \sum_{j \neq i} \frac{(\phi_j - \phi_i)\omega_{ji}}{r_{ji}^2}$$

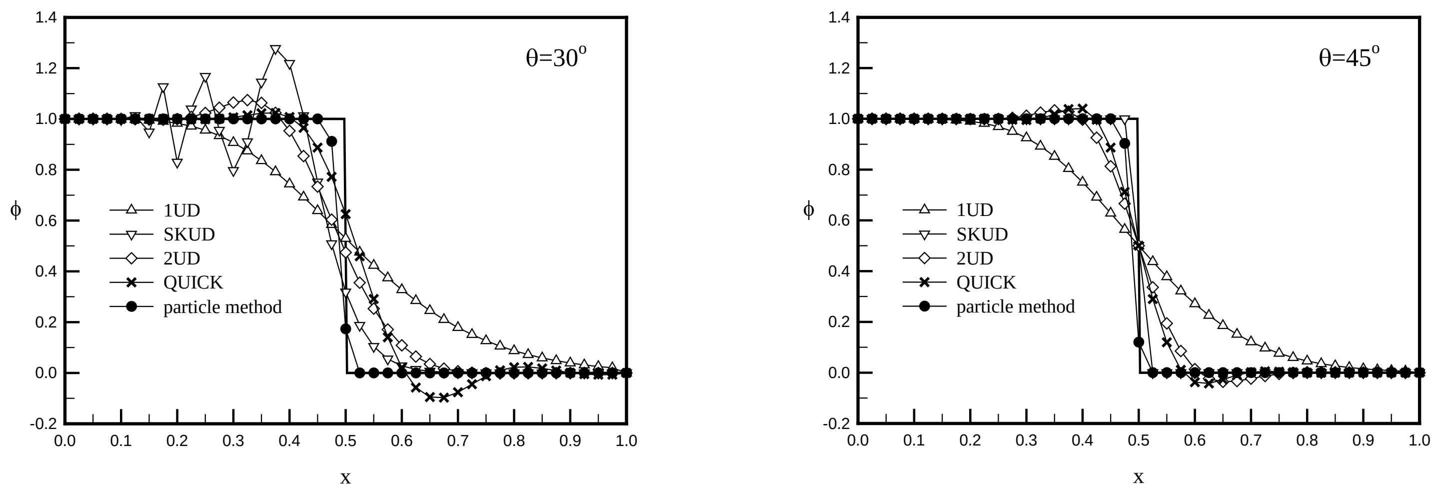


$$\omega(r, r_e) = \begin{cases} \frac{r_e}{r} - 1 & r \leq r_e \\ 0 & \text{otherwise} \end{cases}$$

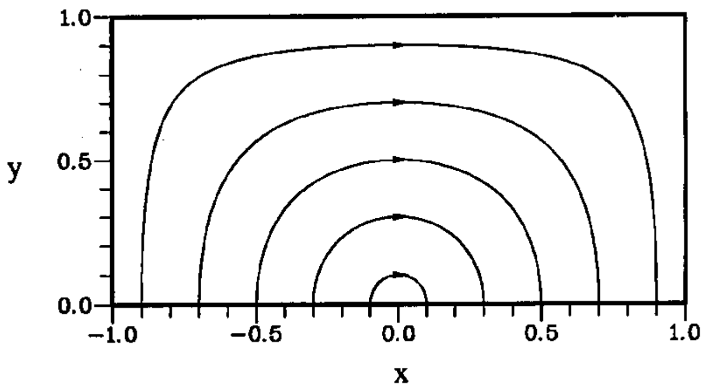
Pure convection

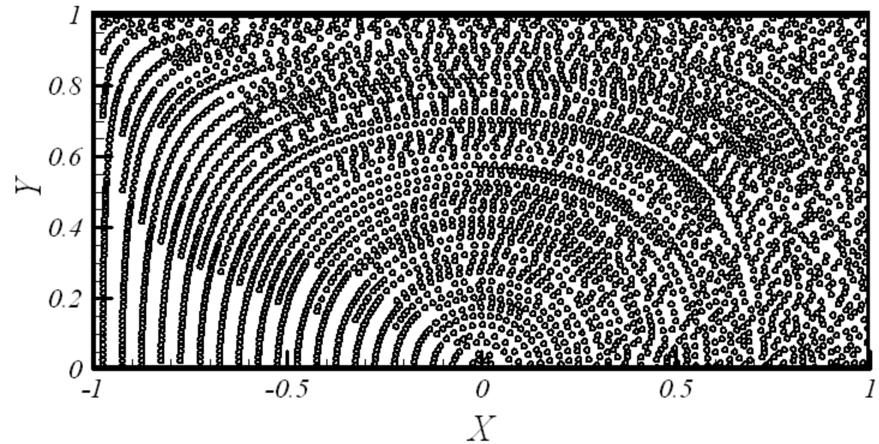


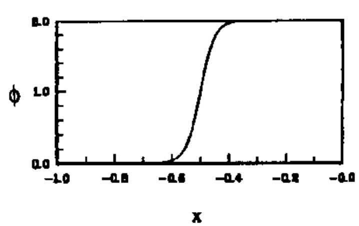


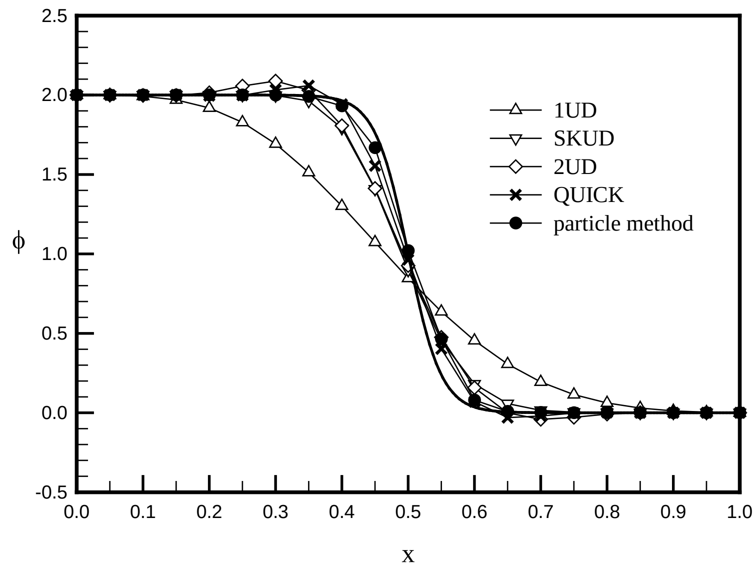


Pure convection

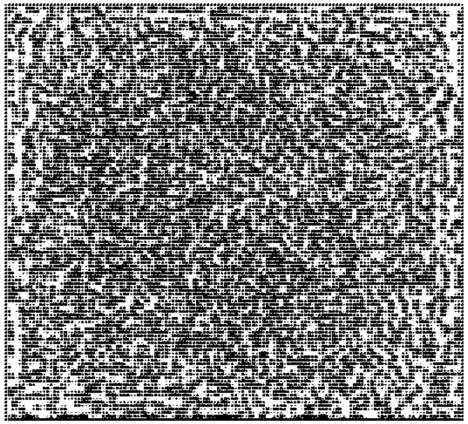




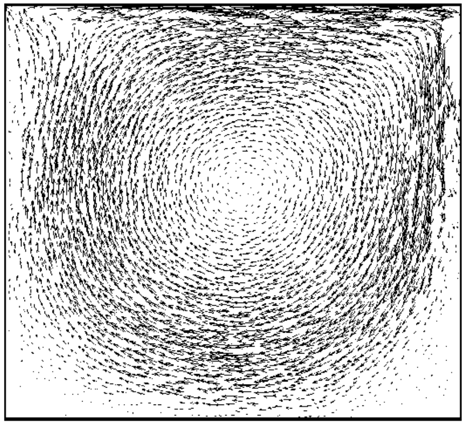




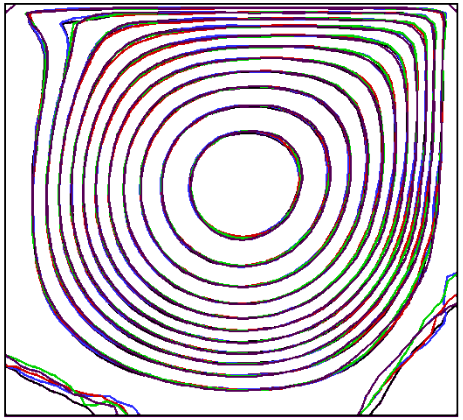
Lid-driven cavity



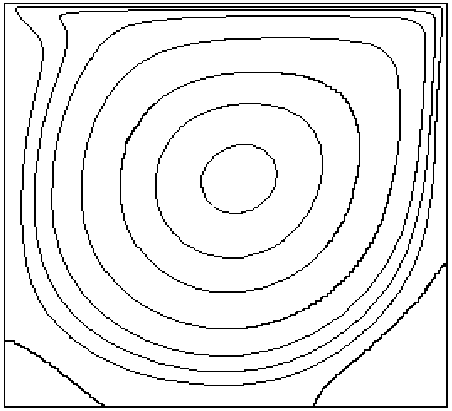
Particle distribution



Particle velocity

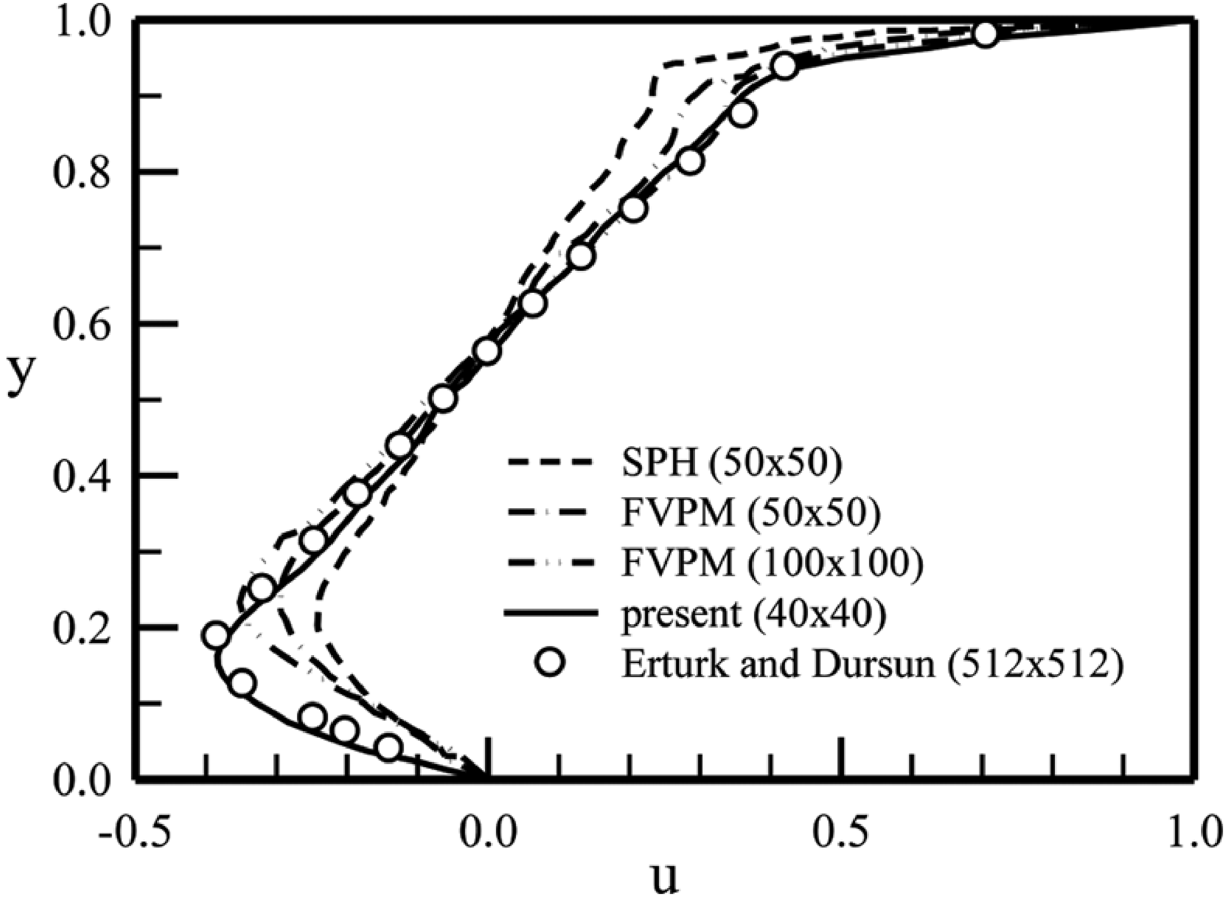


Streamlines at various computation times



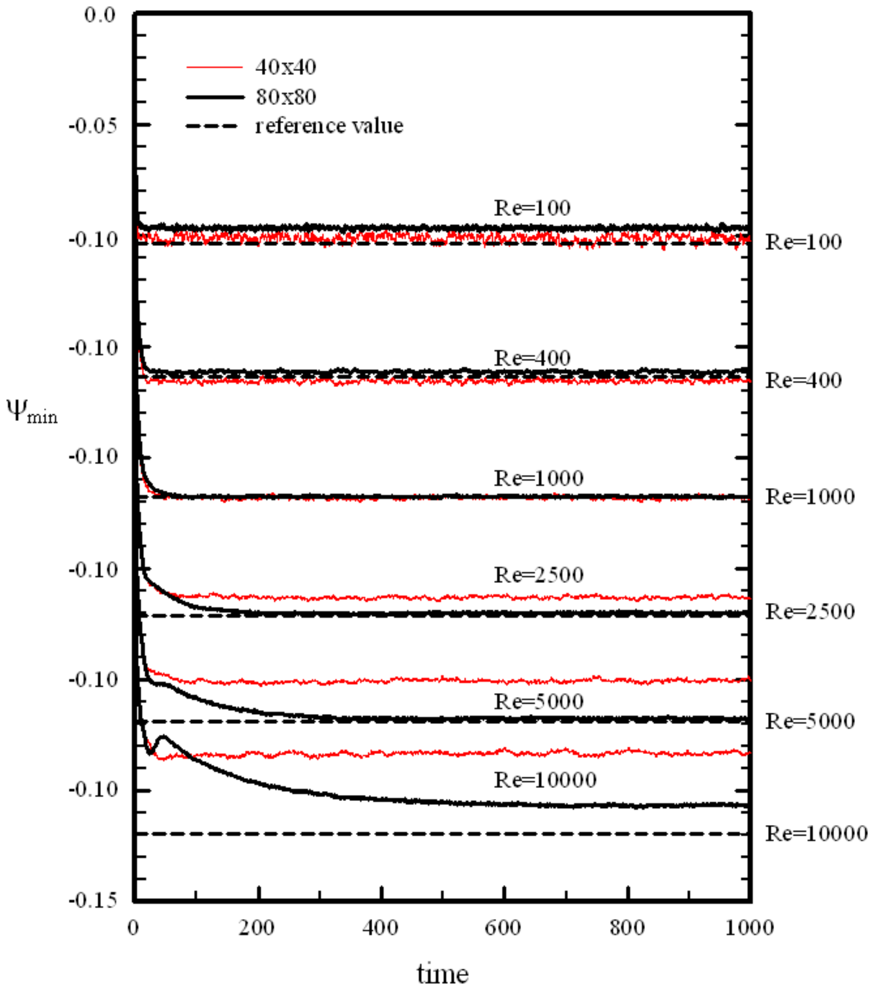
Reference streamlines

Lid-driven cavity



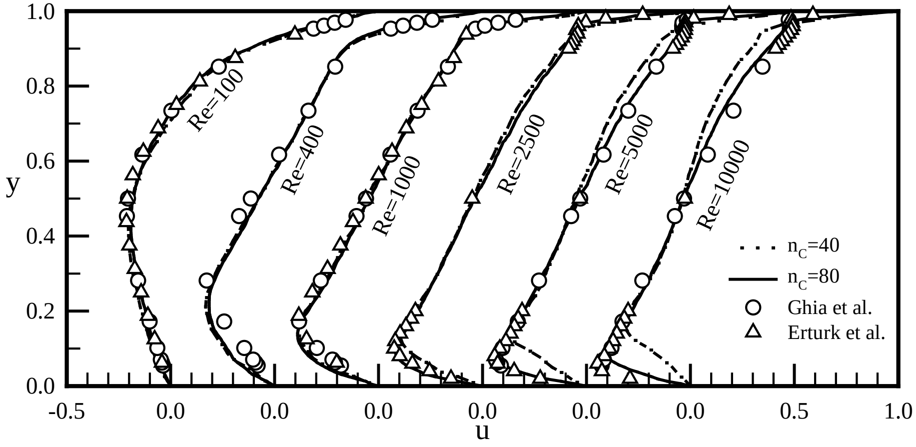
Re=1000

Lid-driven cavity

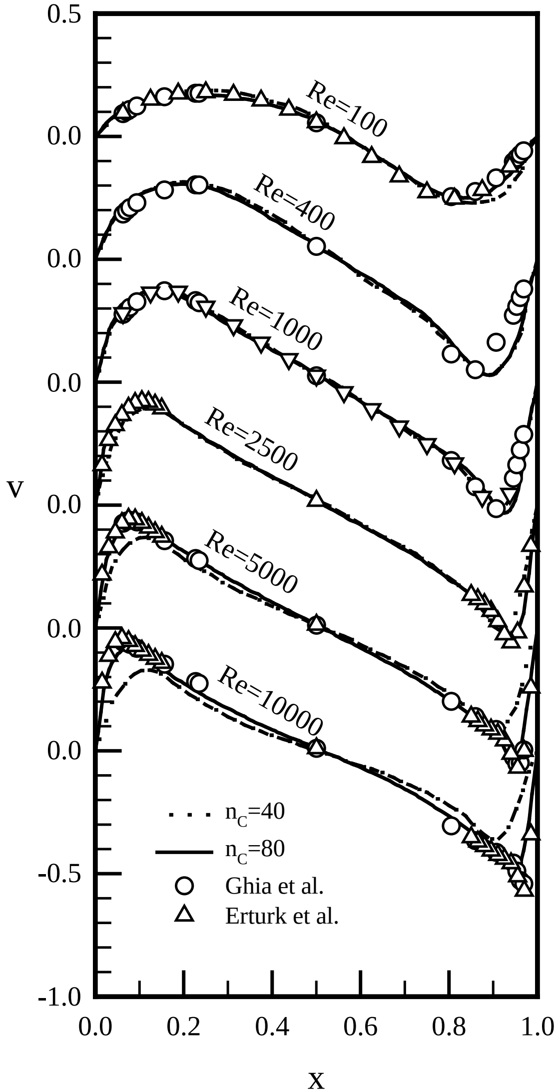


Recirculation flow rate

Lid-driven cavity



Horizontal velocity



Vertical velocity

Weird consequences

- Pronounced solution variations in low-Reynolds number flows
- Inaccurate results with denser particle distribution

Inaccurate diffusion operators?

Analysis of Laplacian operator

- **Inconsistency:** $O(1/\delta)$

$$\frac{4}{\sum_{j\neq i}\omega_{ji}}\sum_{j\neq i}\frac{(\phi_j-\phi_i)\omega_{ji}}{r_{ji}^2}\approx\frac{4}{\sum_{j\neq i}\omega_{ji}}[\phi_x\sum_{j\neq i}\frac{(x_j-x_i)\omega_{ji}}{r_{ji}^2}$$

$$+\phi_y\sum_{j\neq i}\frac{(y_j-y_i)\omega_{ji}}{r_{ji}^2}+\frac{1}{2}\phi_{xx}\sum_{j\neq i}\frac{(x_j-x_i)^2\omega_{ji}}{r_{ji}^2}$$

$$+\phi_{xy}\sum_{j\neq i}\frac{(x_j-x_i)(y_j-y_i)\omega_{ji}}{r_{ji}^2}+\frac{1}{2}\phi_{yy}\sum_{j\neq i}\frac{(y_j-y_i)^2\omega_{ji}}{r_{ji}^2}]$$

Smoothing difference

- **gradient model**

$$\sum_{q\neq p}\frac{\phi_q-\phi_p}{r_{qp}}\left(\frac{x_q-x_p}{r_{qp}}\right)\omega_{qp}=\phi_x\sum_{q\neq p}\frac{(x_q-x_p)^2\omega_{qp}}{r_{qp}^2}+\phi_y\sum_{q\neq p}\frac{(x_q-x_p)(y_q-y_p)\omega_{qp}}{r_{qp}^2}+\frac{1}{2}\phi_{xx}\sum_{q\neq p}\frac{(x_q-x_p)^3\omega_{qp}}{r_{qp}^2}+\phi_{xy}\sum_{q\neq p}\frac{(x_q-x_p)^2(y_q-y_p)\omega_{qp}}{r_{qp}^2}$$
$$+\frac{1}{2}\phi_{yy}\sum_{q\neq p}\frac{(x_q-x_p)(y_q-y_p)^2\omega_{qp}}{r_{qp}^2}+HOT$$

$$\sum_{q\neq p}\frac{\phi_q-\phi_p}{r_{qp}}\left(\frac{y_q-y_p}{r_{qp}}\right)\omega_{qp}=\phi_x\sum_{q\neq p}\frac{(x_q-x_p)(y_q-y_p)\omega_{qp}}{r_{qp}^2}+\phi_y\sum_{q\neq p}\frac{(y_q-y_p)^2\omega_{qp}}{r_{qp}^2}+\frac{1}{2}\phi_{xx}\sum_{q\neq p}\frac{(x_q-x_p)^2(y_q-y_p)\omega_{qp}}{r_{qp}^2}$$
$$+\phi_{xy}\sum_{q\neq p}\frac{(x_q-x_p)(y_q-y_p)^2\omega_{qp}}{r_{qp}^2}+\frac{1}{2}\phi_{yy}\sum_{q\neq p}\frac{(y_q-y_p)^3\omega_{qp}}{r_{qp}^2}+HOT$$

Smoothing difference

- **Laplacian model**

$$\sum_{q\neq p}\frac{\phi_q-\phi_p}{r_{qp}^2}\left(\frac{x_q-x_p}{r_{qp}}\right)^2\omega_{qp}=\phi_x\sum_{q\neq p}\frac{(x_q-x_p)^3\omega_{qp}}{r_{qp}^4}+\phi_y\sum_{q\neq p}\frac{(x_q-x_p)^2(y_q-y_p)\omega_{qp}}{r_{qp}^4}+\frac{1}{2}\phi_{xx}\sum_{q\neq p}\frac{(x_q-x_p)^4\omega_{qp}}{r_{qp}^4}+\phi_{xy}\sum_{q\neq p}\frac{(x_q-x_p)^3(y_q-y_p)\omega_{qp}}{r_{qp}^4}$$
$$+\frac{1}{2}\phi_{yy}\sum_{q\neq p}\frac{(x_q-x_p)^2(y_q-y_p)^2\omega_{qp}}{r_{qp}^4}+HOT$$

$$\sum_{q\neq p}\frac{\phi_q-\phi_p}{r_{qp}^2}\left(\frac{x_q-x_p}{r_{qp}}\right)\left(\frac{y_q-y_p}{r_{qp}}\right)\omega_{qp}=\phi_x\sum_{q\neq p}\frac{(x_q-x_p)^2(y_q-y_p)\omega_{qp}}{r_{qp}^4}+\phi_y\sum_{q\neq p}\frac{(x_q-x_p)(y_q-y_p)^2\omega_{qp}}{r_{qp}^4}$$
$$+\frac{1}{2}\phi_{xx}\sum_{q\neq p}\frac{(x_q-x_p)^3(y_q-y_p)\omega_{qp}}{r_{qp}^4}+\phi_{xy}\sum_{q\neq p}\frac{(x_q-x_p)^2(y_q-y_p)^2\omega_{qp}}{r_{qp}^4}$$
$$+\frac{1}{2}\phi_{yy}\sum_{q\neq p}\frac{(x_q-x_p)(y_q-y_p)^3\omega_{qp}}{r_{qp}^4}+HOT$$

$$\sum_{q\neq p}\frac{\phi_q-\phi_p}{r_{qp}^2}\left(\frac{y_q-y_p}{r_{qp}}\right)^2\omega_{qp}=\phi_x\sum_{q\neq p}\frac{(x_q-x_p)(y_q-y_p)^2\omega_{qp}}{r_{qp}^4}+\phi_y\sum_{q\neq p}\frac{(y_q-y_p)^3\omega_{qp}}{r_{qp}^4}+\frac{1}{2}\phi_{xx}\sum_{q\neq p}\frac{(x_q-x_p)^2(y_q-y_p)^2\omega_{qp}}{r_{qp}^4}$$
$$+\phi_{xy}\sum_{q\neq p}\frac{(x_q-x_p)(y_q-y_p)^3\omega_{qp}}{r_{qp}^4}+\frac{1}{2}\phi_{yy}\sum_{q\neq p}\frac{(y_q-y_p)^4\omega_{qp}}{r_{qp}^4}+HOT$$

Smoothing difference

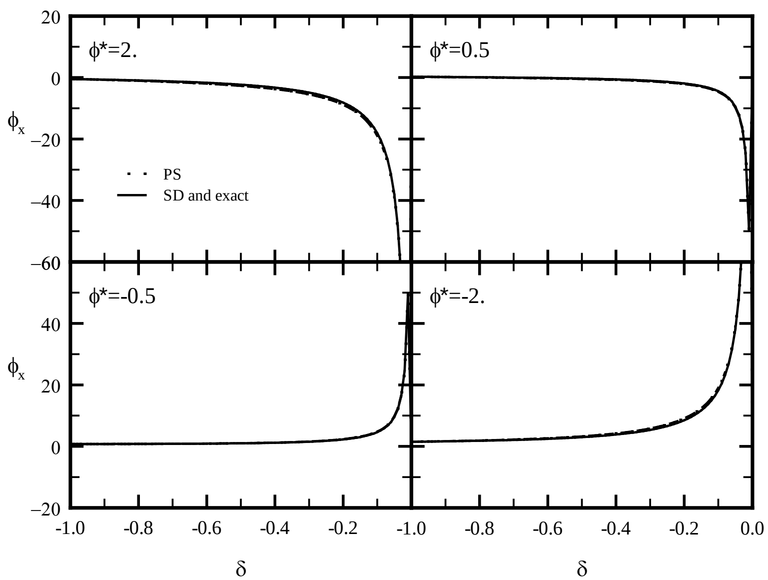
- **difference equation:** $O(\delta)$

$$\mathbf{Cq} = \mathbf{b}$$

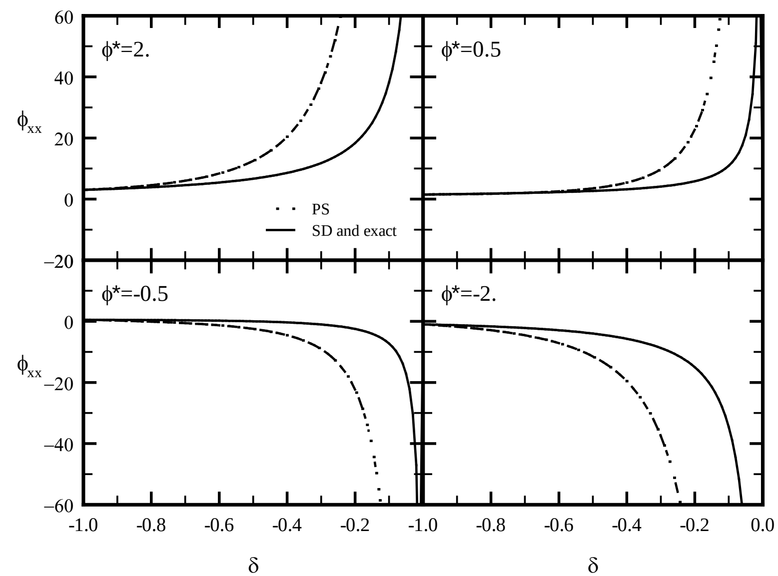
$$\mathbf{q} = (\phi_x, \phi_y, \phi_{xx}, \phi_{xy}, \phi_{yy})^T$$

Three-point stencil

$$\Phi(0) = 0, \quad \Phi(1) = 1, \quad \Phi(\delta) = \phi^*$$



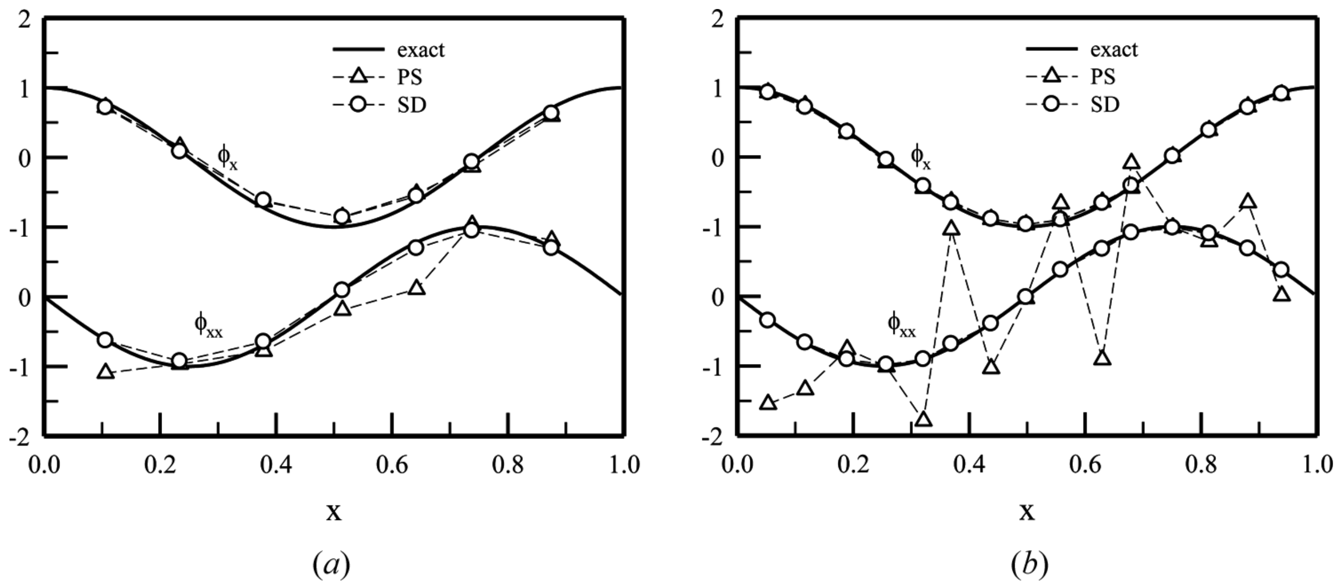
Gradient



Laplacian

Analysis of Laplacian operator

- **One-dimensional case** $$x_i = \frac{(i-1)+\alpha(\xi-0.5)}{n_C}$$



a=0.3: (a) n_C=8; (b) n_C=16.

Analysis of Laplacian operator

(a) Error norm with $\alpha = 0.3$ in one-dimensional model problem. (b) Error norm with $n_C = 128$ in one-dimensional model problem

a

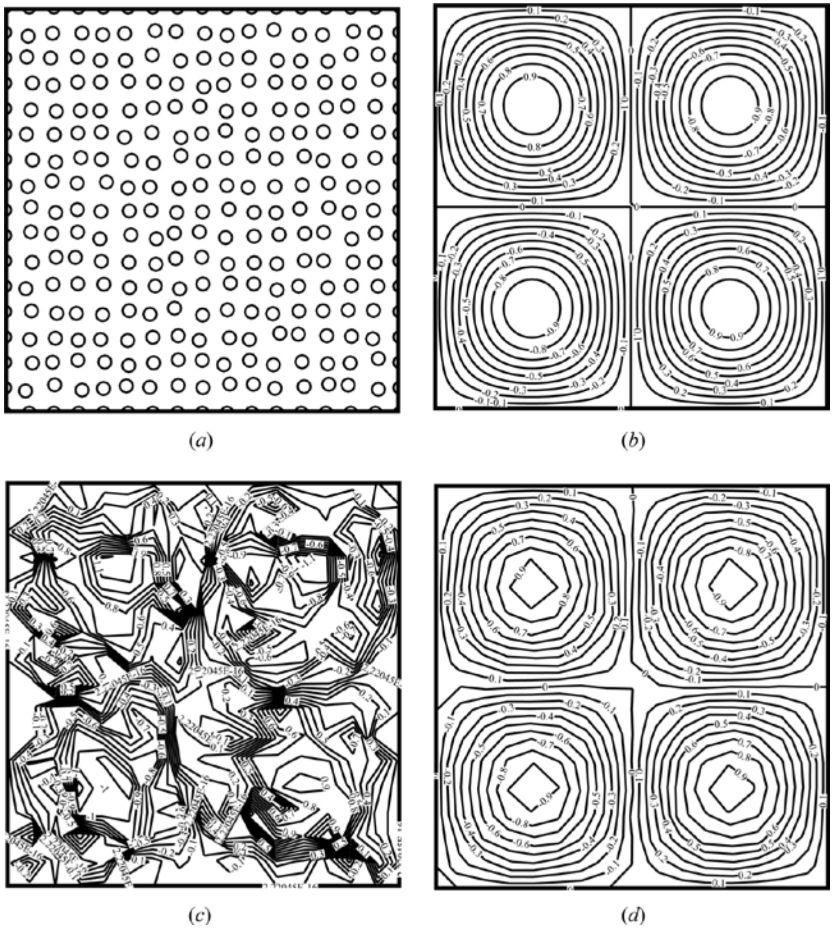
	PS		SD	
n_C	ϕ_x	ϕ_{xx}	ϕ_x	ϕ_{xx}
8	9.5199e-2	3.3329e-1	8.0039e-2	4.9107e-2
16	2.6480e-2	8.6628e-1	2.1978e-2	2.2173e-2
32	1.0839e-2	1.9378e0	5.7820e-3	1.1400e-2
64	4.1440e-3	3.9166e0	1.4675e-3	5.5280e-3
128	2.2525e-3	7.5399e0	3.6937e-4	2.6748e-3
256	1.1827e-3	1.5861e+1	9.2667e-5	1.4120e-3

b

	PS		SD	
α	ϕ_x	ϕ_{xx}	ϕ_x	ϕ_{xx}
0	2.8706e-4	6.2975e-3	2.8693e-4	1.4477e-4
0.1	1.0087e-3	2.7300e0	3.2164e-4	7.6187e-4
0.2	1.7123e-3	5.1957e0	3.4829e-4	1.6613e-3
0.3	2.2525e-3	7.5399e0	3.6937e-4	2.6748e-3
0.4	2.6608e-3	9.8841e0	3.8636e-4	3.7777e-3
0.5	2.9643e-3	1.2352e+1	4.0019e-4	4.9698e-3

Analysis of Laplacian operator

- **Two-dimensional case**



(a)particle; (b)exact (c)PS; (d)SD.

Analysis of Laplacian operator

(a). Error norm with $\alpha = 0.3$ in two-dimensional model problem. (b) Error norm with $n_C = 128$ in two-dimensional model problem

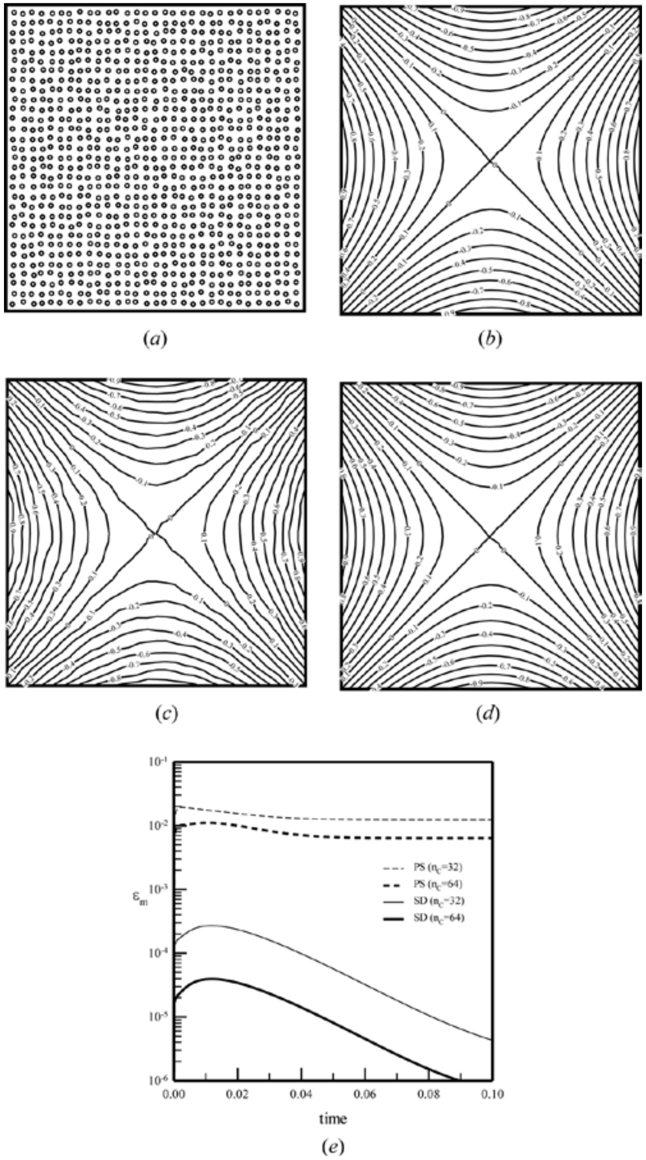
a

	PS			SD		
n_C	ϕ_x	ϕ_y	$\nabla^2\phi$	ϕ_x	ϕ_y	$\nabla^2\phi$
8	1.090e-1	1.130e-1	2.105e-1	1.033e-1	1.046e-1	7.17e-2
16	4.996e-2	5.209e-2	3.054e-1	2.912e-2	2.893e-2	2.086e-2
32	4.191e-2	4.014e-2	6.641e-1	7.478e-3	7.476e-3	8.547e-3
64	4.051e-2	4.167e-2	1.389e0	1.893e-3	1.893e-3	4.069e-3
128	4.095e-2	4.079e-2	2.793e0	4.739e-4	4.747e-4	1.999e-3
256	4.071e-2	4.072e-2	5.570e0	1.188e-4	1.189e-4	9.948e-4

b

	PS			SD		
α	ϕ_x	ϕ_y	$\nabla^2\phi$	ϕ_x	ϕ_y	$\nabla^2\phi$
0	4.006e-4	4.006e-4	2.240e-3	3.794e-4	3.794e-4	1.912e-4
0.1	1.472e-2	1.478e-2	1.015e0	3.999e-4	3.999e-4	7.240e-4
0.2	2.827e-2	2.825e-2	1.946e0	4.279e-4	4.283e-4	1.345e-3
0.3	4.095e-2	4.079e-2	2.793e0	4.739e-4	4.747e-4	1.999e-3
0.4	5.333e-2	5.293e-2	3.576e0	5.170e-4	5.178e-4	2.750e-3
0.5	6.556e-2	6.489e-2	4.388e0	5.536e-4	5.547e-4	3.551e-3

Pure conduction

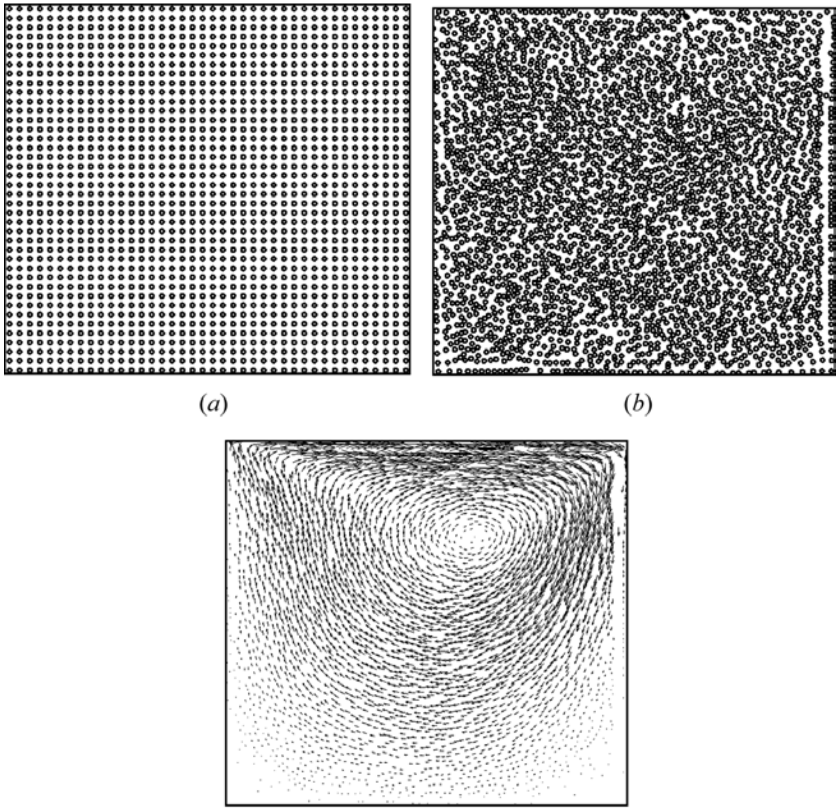


Pure conduction

(a). Error norm at t = 0.1 with $\alpha = 0.3$ and $C_D = 1$ in pure conduction problem. (b) Error norm at t = 0.1 with $n_C = 32$ and $C_D = 1$ in pure conduction problem. (c) Error norm at t = 0.1 with $\alpha = 0.3$ and $n_C = 32$ in pure conduction problem

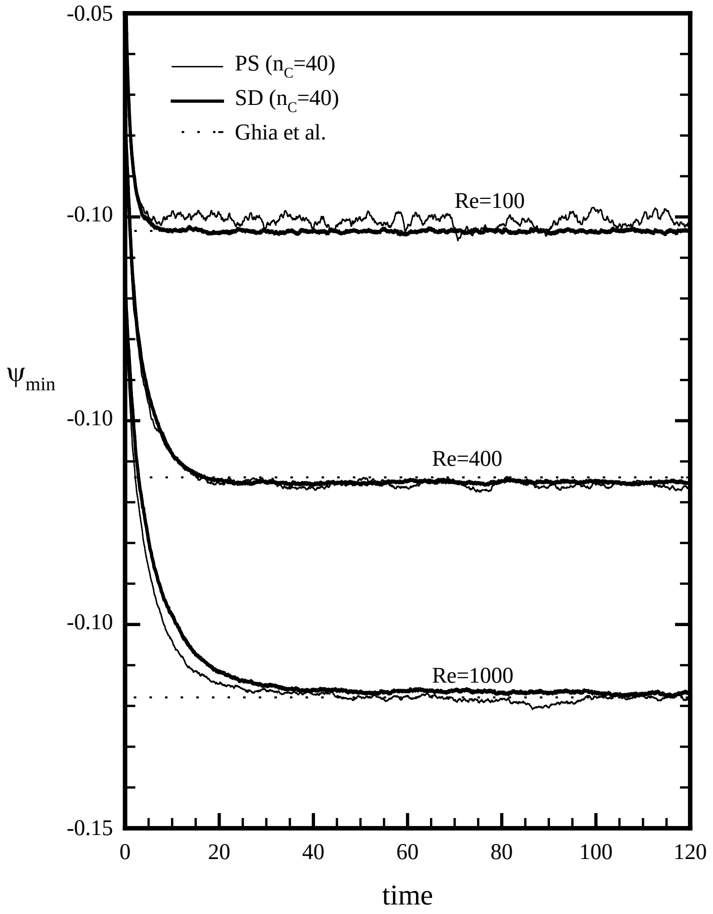
n_C	8		16	32		64
PS	4.009e-2		2.475e-2	1.235e-2		6.362e-3
SD	6.296e-5		2.219e-5	4.290e-6		7.048e-7
α	0.	0.1	0.2	0.3	0.4	0.5
PS	1.015e-4	4.418e-3	8.550e-3	1.235e-2	1.549e-2	1.814e-2
SD	6.781e-6	6.466e-6	4.698e-6	4.290e-6	7.780e-6	1.339e-5
C_D	0.25	0.5	0.75	1.0	1.5	2
PS	1.235e-2	1.235e-2	1.235e-2	1.235e-2	7.866e-2	9.662e-1
SD	9.024e-6	6.078e-6	4.025e-6	4.290e-6	9.408e-6	9.686e-1

Cavity flow

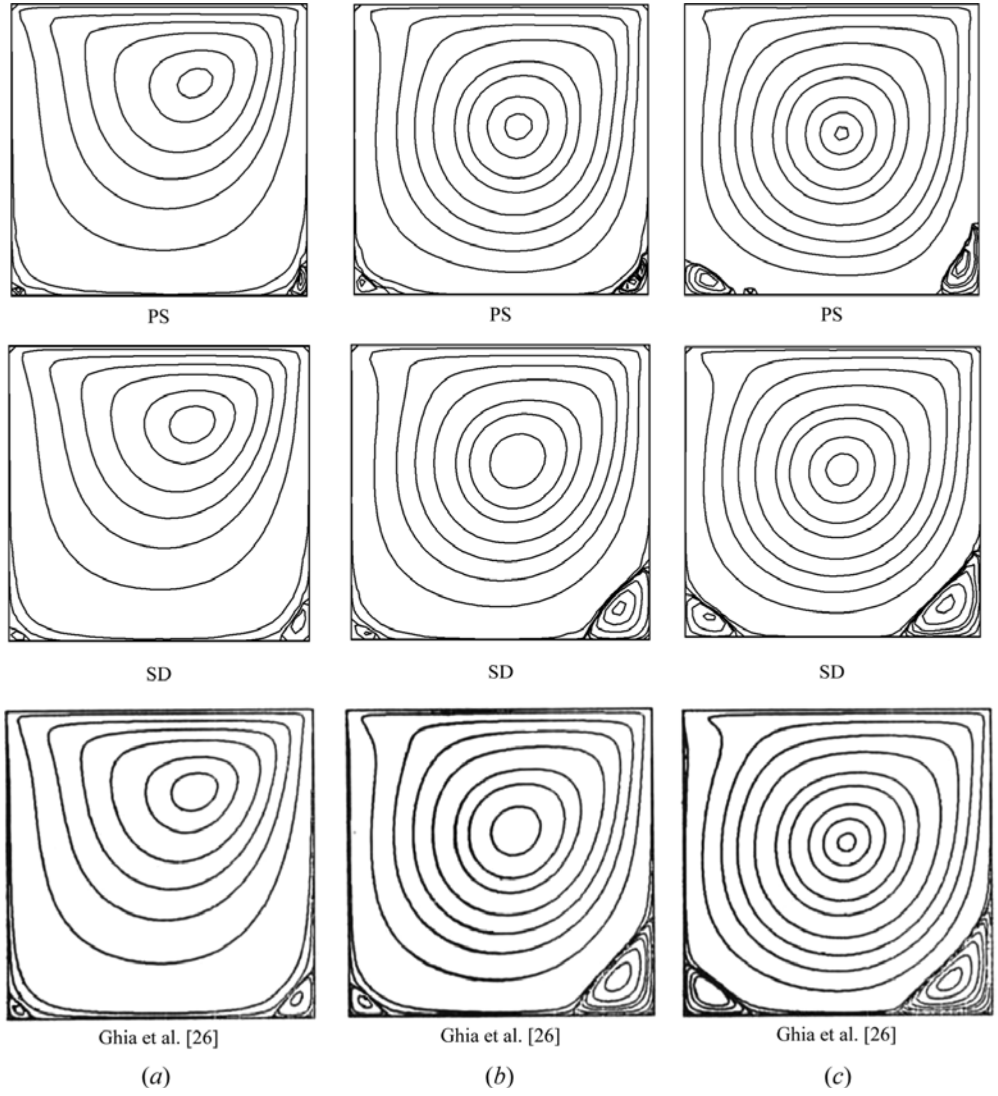


(a) (b)

(c)

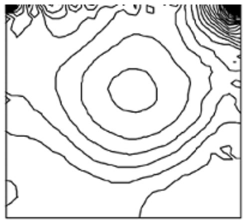


Cavity flow

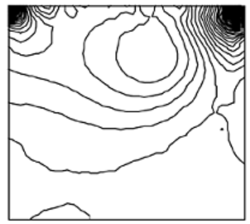


Cavity flow

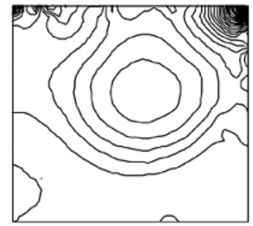
Re=100



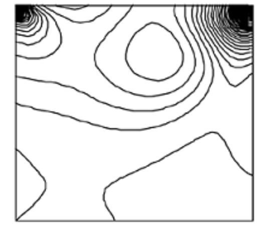
Re=400



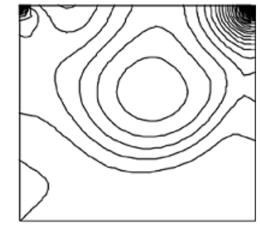
Re=100



Re=400

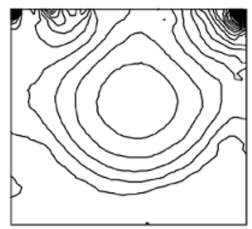


Re=100

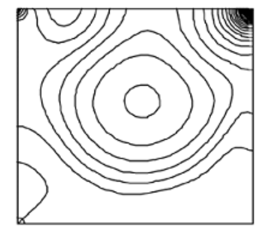


Re=400

Re=1000



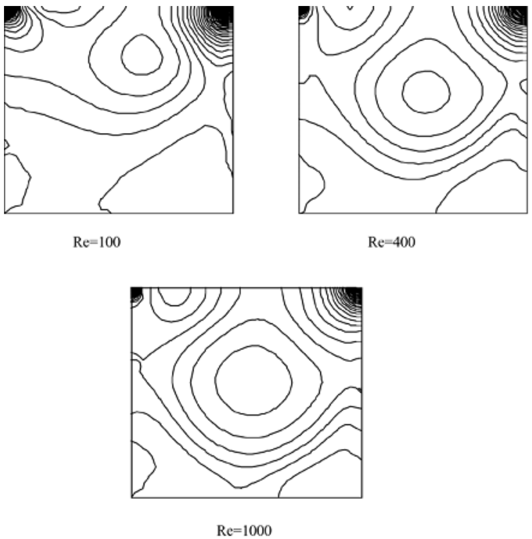
Re=1000



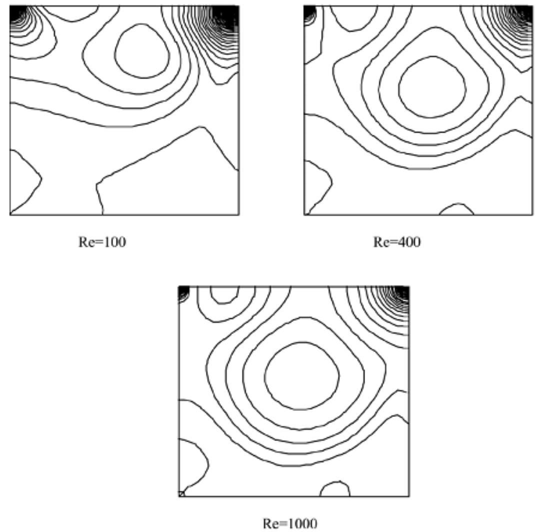
Re=1000

PS $(t = 100)$ SD $(t = 100)$ FV

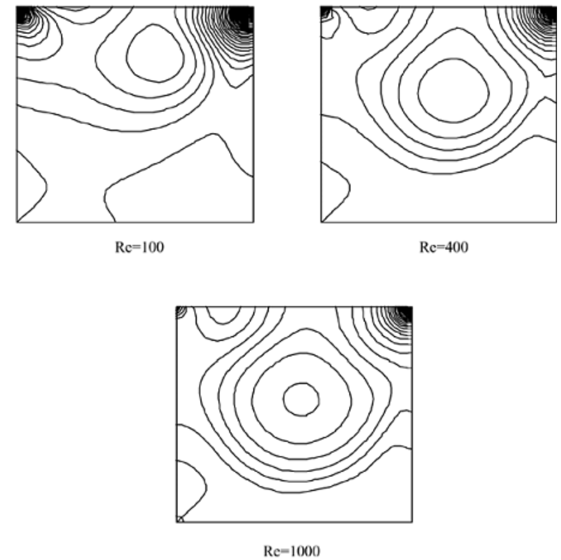
Cavity flow



PS $(t = 100 - 105)$

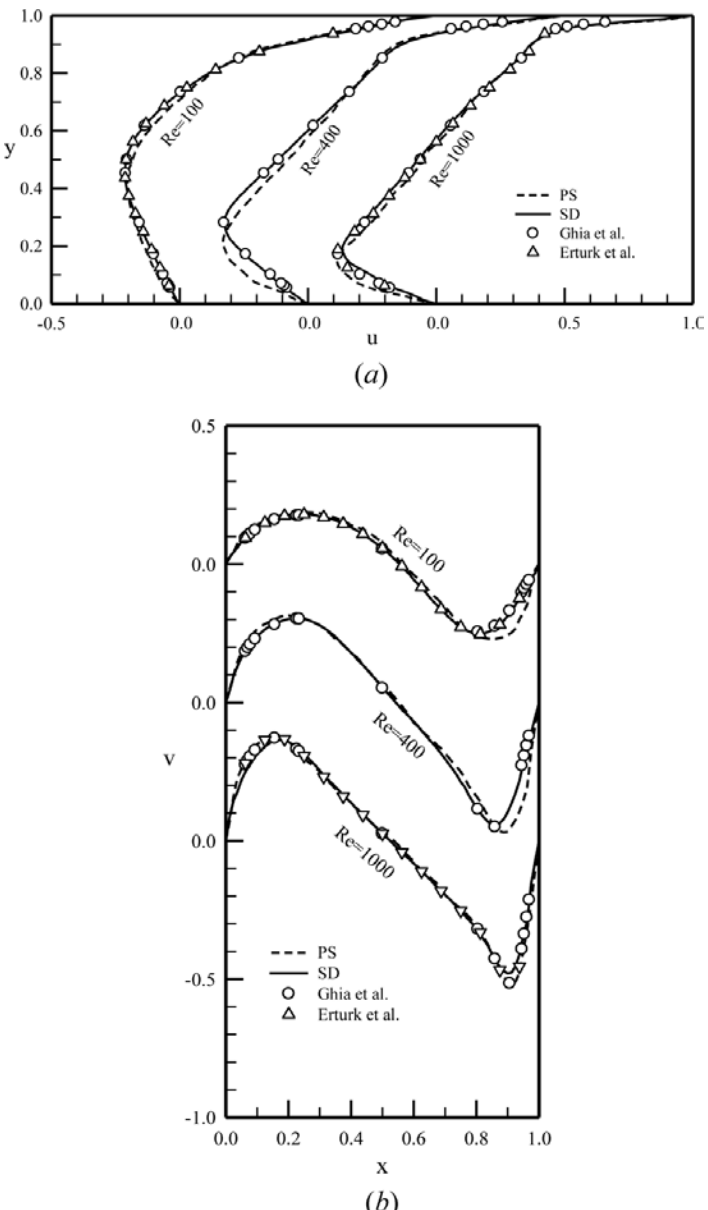


SD $(t = 100 - 105)$

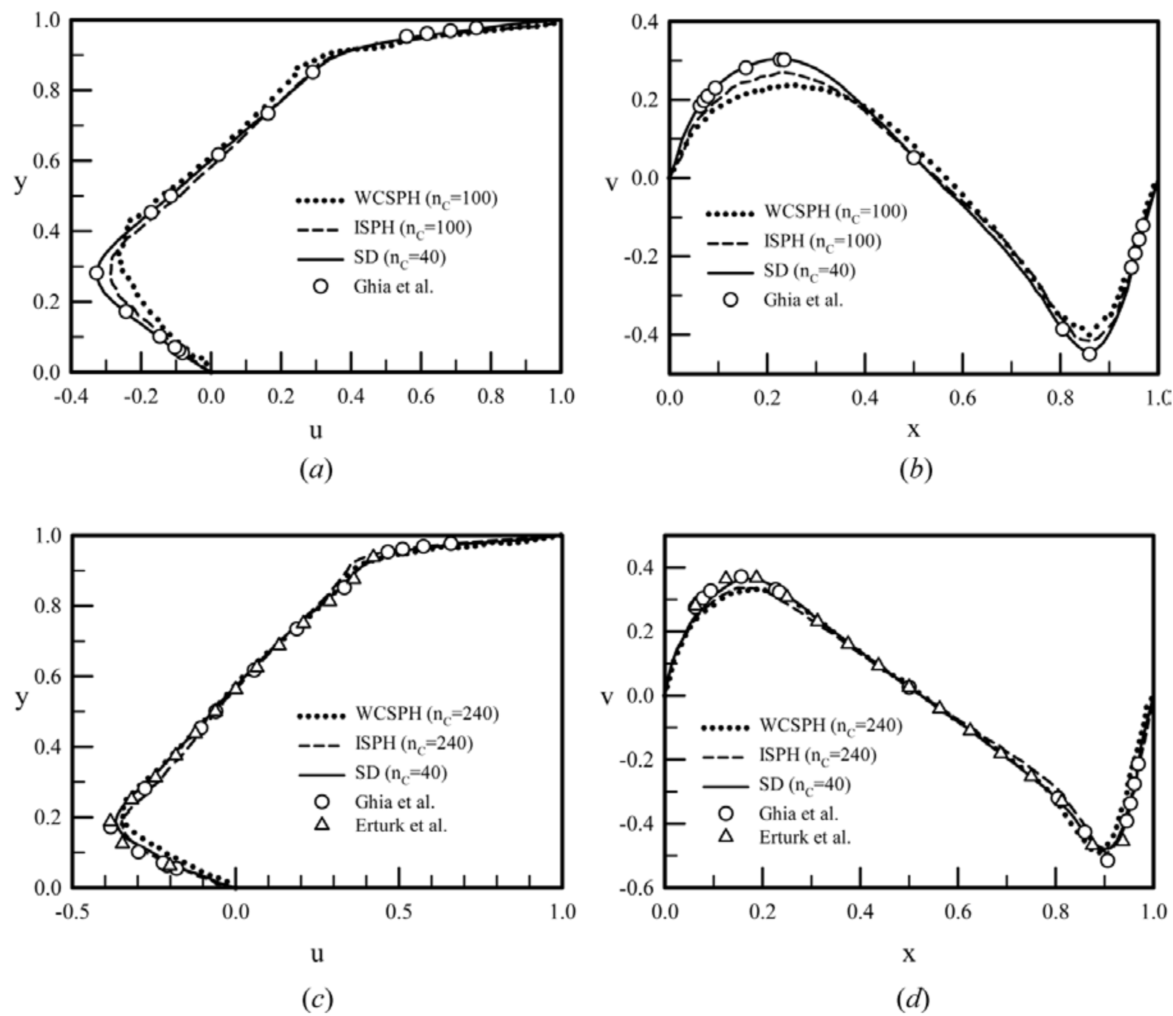


FV

Cavity flow

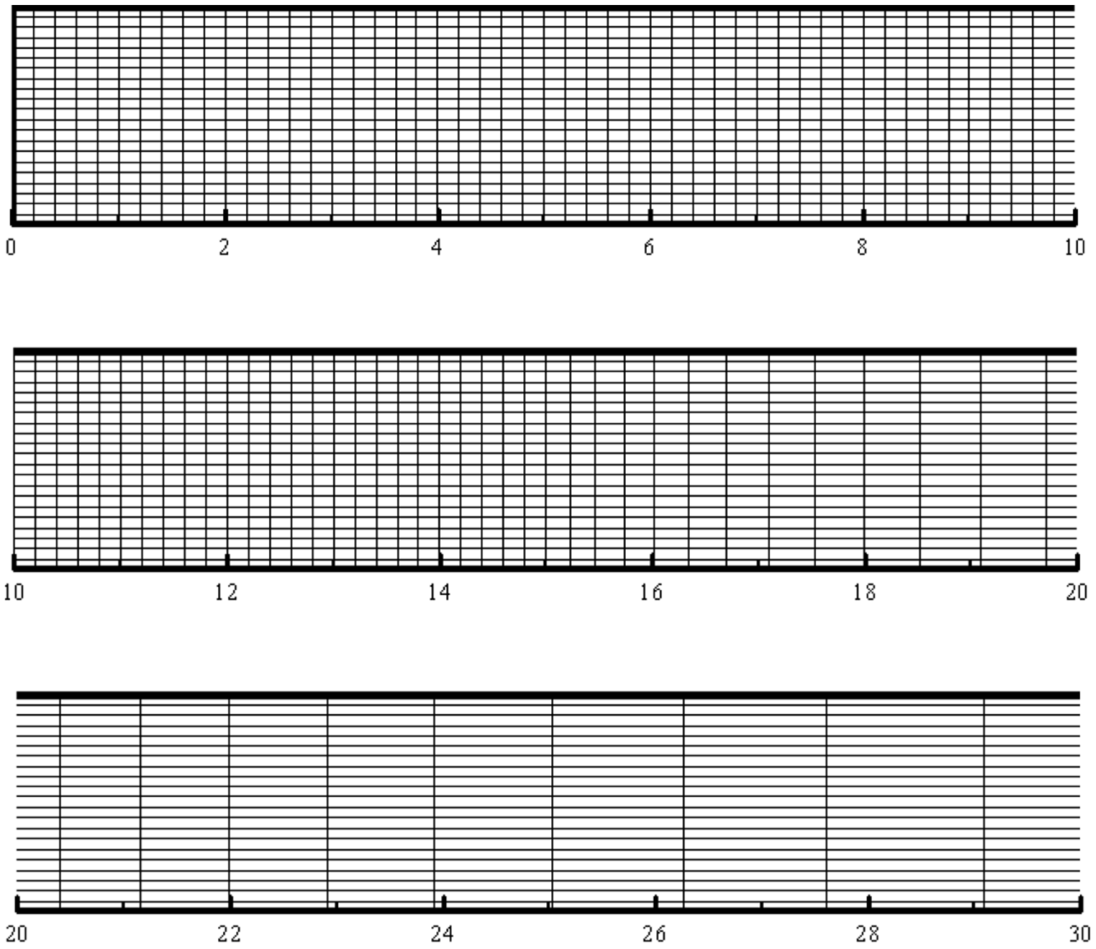


Re=100
Re=400
Re=1000
PS
SD
Ghia et al.
Erturk et al.
y
u
(a)
v
x
(b)

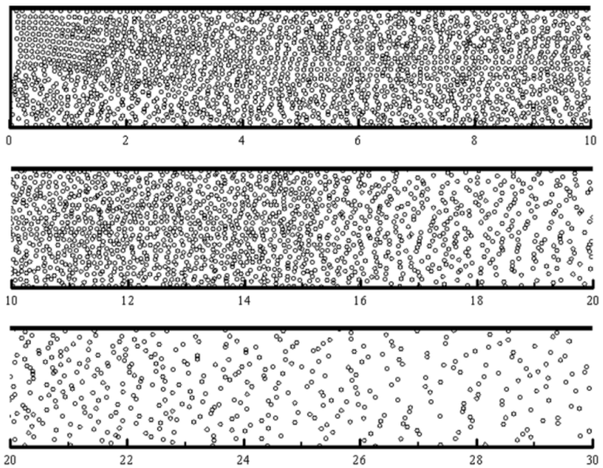


WCSPH (n_c=100)
ISPH (n_c=100)
SD (n_c=40)
Ghia et al.
y
u
(a)
v
x
(b)
WCSPH (n_c=240)
ISPH (n_c=240)
SD (n_c=40)
Ghia et al.
Erturk et al.
y
u
(c)
v
x
(d)

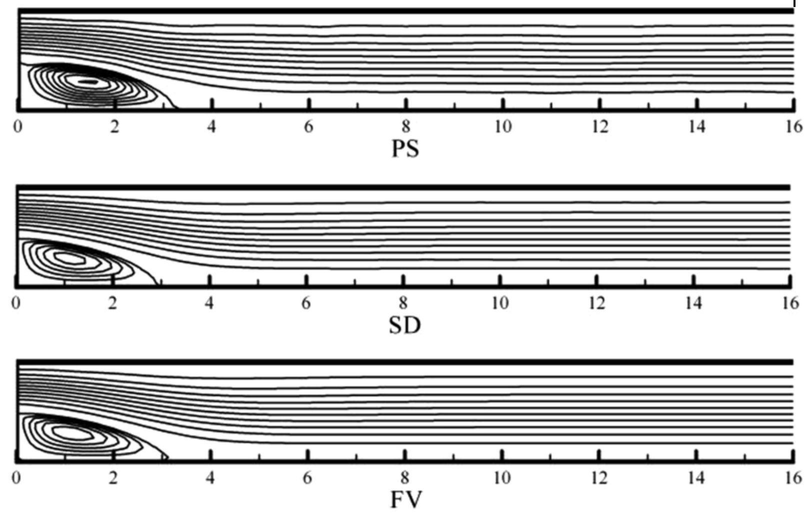
Backward facing step flow



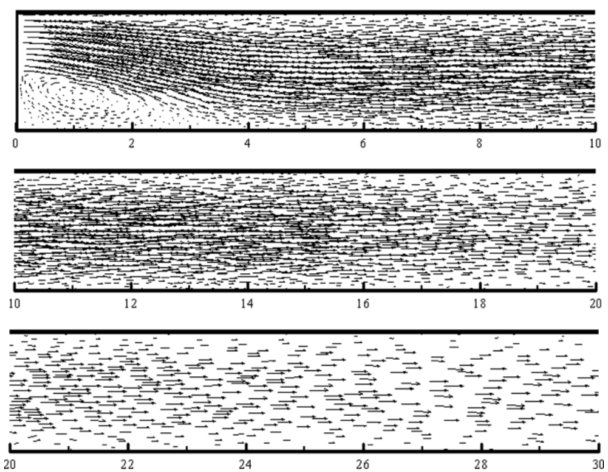
Backward facing step flow



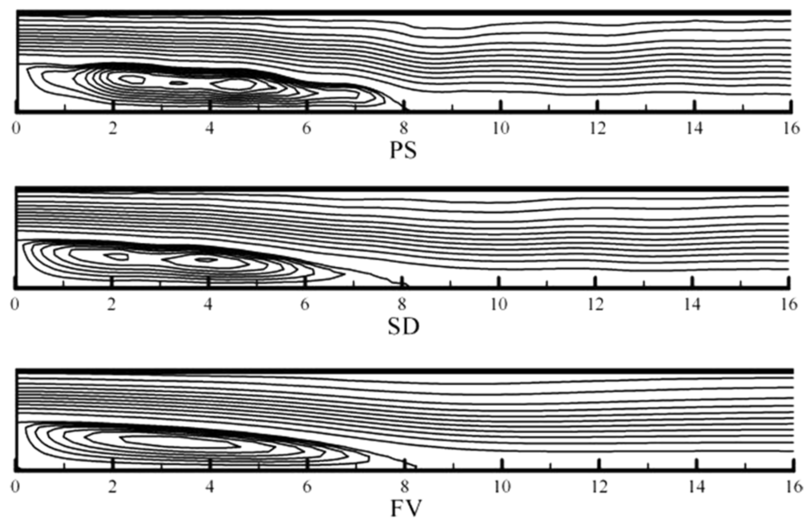
(*a*)



(*a*)

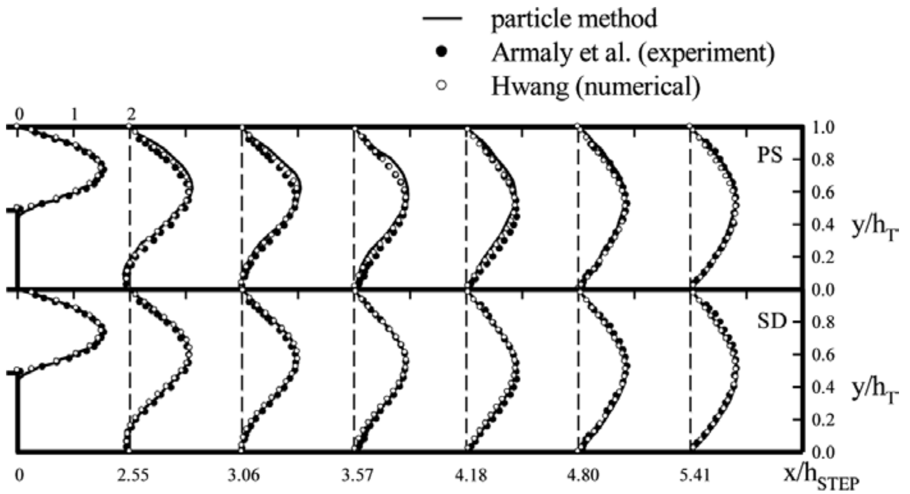


(*b*)

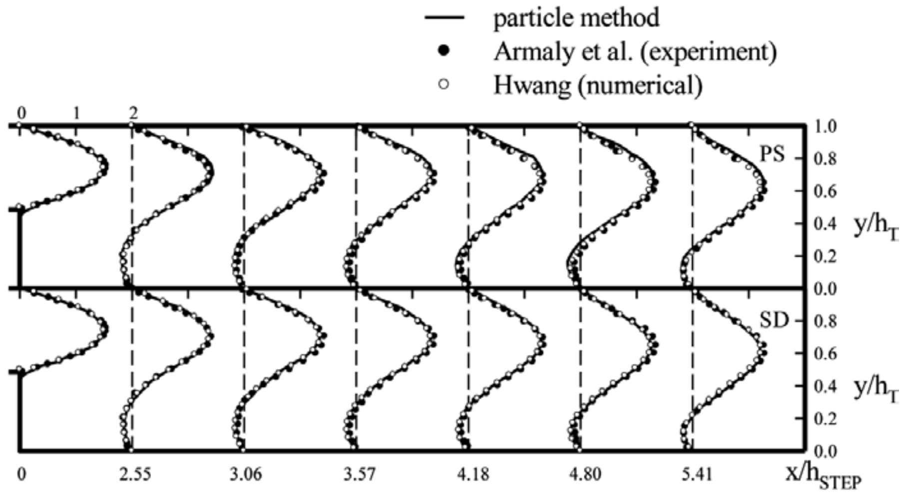


(*b*)

Backward facing step flow

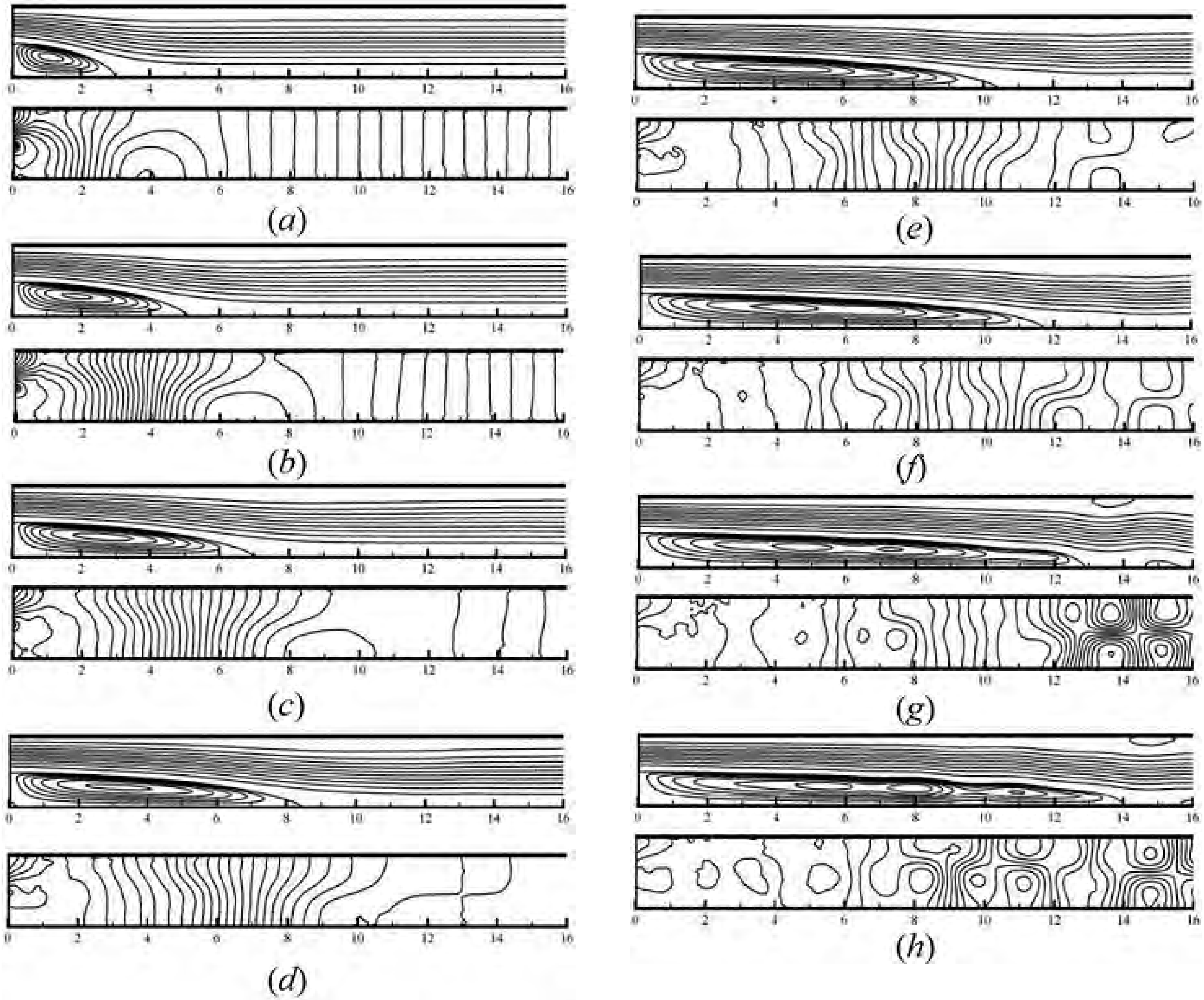


(a)

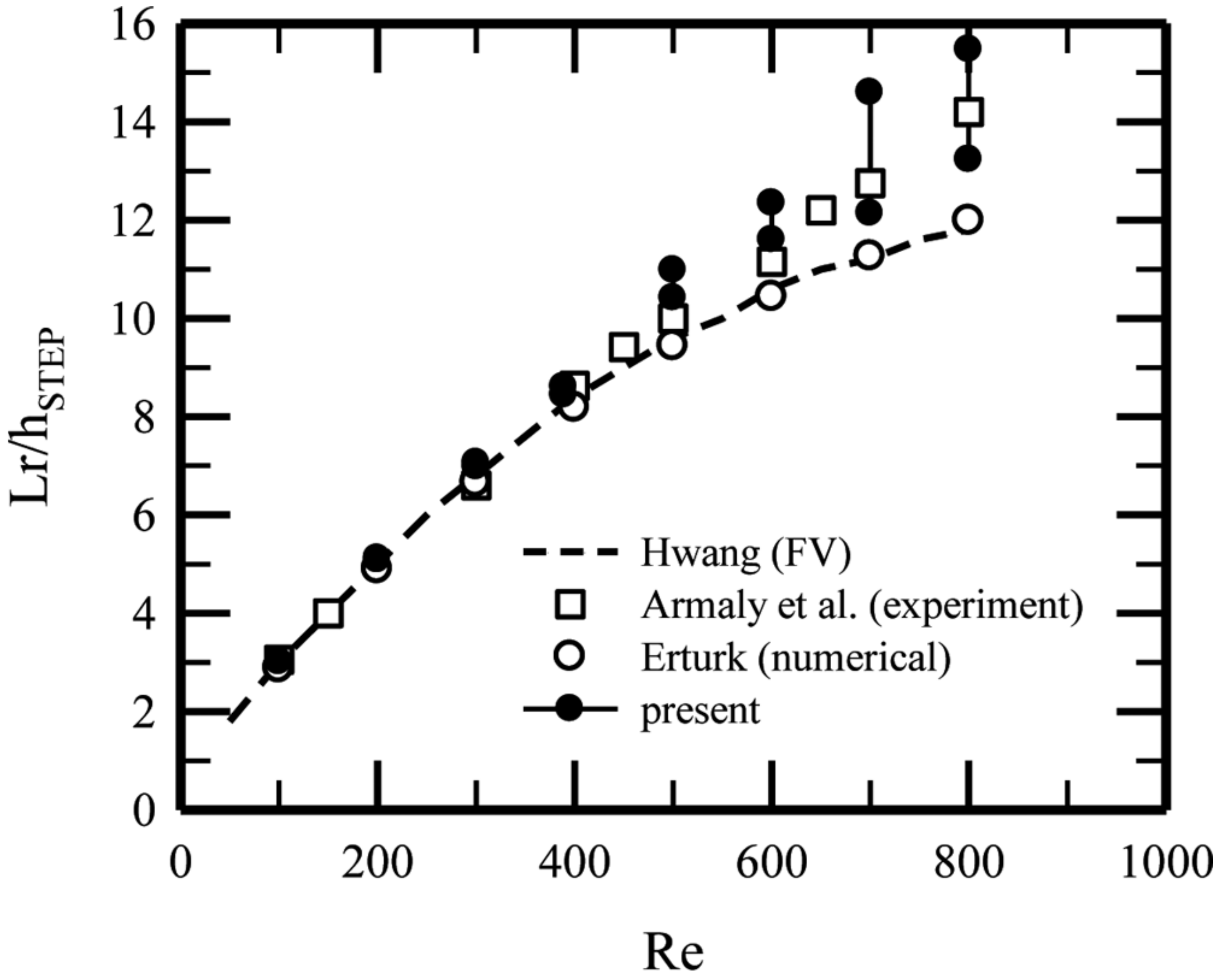


(b)

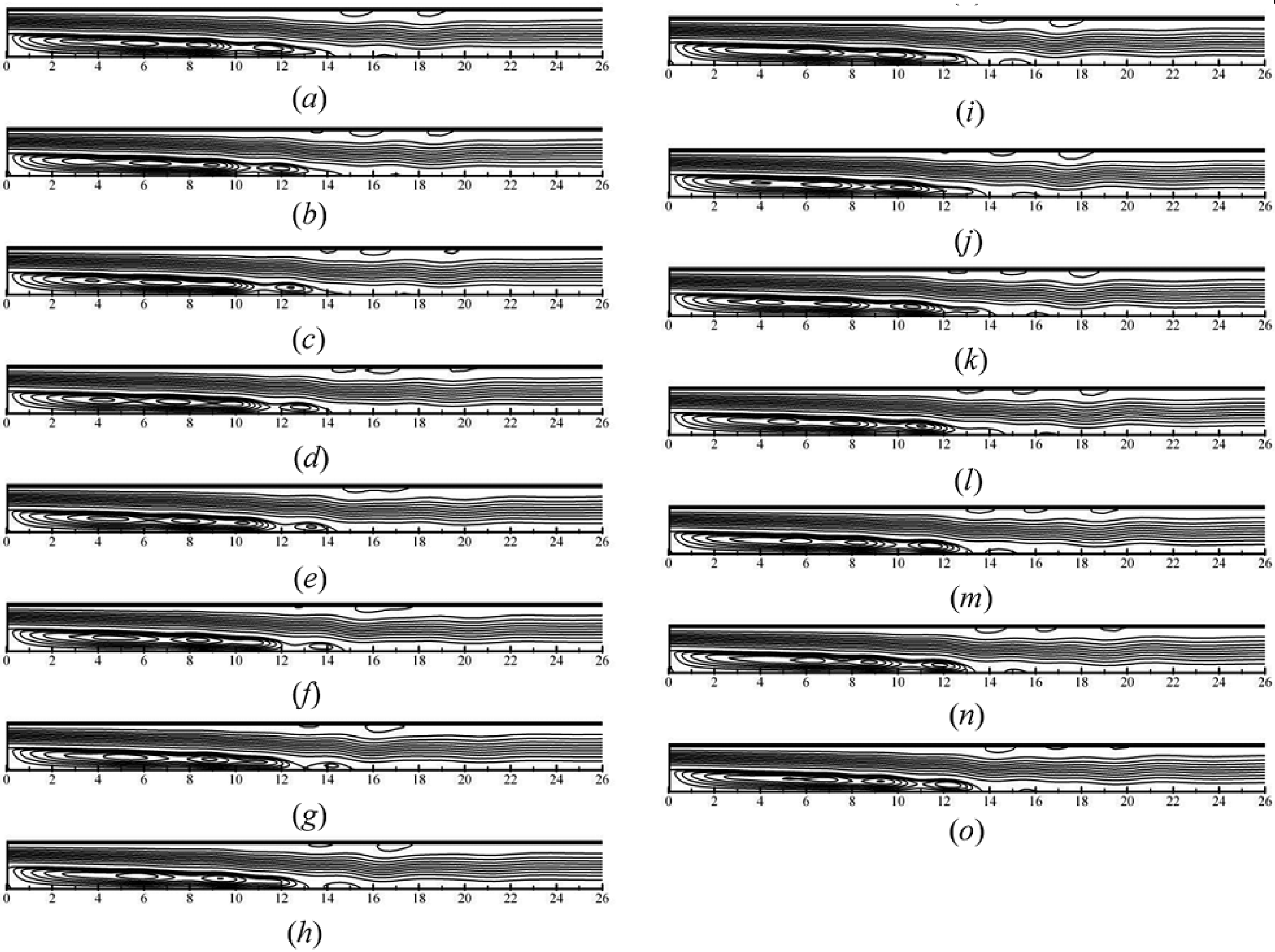
Backward facing step flow



Backward facing step flow



Backward facing step flow



CPU time

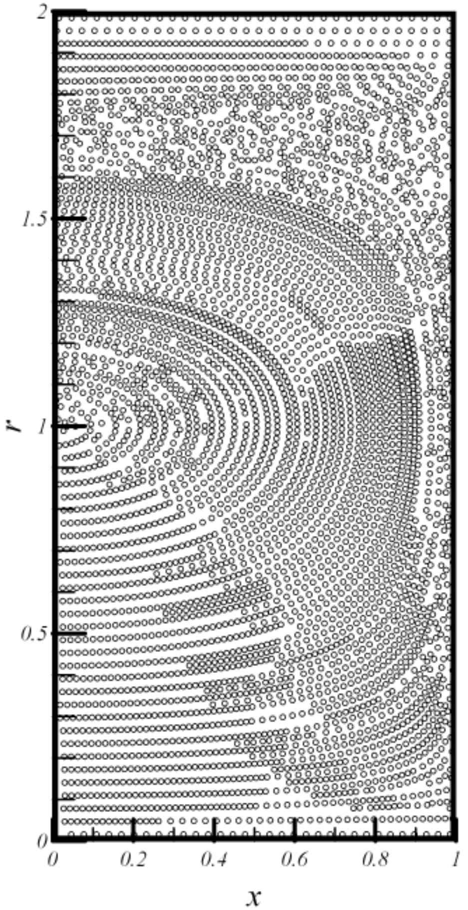
CPU time (s) of diffusion operator in cavity flow at $t = 100$

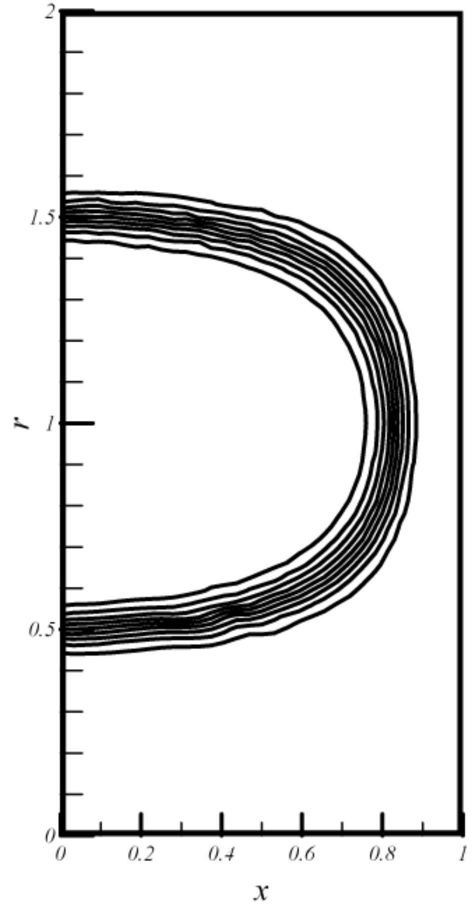
Re	PS	SD
100	3.777×10^2	1.049×10^4
400	3.714×10^2	1.087×10^4
1000	3.745×10^2	1.134×10^4

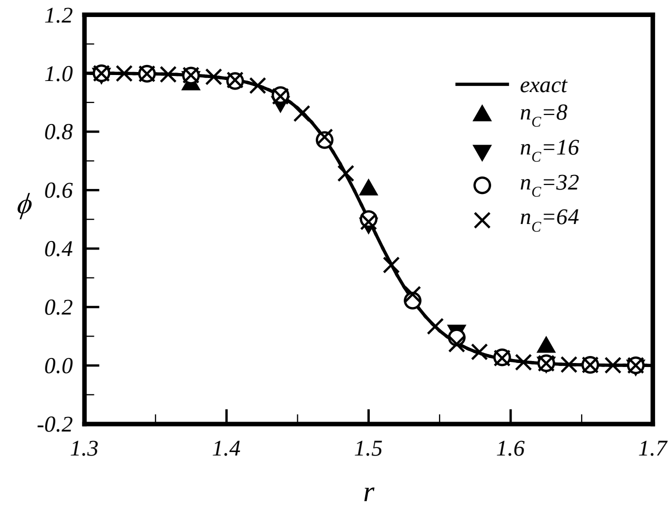
CPU time (s) of diffusion operator in backward-facing step flow at $t = 100$

Re	PS	SD
100	1.540×10^3	3.206×10^4
389	1.534×10^3	3.405×10^4

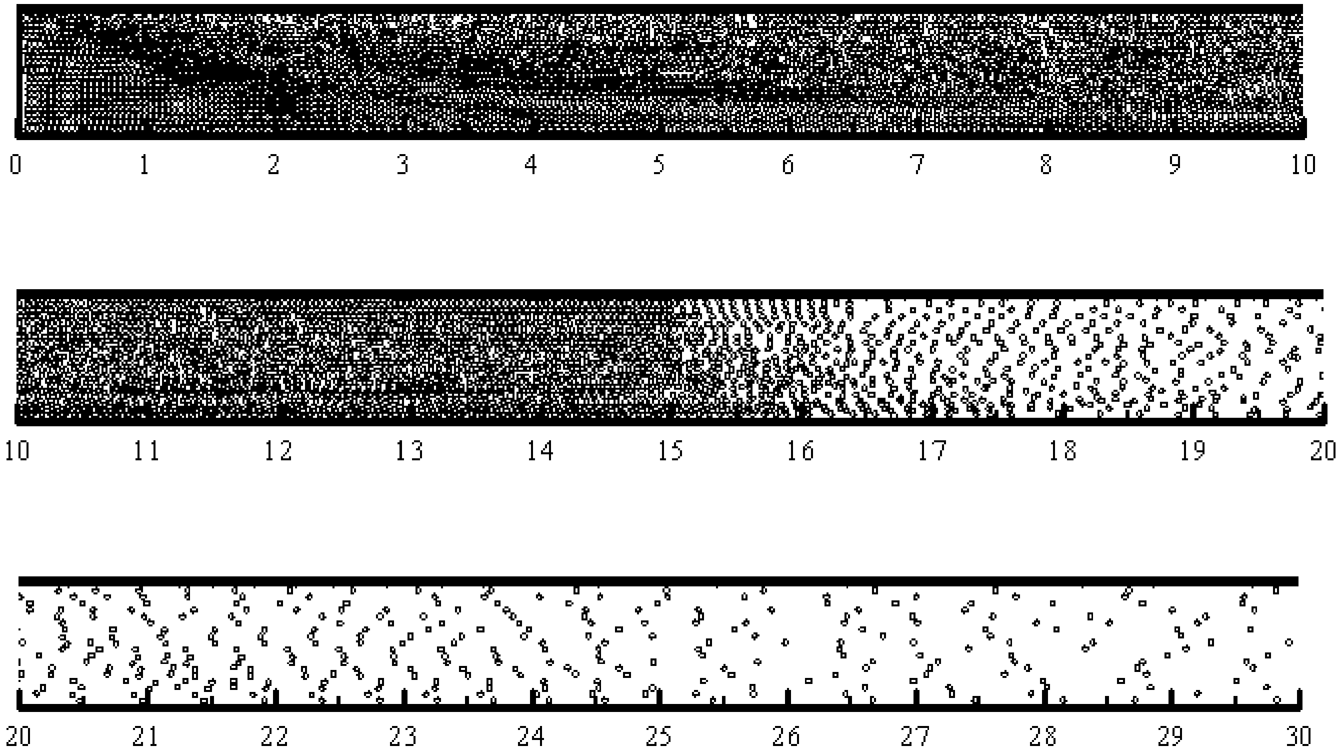
Axisymmetric: *pure convection*



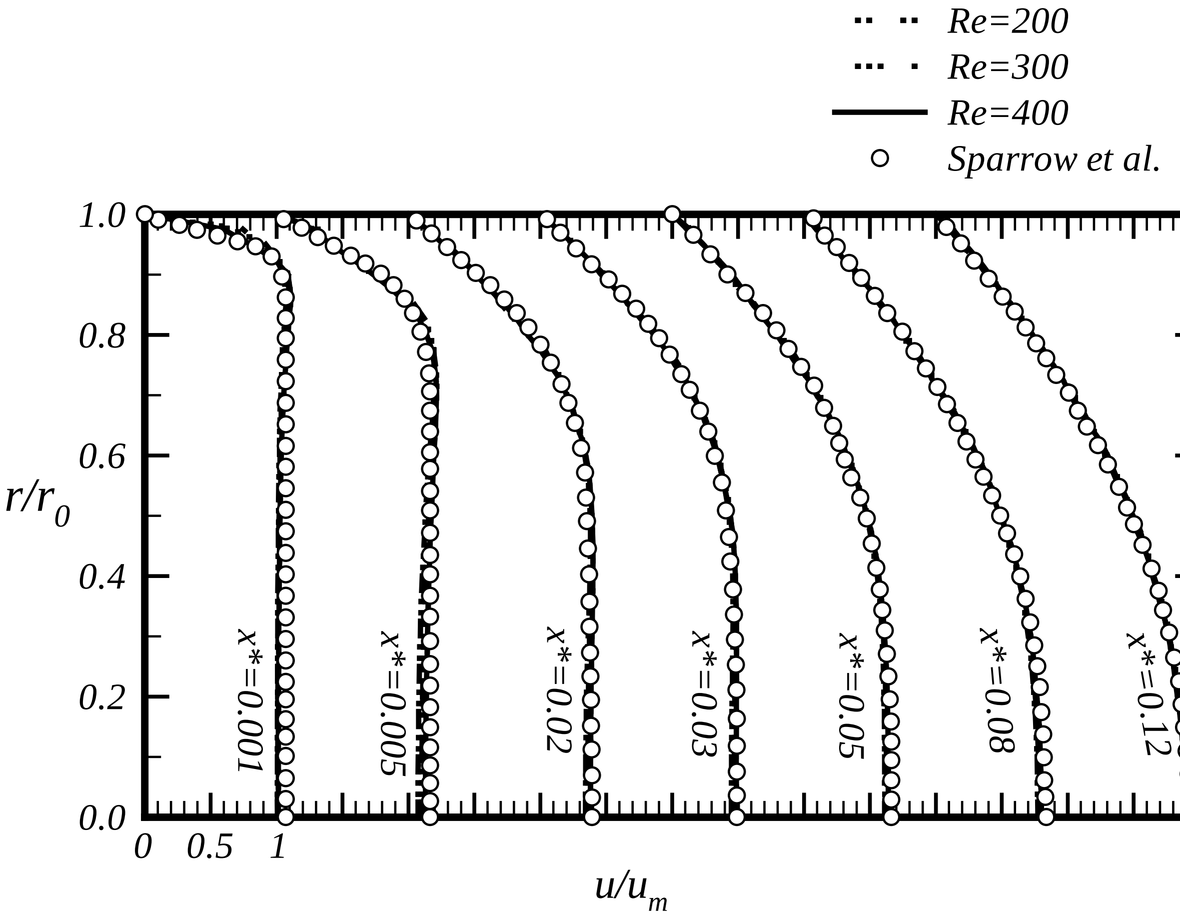




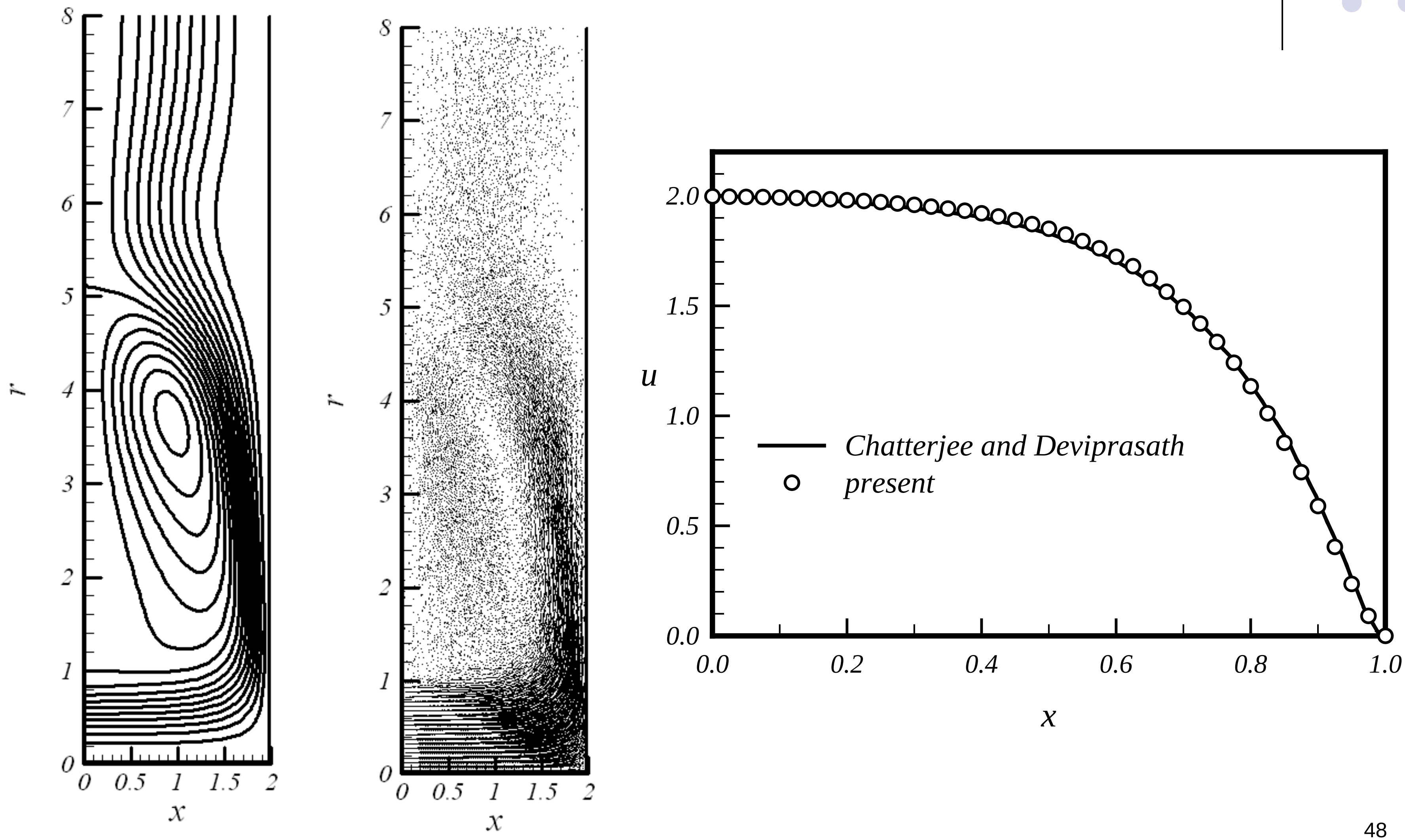
Axisymmetric: ***developing pipe flow***



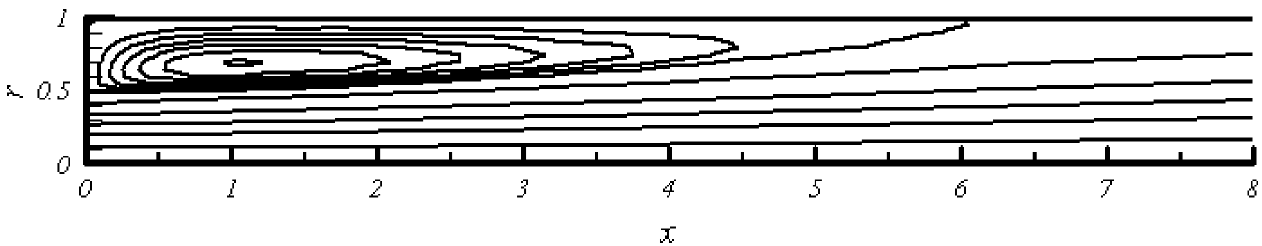
Axisymmetric: *developing pipe flow*

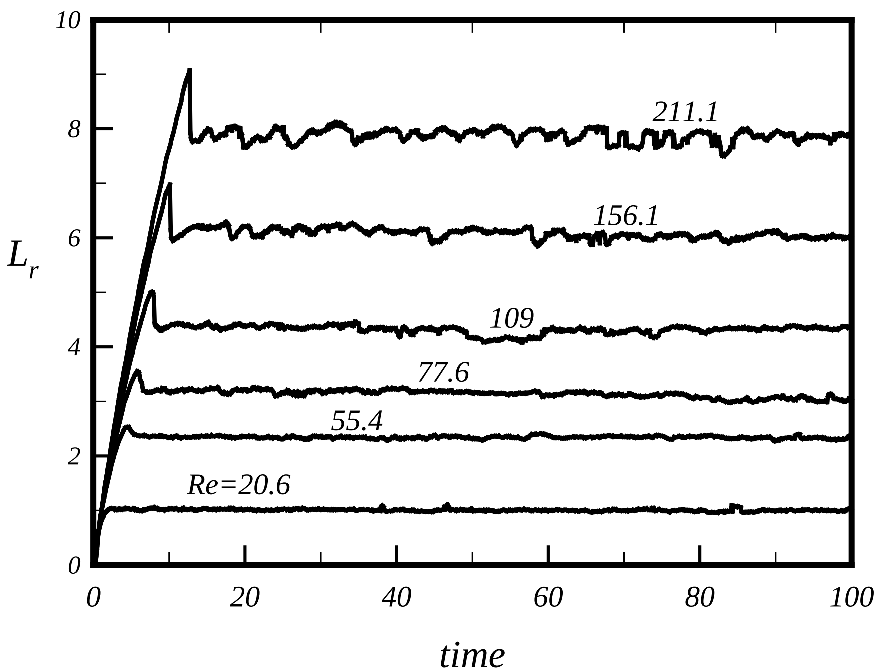


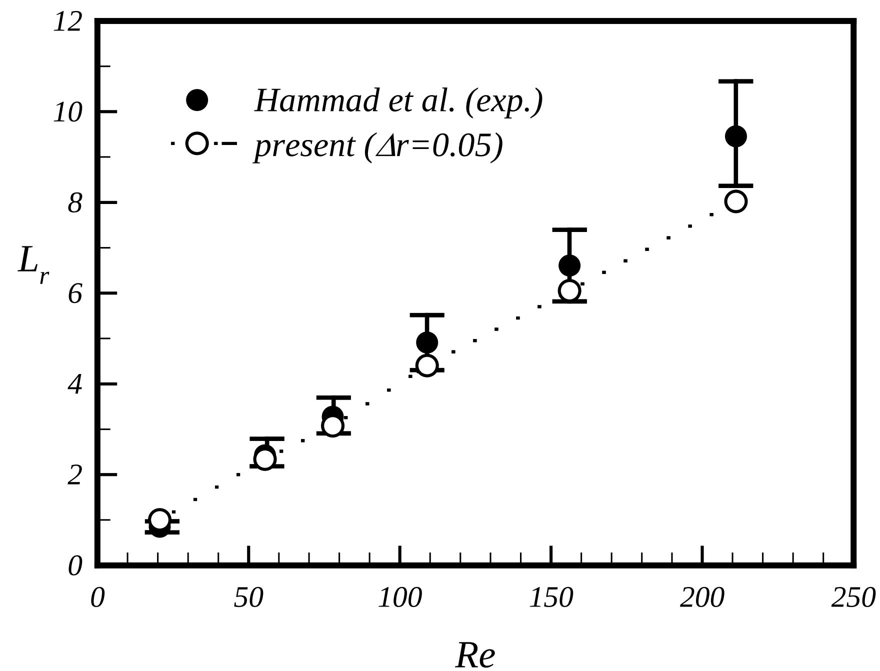
Axisymmetric flow: *confined impinging jet*



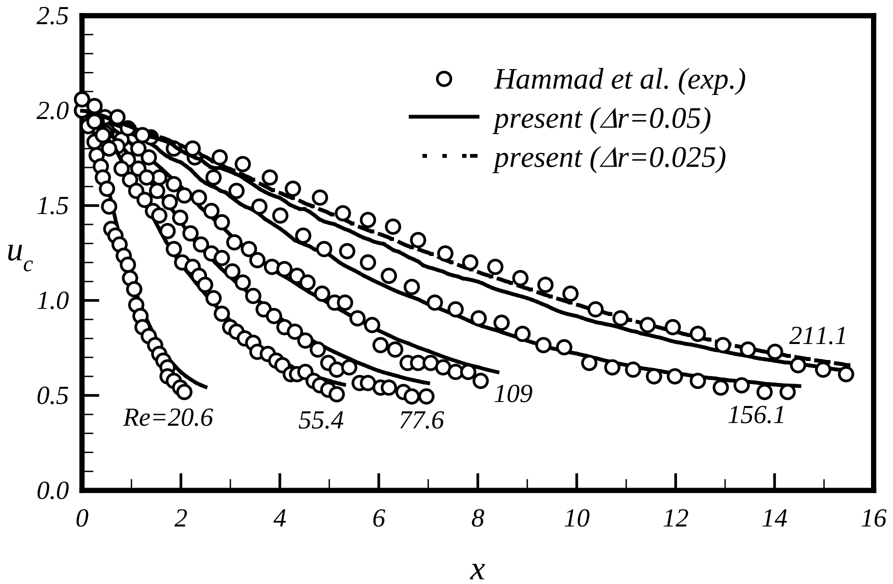
Axisymmetric flow: *sudden expansion*

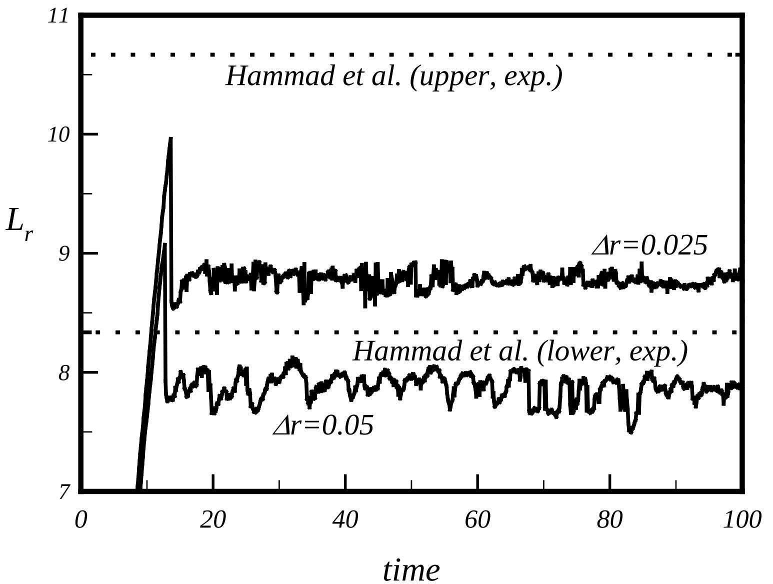






Axisymmetric flow: *sudden expansion*

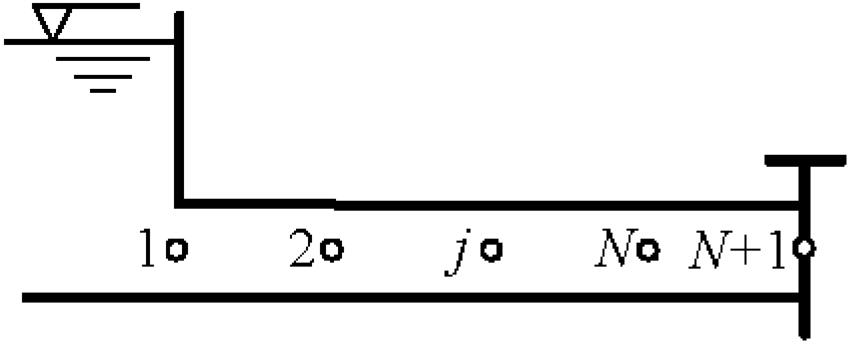




Characteristic particle method (CPM) – Hyperbolic Equation Systems

緣起

- **Water hammer project**
- **Godunov-type finite-volume method**
- **Particle method**



Governing Equations (1)

$$\frac{\partial H}{\partial \tau} + V\frac{\partial H}{\partial X} + \frac{a^2}{g}\frac{\partial V}{\partial X} = 0$$

$$\frac{\partial V}{\partial \tau} + V\frac{\partial V}{\partial X} + g\frac{\partial H}{\partial X} + J = 0$$

$$a = \sqrt{\frac{K/\rho}{1+(K/E)(D/e)}} \qquad J = \frac{f\left|V\right|}{2D}V = \Lambda V$$

Governing Equations (2)

$$\frac{\partial h}{\partial t} + v\frac{\partial h}{\partial x} + \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{\partial h}{\partial x} = -\lambda v$$

Governing Equations (3)

$$\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{A}\frac{\partial \boldsymbol{V}}{\partial x} = \boldsymbol{S}$$

$$\boldsymbol{V} = \begin{pmatrix} h \\ v \end{pmatrix} \qquad \boldsymbol{A} = \begin{pmatrix} v & 1 \\ 1 & v \end{pmatrix} \qquad \boldsymbol{S} = \begin{pmatrix} 0 \\ -\lambda v \end{pmatrix}$$

Characteristic Analyses(1)

$$\boldsymbol{A} = \boldsymbol{R}\boldsymbol{\Lambda}\boldsymbol{R}^{-1} \qquad \boldsymbol{R} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \boldsymbol{\Lambda} = \begin{pmatrix} v+1 & 0 \\ 0 & v-1 \end{pmatrix}$$

$$\boldsymbol{R}^{-1}\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{\Lambda}\boldsymbol{R}^{-1}\frac{\partial \boldsymbol{V}}{\partial x} = \boldsymbol{R}^{-1}\boldsymbol{S}$$

$$\boldsymbol{W} = \begin{pmatrix} w^{+} \\ w^{-} \end{pmatrix} = \boldsymbol{R}^{-1}\boldsymbol{V} = \begin{pmatrix} (h+v)/2 \\ (h-v)/2 \end{pmatrix}$$

$$\boldsymbol{V} = \begin{pmatrix} h \\ v \end{pmatrix} = \boldsymbol{R}\boldsymbol{W} = \begin{pmatrix} w^{+} + w^{-} \\ w^{+} - w^{-} \end{pmatrix}$$

Characteristic Analyses(2)

$$\frac{\partial \boldsymbol{W}}{\partial t} + \boldsymbol{\Lambda} \frac{\partial \boldsymbol{W}}{\partial x} = \boldsymbol{R}^{-1}\boldsymbol{S}$$

$$\boldsymbol{R}^{-1}\boldsymbol{S} = \begin{pmatrix} -\lambda v/2 \\ \lambda v/2 \end{pmatrix} = \begin{pmatrix} -\lambda(w^{+} - w^{-})/2 \\ \lambda(w^{+} - w^{-})/2 \end{pmatrix}$$

$$\frac{\partial w^{+}}{\partial t} + (v+1)\frac{\partial w^{+}}{\partial x} = -\lambda(w^{+} - w^{-})/2$$

$$\frac{\partial w^{-}}{\partial t} + (v-1)\frac{\partial w^{-}}{\partial x} = \lambda(w^{+} - w^{-})/2$$

Characteristic Analyses(3)

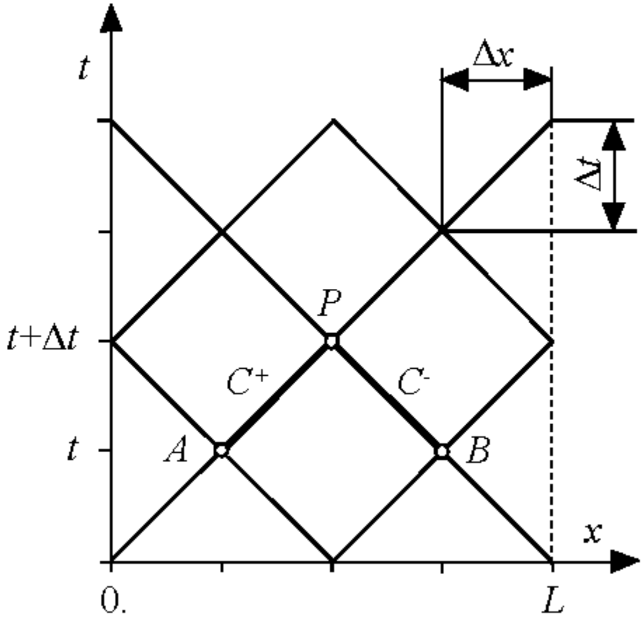
(i) Along the characteristic line of $\dfrac{dx}{dt} = v + 1$

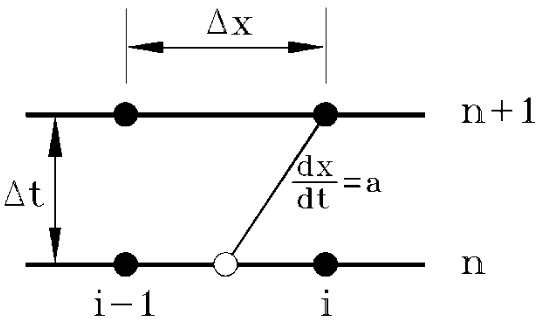
$$\frac{dw^{+}}{dt} + \frac{\lambda}{2} w^{+} = \frac{\lambda}{2} w^{-}$$

(ii) Along the characteristic line of $\dfrac{dx}{dt} = v - 1$

$$\frac{dw^{-}}{dt} + \frac{\lambda}{2} w^{-} = \frac{\lambda}{2} w^{+}$$

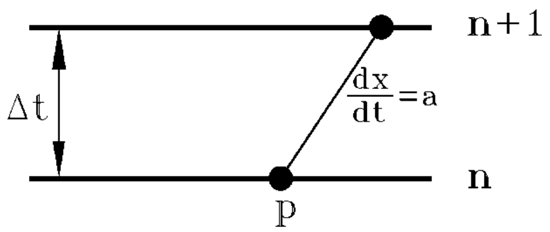
特徵線法MOC





Particle Method

- Moving along characteristic line
- No convection term error
- Adaptive
- Dual particles
- Particle interactions



Regular Particles (Transmitted)

1. **For the right-running characteristic,**

$$x_{p^+}^{n+1} = x_{p^+}^{n} + (v_{p^+}^{n} + 1)\Delta t$$

$$w_{p^+}^{+,n+1} = (w_{p^+}^{+,n} - w_{p^+}^{-,n}) e^{-\lambda \Delta t / 2} + w_{p^+}^{-,n}$$

2. **For the left-running characteristic,**

$$x_{p^-}^{n+1} = x_{p^-}^{n} + (v_{p^-}^{n} - 1)\Delta t$$

$$w_{p^-}^{-,n+1} = (w_{p^-}^{-,n} - w_{p^-}^{+,n}) e^{-\lambda \Delta t / 2} + w_{p^-}^{+,n}$$

Regular Particles (Interpolated)

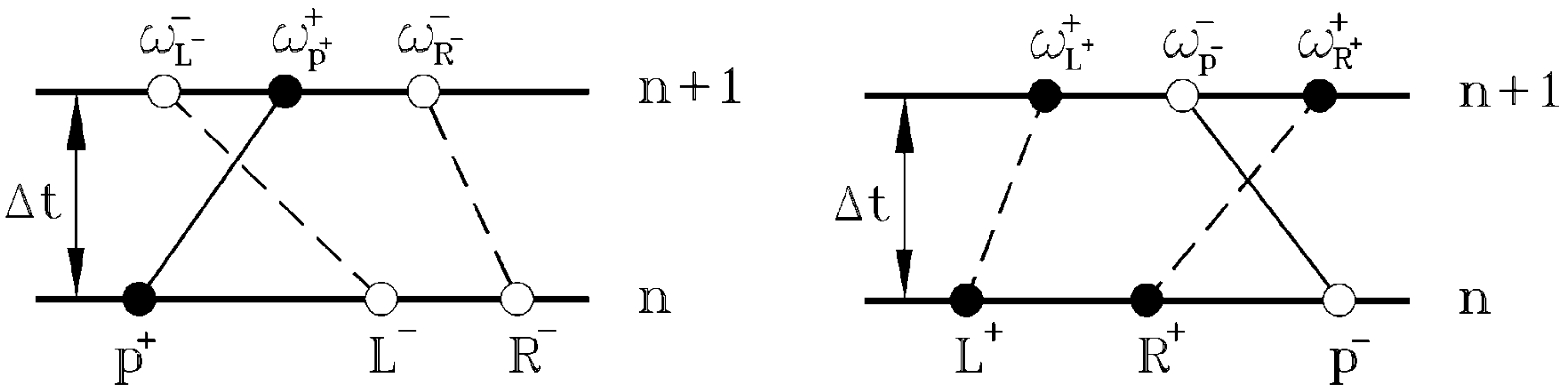
(i) For the right-running characteristic,

$$w_{p^+}^{-,n+1} = \frac{x_{R^-}^{n+1} - x_{p^+}^{n+1}}{x_{R^-}^{n+1} - x_{L^-}^{n+1}} w_{L^-}^{-,n+1} + \frac{x_{p^+}^{n+1} - x_{L^-}^{n+1}}{x_{R^-}^{n+1} - x_{L^-}^{n+1}} w_{R^-}^{-,n+1}$$

(ii) For the left-running characteristic,

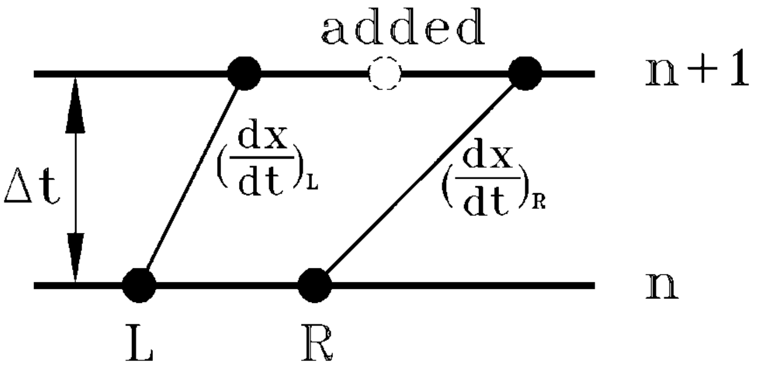
$$w_{p^-}^{+,n+1} = \frac{x_{R^+}^{n+1} - x_{p^-}^{n+1}}{x_{R^+}^{n+1} - x_{L^+}^{n+1}} w_{L^+}^{+,n+1} + \frac{x_{p^-}^{n+1} - x_{L^+}^{n+1}}{x_{R^+}^{n+1} - x_{L^+}^{n+1}} w_{R^+}^{+,n+1}$$

Regular Particles

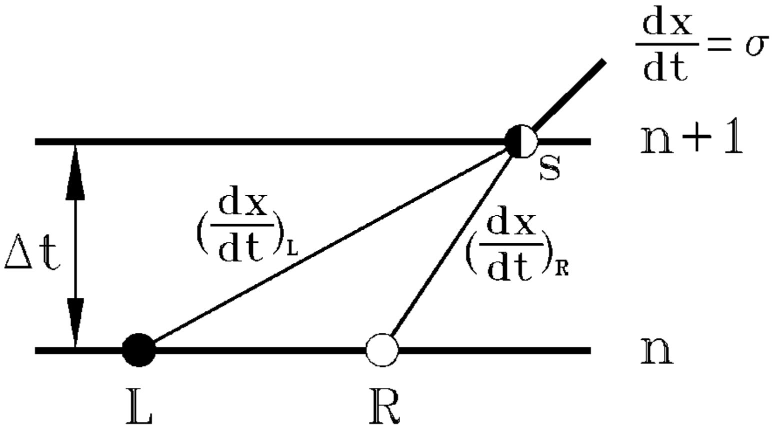


Particle Management

(i) Addition,



(ii) Merge,



Shock Particle (1)

(i) Conservation form

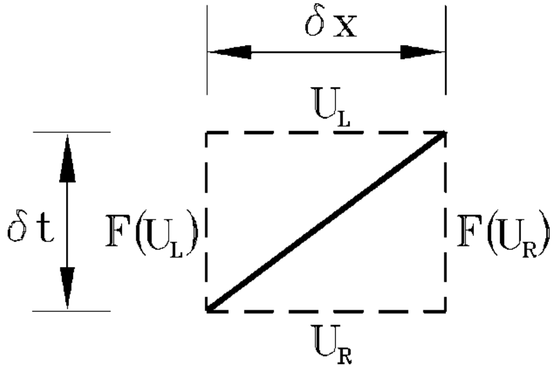
$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} = e^h \boldsymbol{S} \qquad \boldsymbol{U} = \begin{pmatrix} e^h \\ e^h v \end{pmatrix} \qquad \boldsymbol{F} = \begin{pmatrix} e^h v \\ e^h v^2 + e^h \end{pmatrix}$$

(ii) Rankine-Hugoniot relation

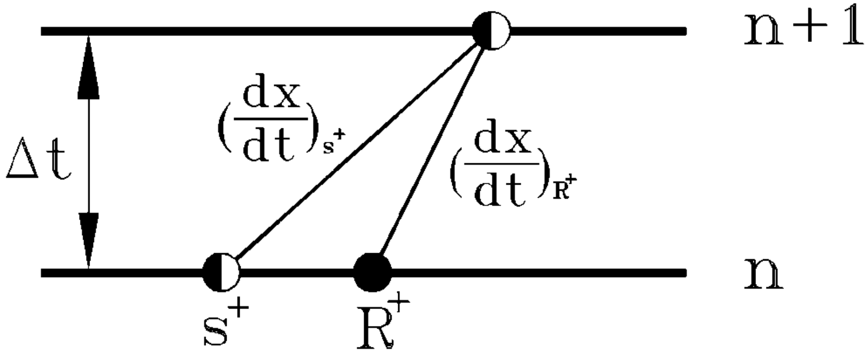
$$(\boldsymbol{U}_L - \boldsymbol{U}_R)\delta x + [\boldsymbol{F}(\boldsymbol{U}_R) - \boldsymbol{F}(\boldsymbol{U}_L)]\delta t$$

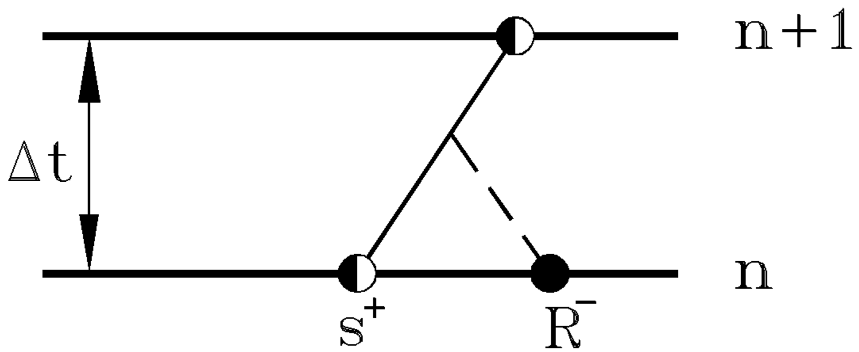
$$\sigma(\boldsymbol{U}_L - \boldsymbol{U}_R) = \boldsymbol{F}(\boldsymbol{U}_L) - \boldsymbol{F}(\boldsymbol{U}_R)$$

$$\sigma = \delta x / \delta t$$

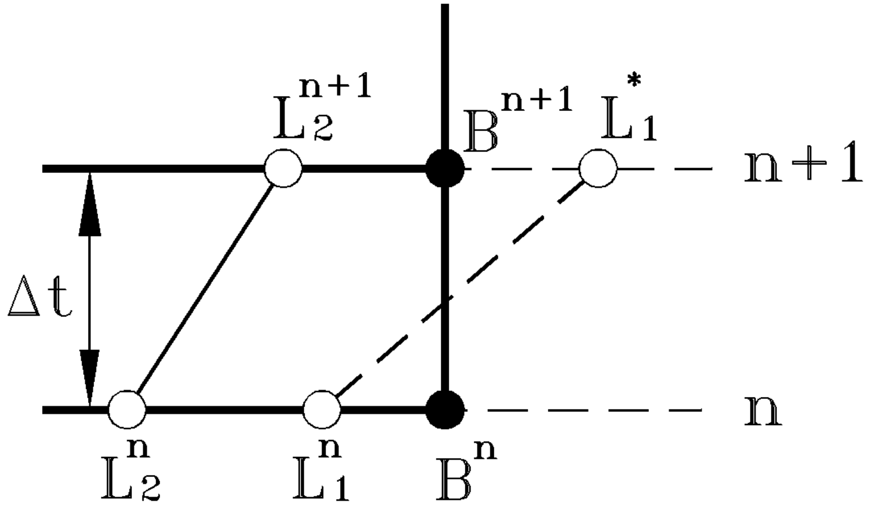


Removal of a particle



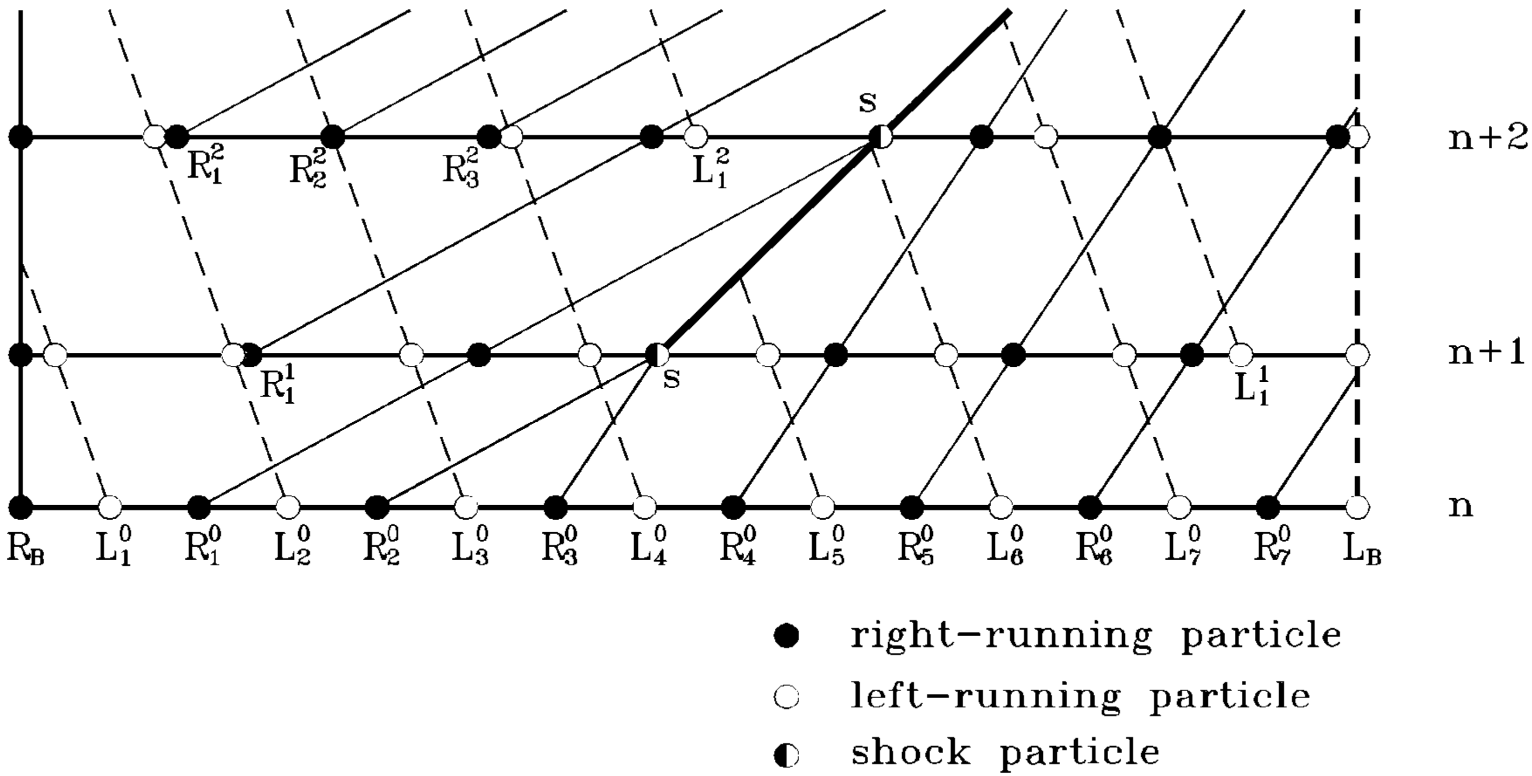


Boundary Condition



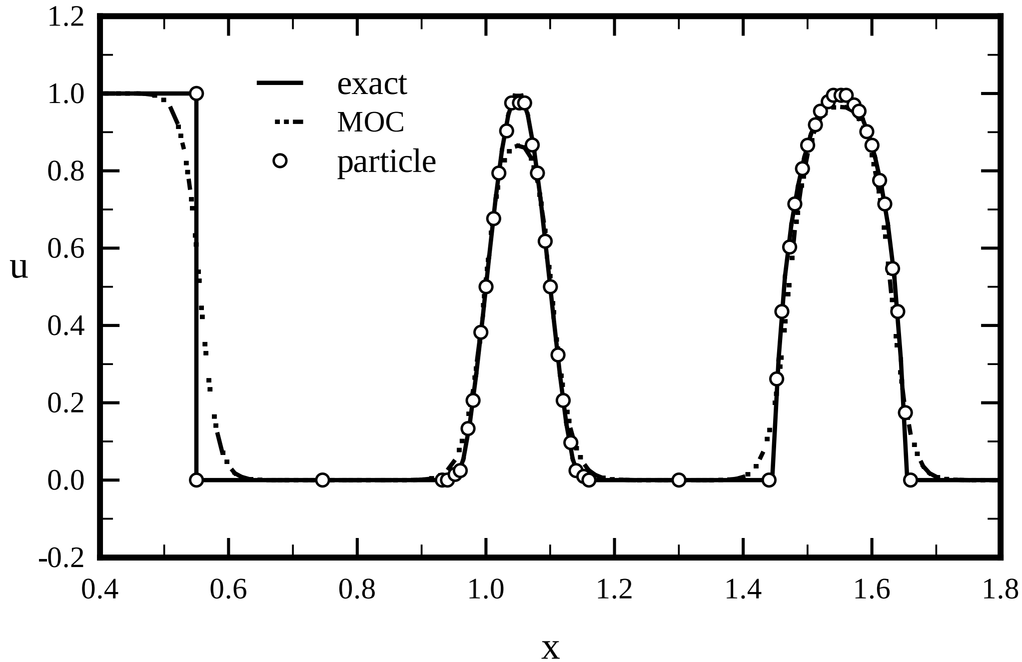
$$w_B^+ = \frac{(x_B^{n+1} - x_{L_2}^{n+1})w_{L_1}^+ + (x_{L_1}^* - x_B^{n+1})w_{L_2}^+}{(x_{L_1}^* - x_{L_2}^{n+1})}$$

Particle Trajectory

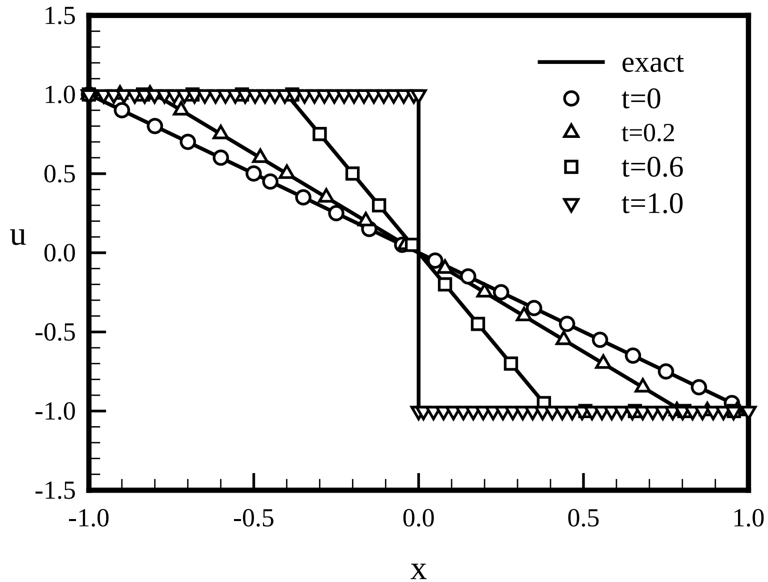


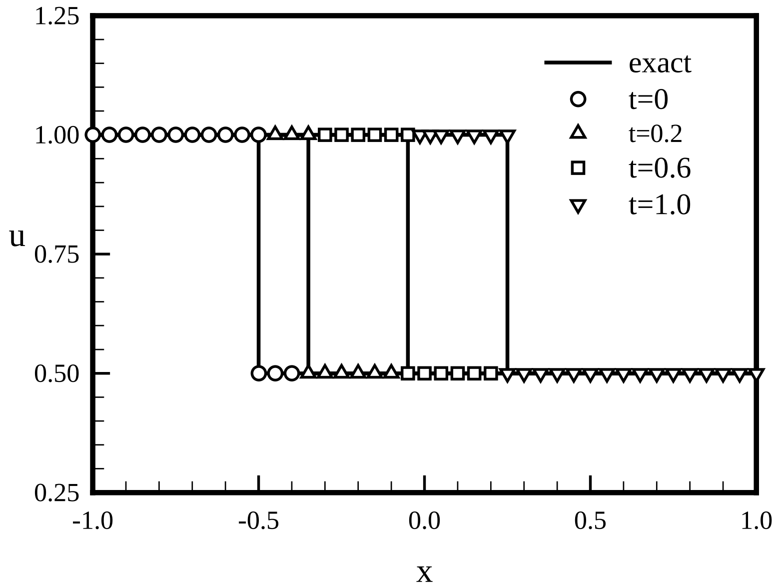
Linear Problem

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad u(x,t) = u(x - a\Delta t, t - \Delta t)$$



Burger's Equation (1)

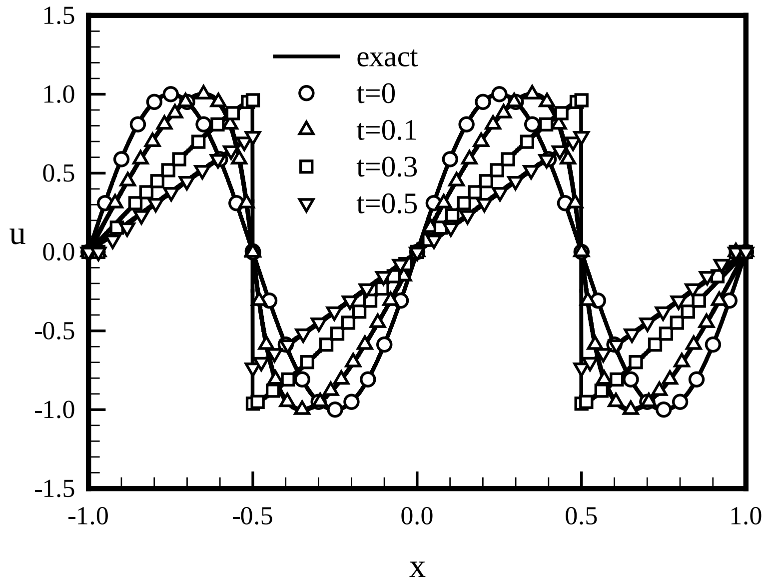


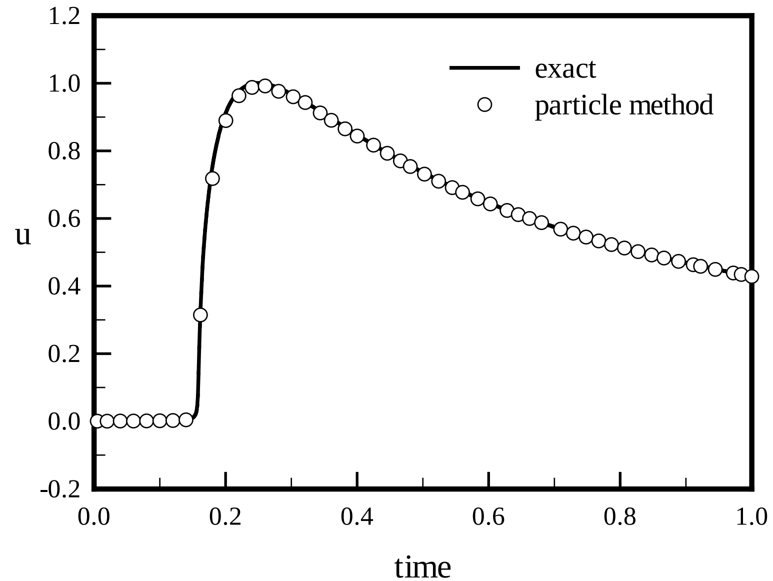


$$u(x,0) = -x$$

$$u(x,0) = \begin{cases} 1.0 & x < -0.5 \\ 0.5 & x > -0.5 \end{cases}$$

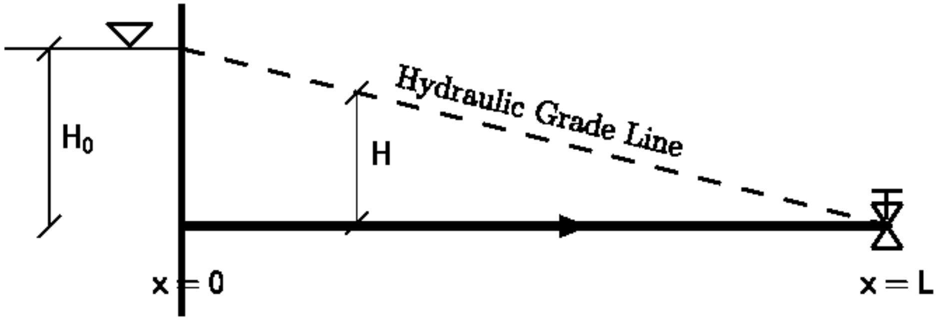
Burger's Equation (2)





$$u(x,0) = sin(2\pi x)$$

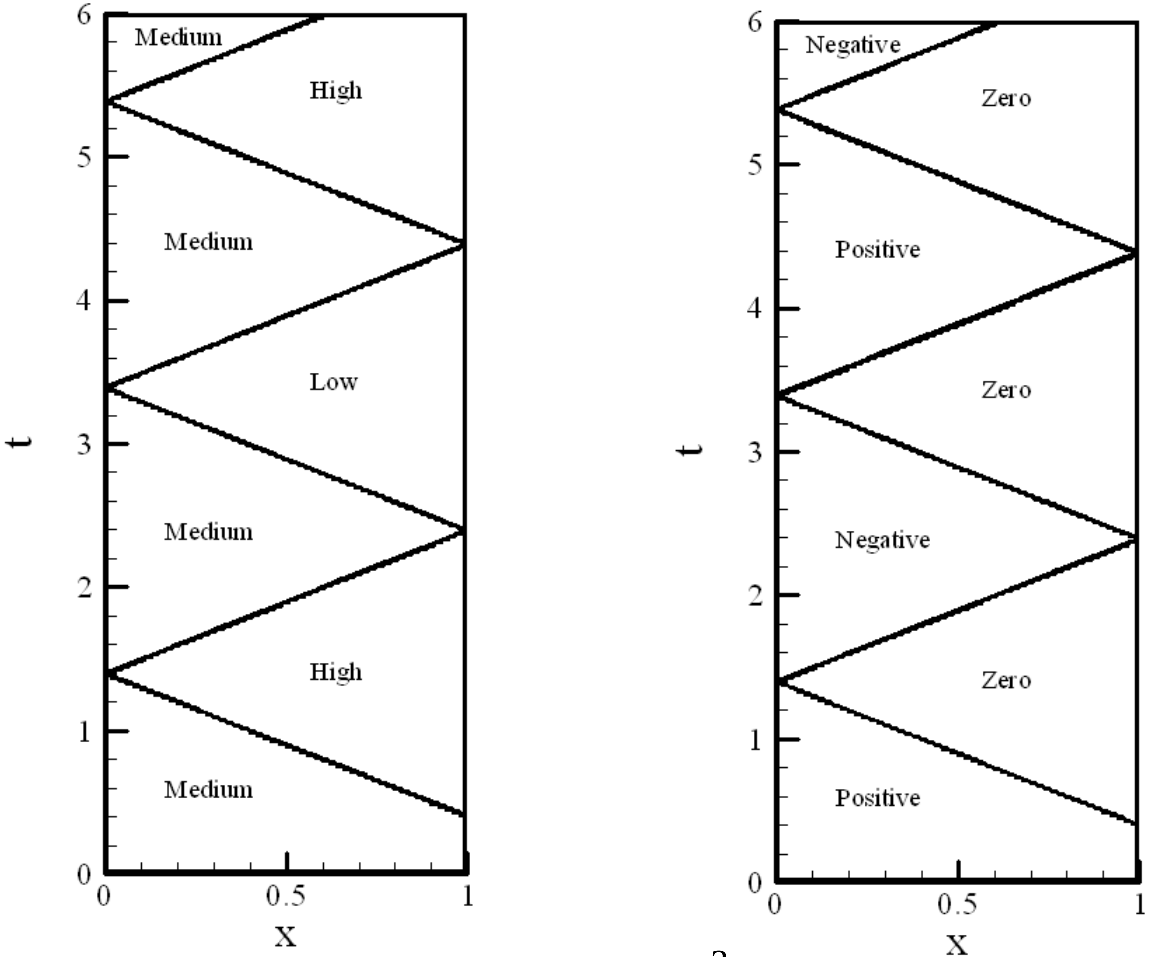
Sudden Valve Closure (1)



IC: $h(x,0) = h_0(1-x)$ $\qquad v(x,0) = v_0$

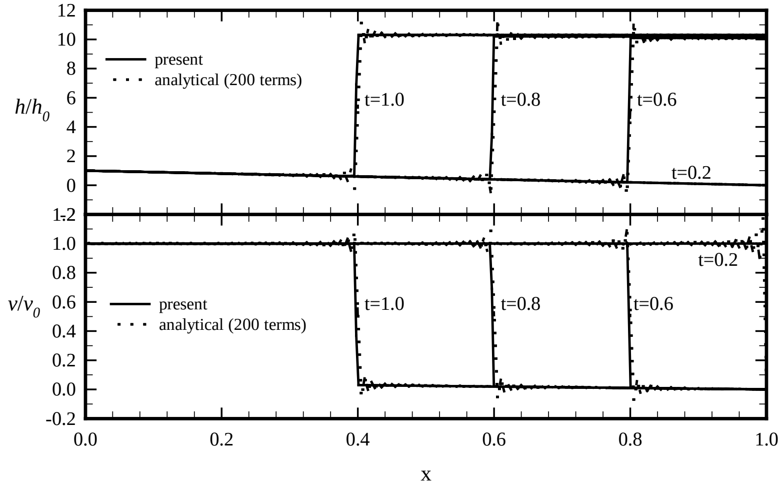
BC: $h(0,t) = h_0$ $\qquad v(1,t) = \begin{cases} v_0 & t < t_0 \\ 0 & t \geq t_0 \end{cases}$

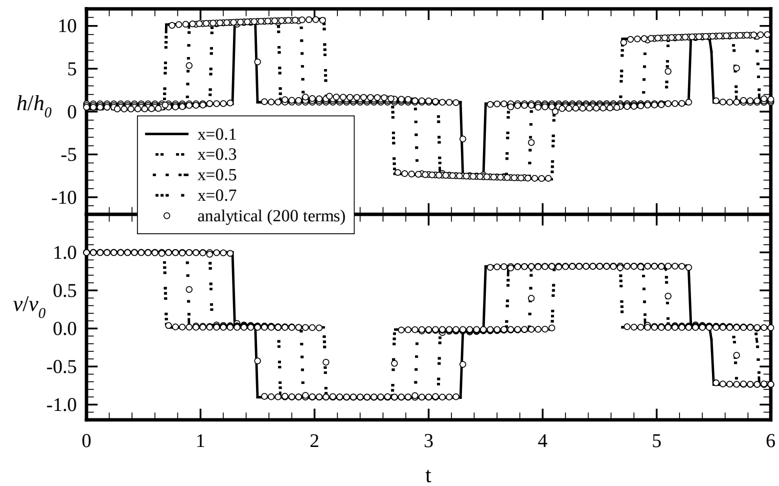
Sudden Valve Closure (2)



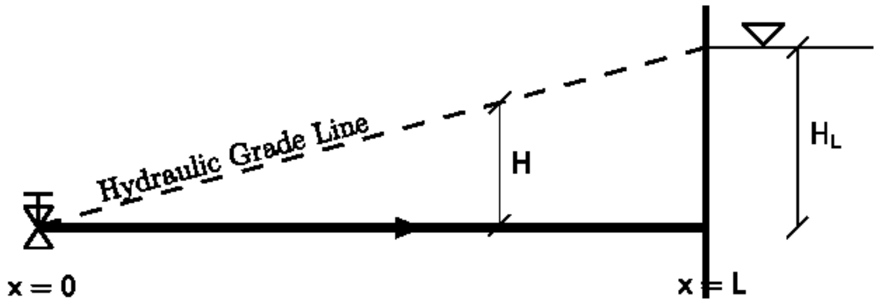
$$v_0 = 10^{-3}$$

Sudden Valve Closure (3)





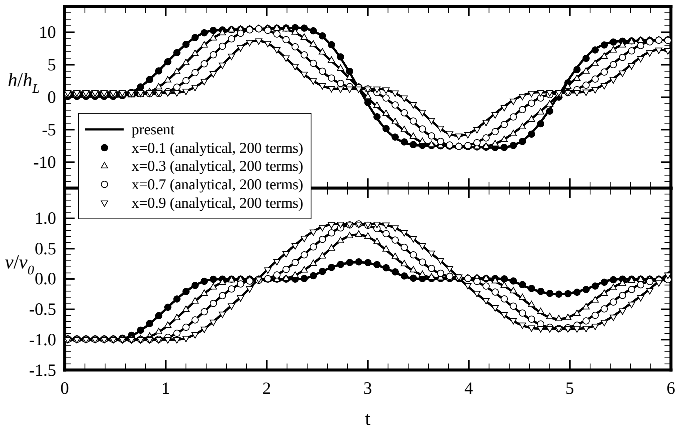
Valve Closure Over Finite Time (1)



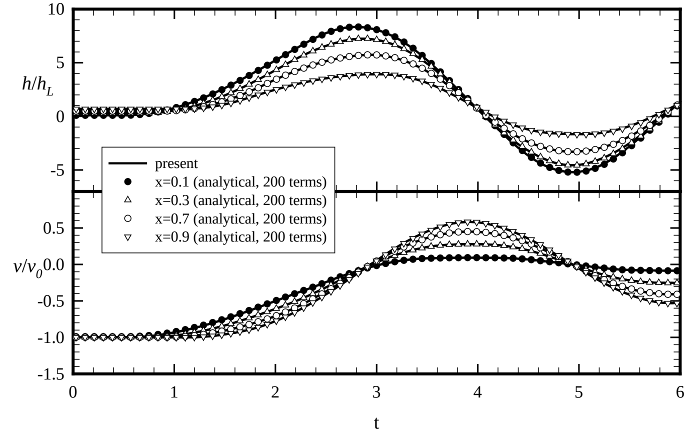
IC: $h(x,0) = h_L x \qquad v(x,0) = v_0$

BC: $$v(0,t) = \begin{cases} v_0 & \text{for } t \le t_0 \\ \dfrac{v_0}{2}\left[1 + \cos\dfrac{\pi(t - t_0)}{t_C}\right] & \text{for } t_0 < t < t_0 + t_C \\ 0 & \text{for } t \ge t_0 + t_C \end{cases}$$

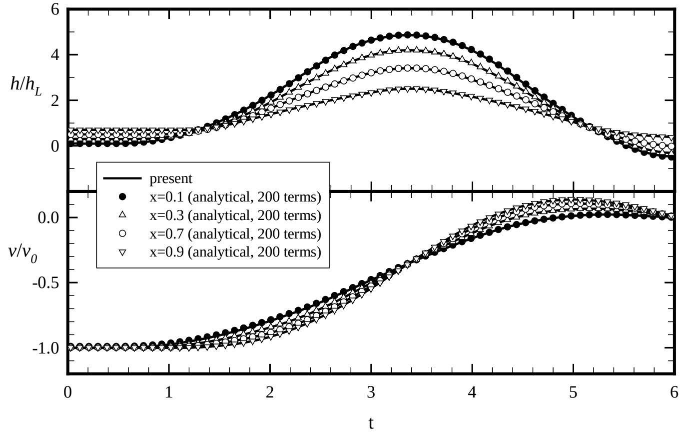
Valve Closure Over Finite Time (2)



t_C=1.0

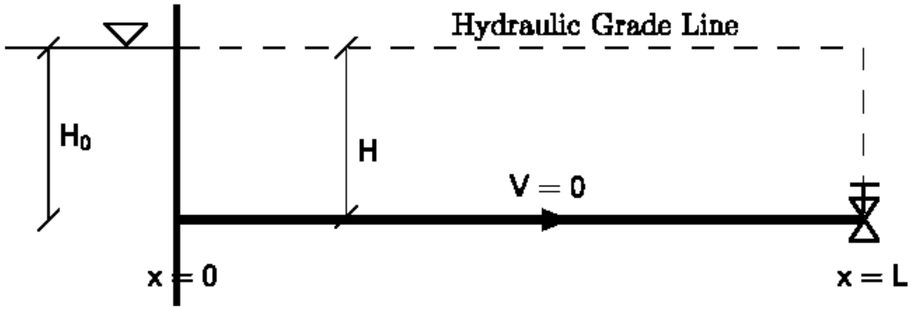


t_C=3.0



t_C=5.0

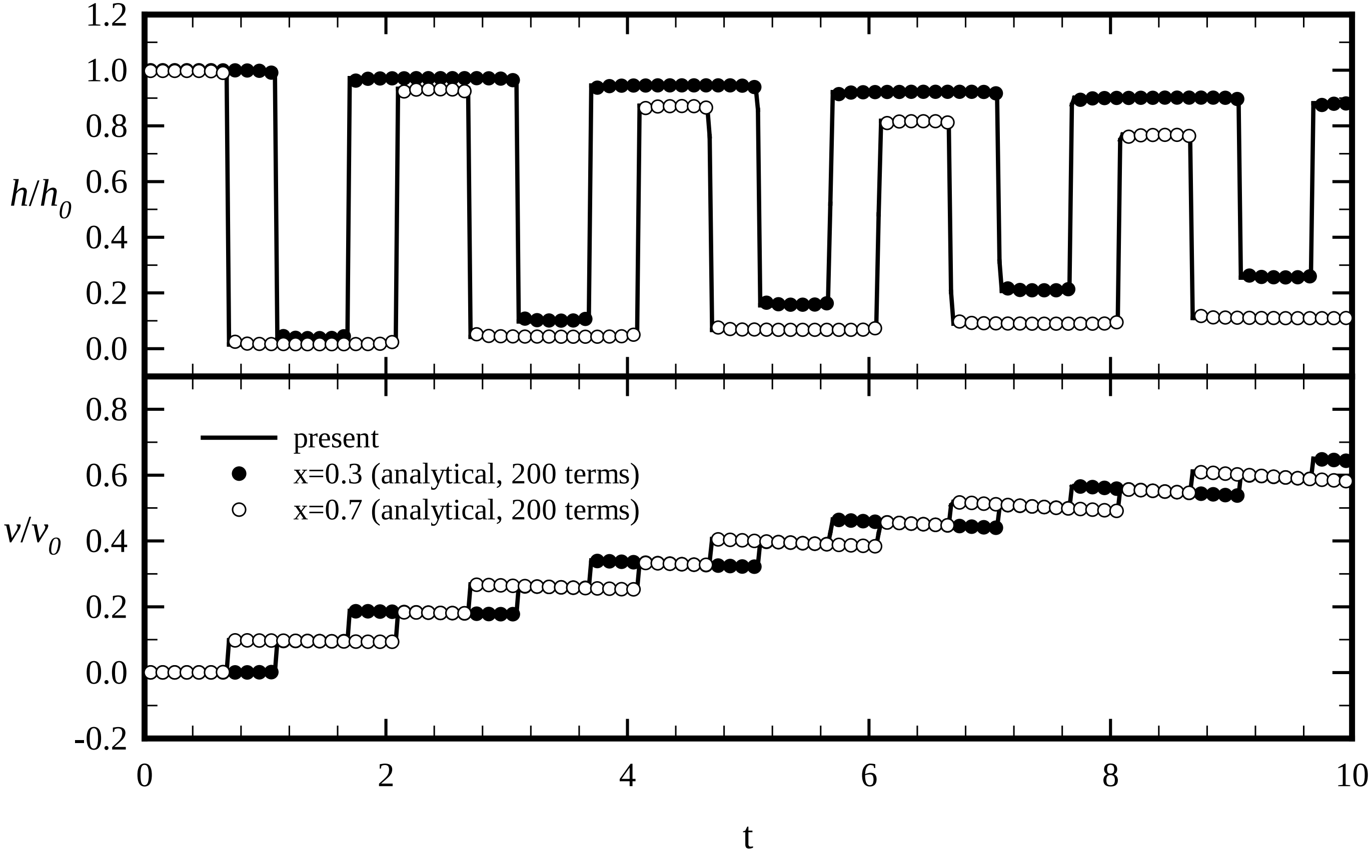
Start-up and Evolution to Steady State(1)



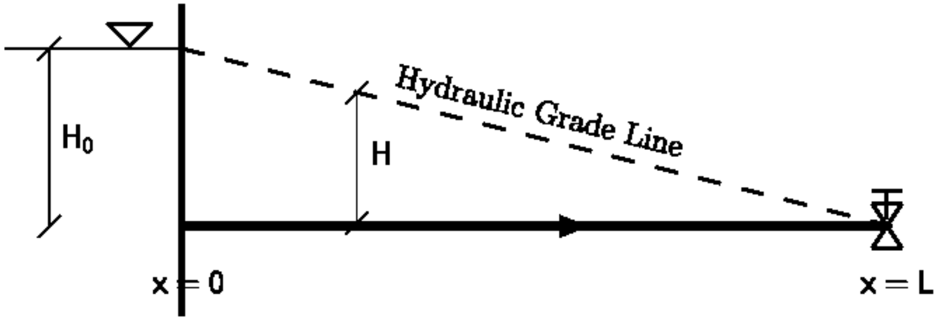
IC: $h(x,0) = h_0$ $\quad v(x,0) = 0$

BC: $h(0,t) = h_0$ $\quad h(1,t) = \begin{cases} h_0 & t < t_0 \\ 0 & t \geq t_0 \end{cases}$

Start-up and Evolution to Steady State(2)



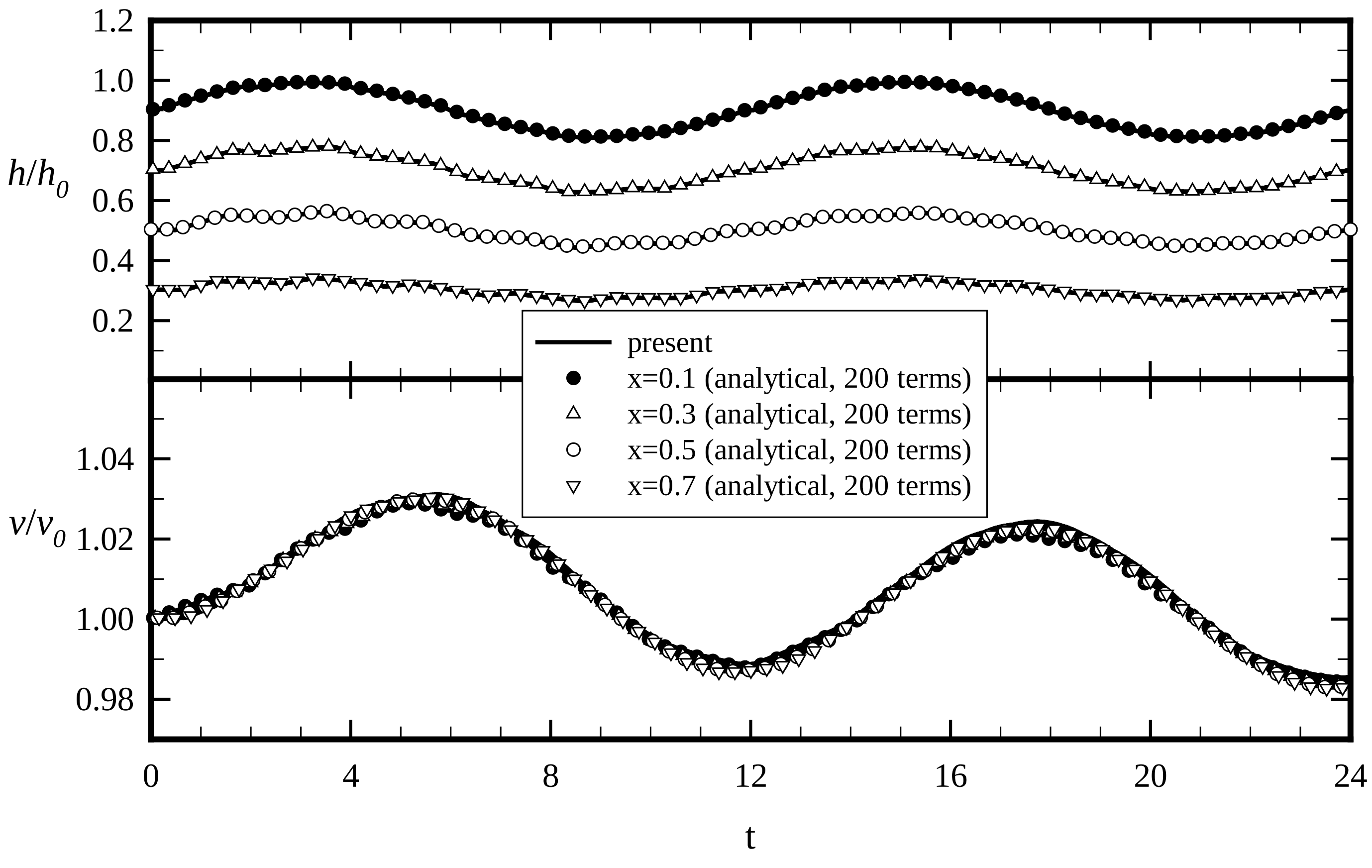
Periodic Forcing (1)



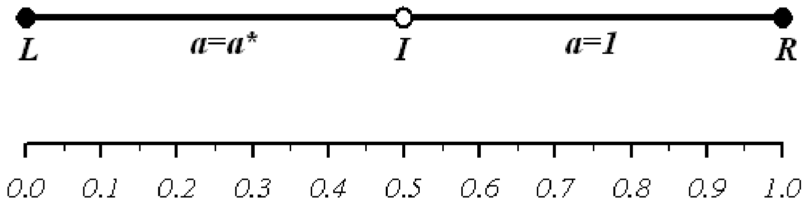
IC: $h(x,0) = h_0(1 - x)$ $\quad v(x,0) = v_0$

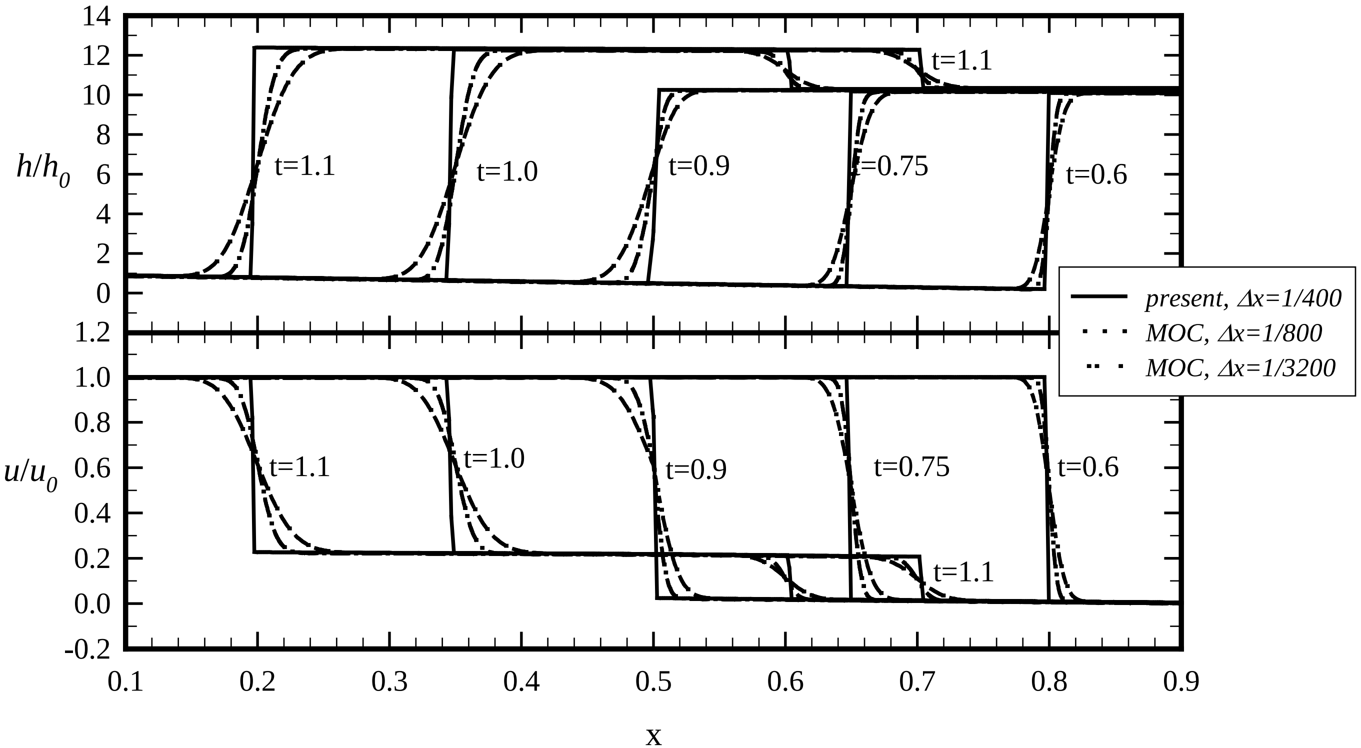
BC: $h(0,t) = h_0 + a_0 \sin(\omega t)$ $\quad h(1,t) = 0$

Periodic Forcing (2)

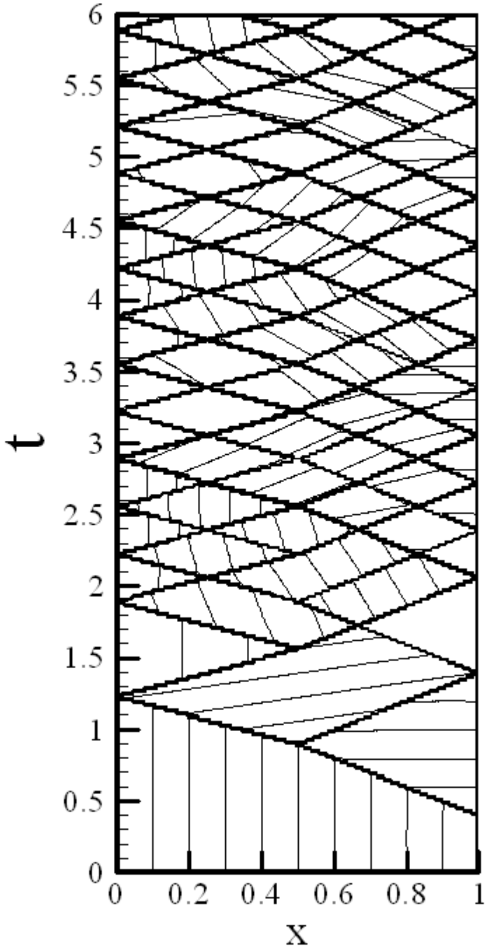


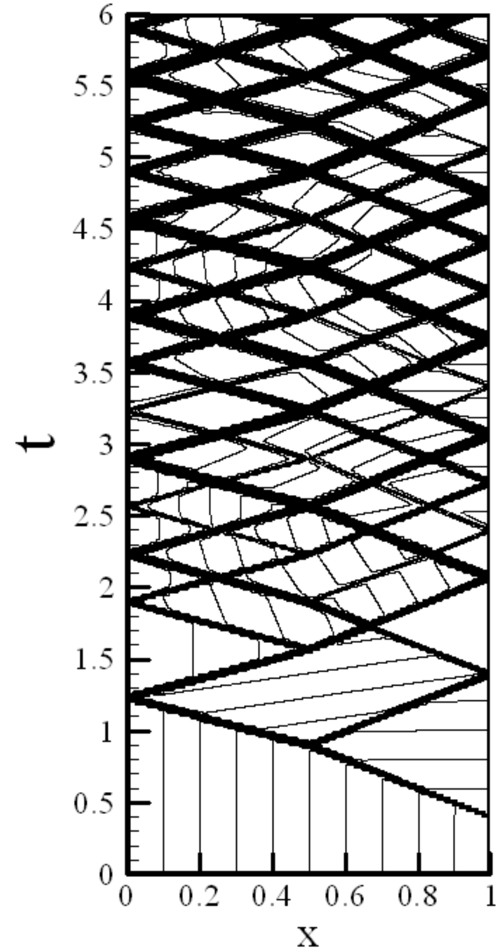
Sound speed variations (1)



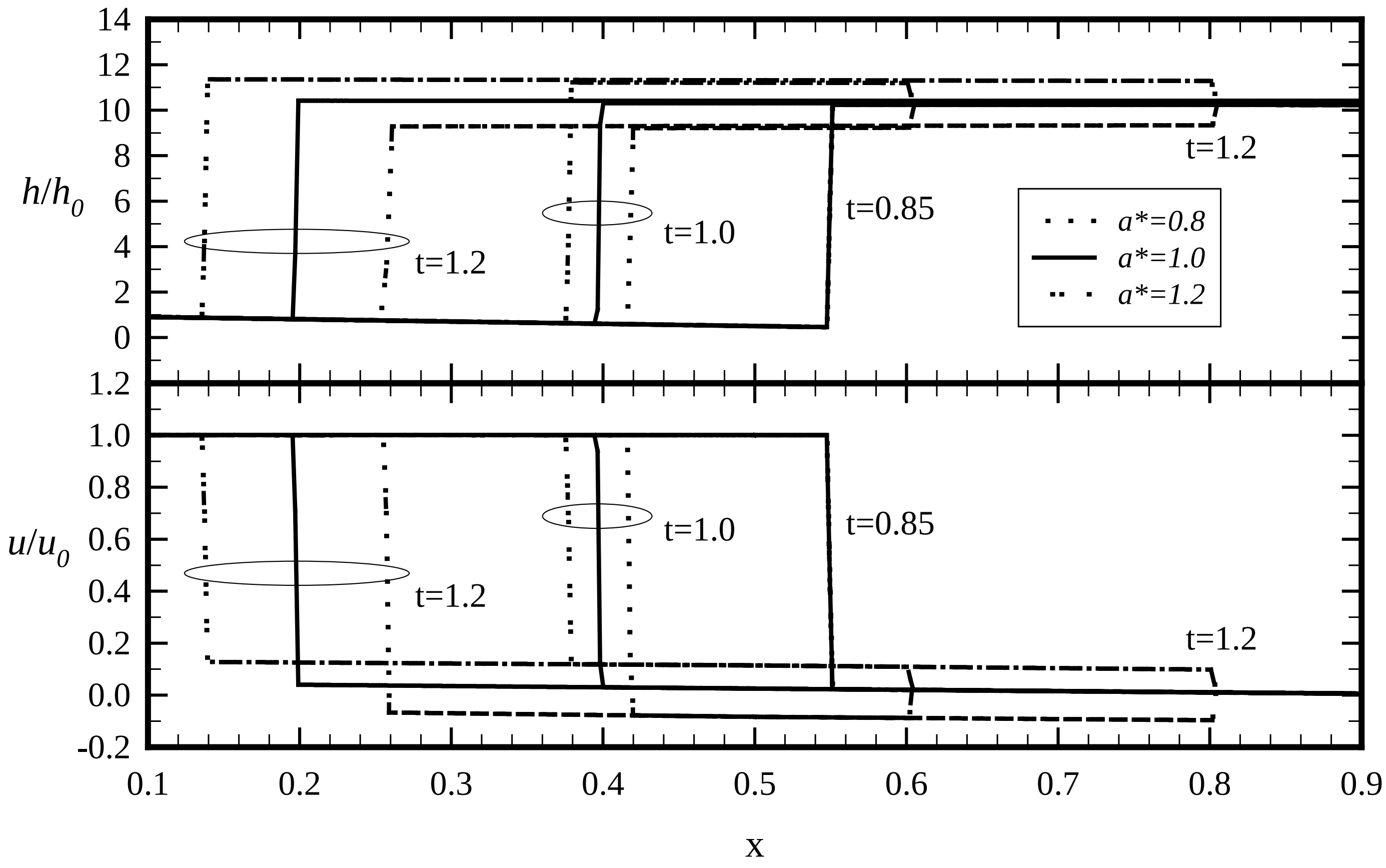


Sound speed variations (2)

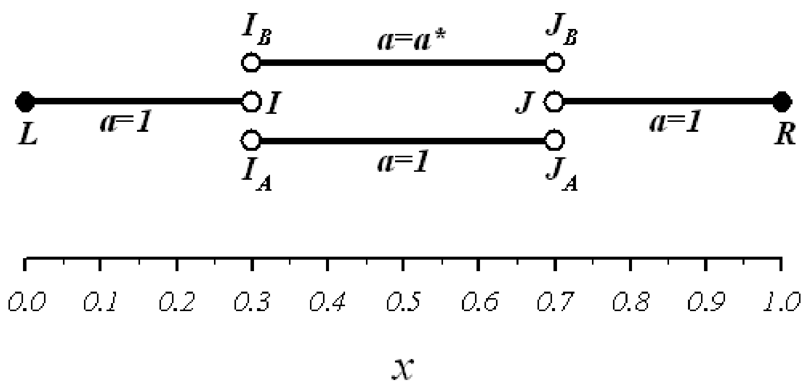




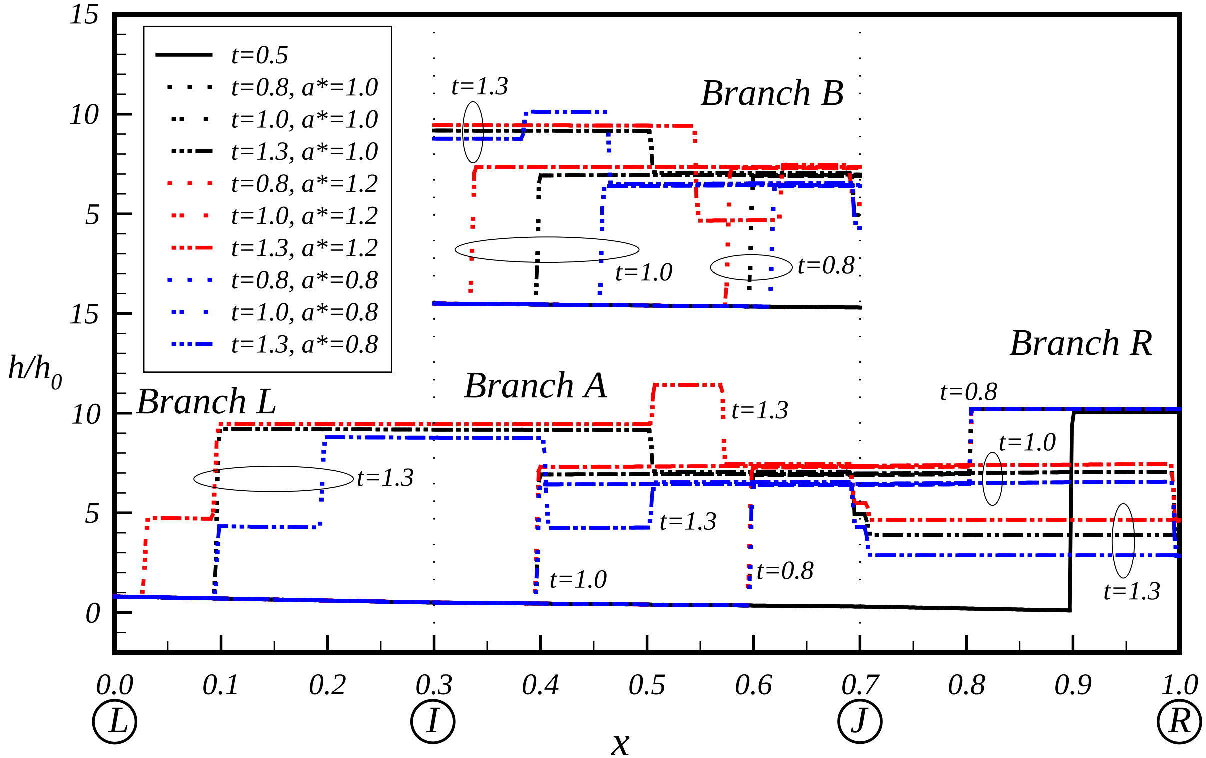
Sound speed variations (3)



Branch

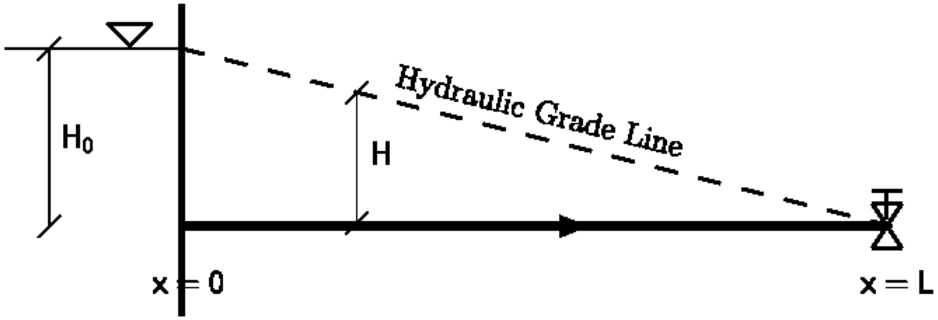


I_B
$a=a^*$
J_B
L
$a=1$
I
J
$a=1$
R
I_A
$a=1$
J_A
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
x



t=0.5
t=0.8, a*=1.0
t=1.0, a*=1.0
t=1.3, a*=1.0
t=0.8, a*=1.2
t=1.0, a*=1.2
t=1.3, a*=1.2
t=0.8, a*=0.8
t=1.0, a*=0.8
t=1.3, a*=0.8
Branch B
Branch A
Branch L
Branch R
t=1.3
t=1.0
t=0.8
h/h_0
15
10
5
0
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
L
I
J
R
x

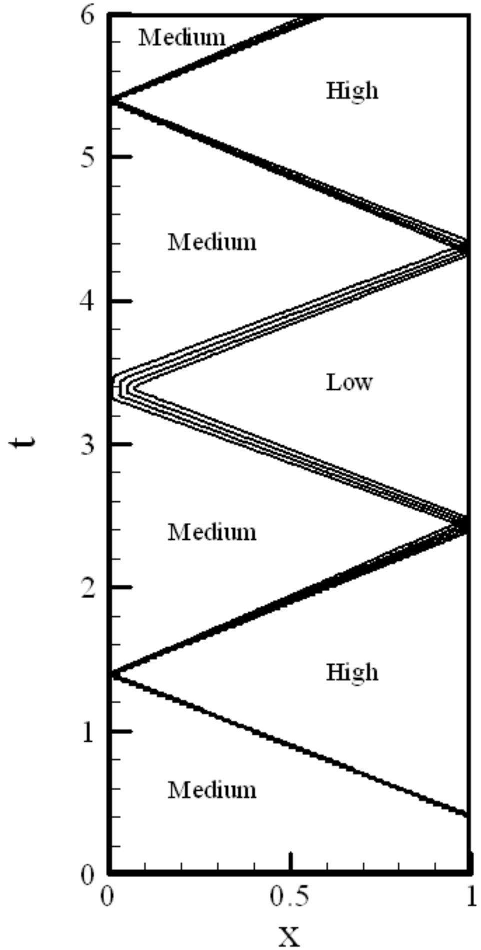
Sudden Valve Closure (1)

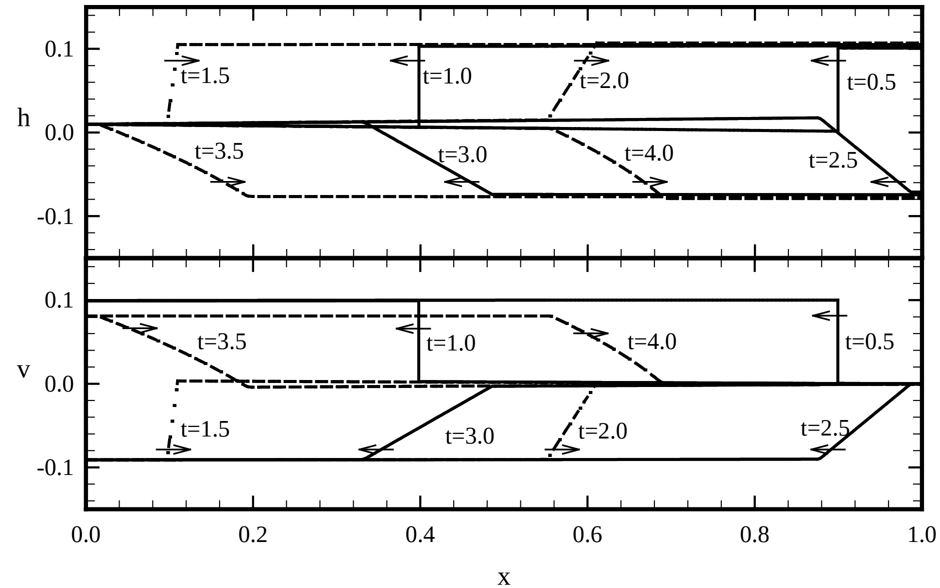


IC: $h(x,0) = h_0(1-x)$ $v(x,0) = v_0$

BC: $h(0,t) = h_0$ $v(1,t) = \begin{cases} v_0 & t < t_0 \\ 0 & t \geq t_0 \end{cases}$

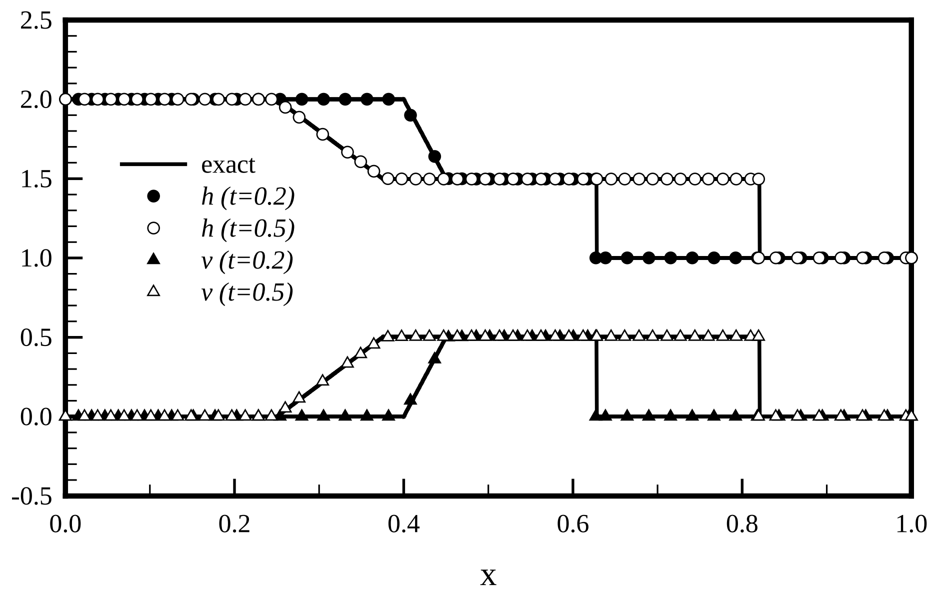
Sudden Valve Closure (2)





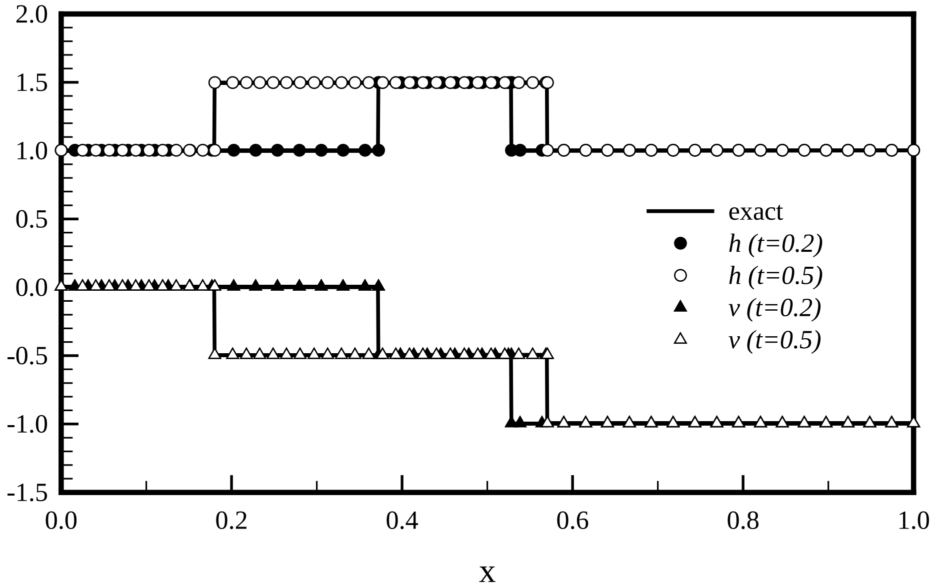
$$v_0 = 10^{-1}$$

Shock Tube Problem (1)



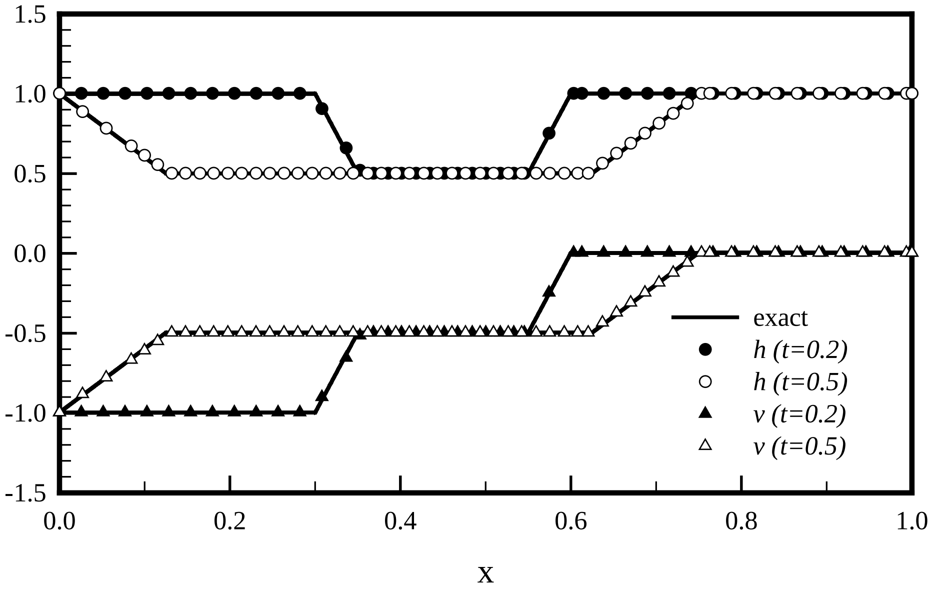
$$(h_L, v_L) = (2, 0) \qquad (h_R, v_R) = (1, 0)$$

Shock Tube Problem (2)



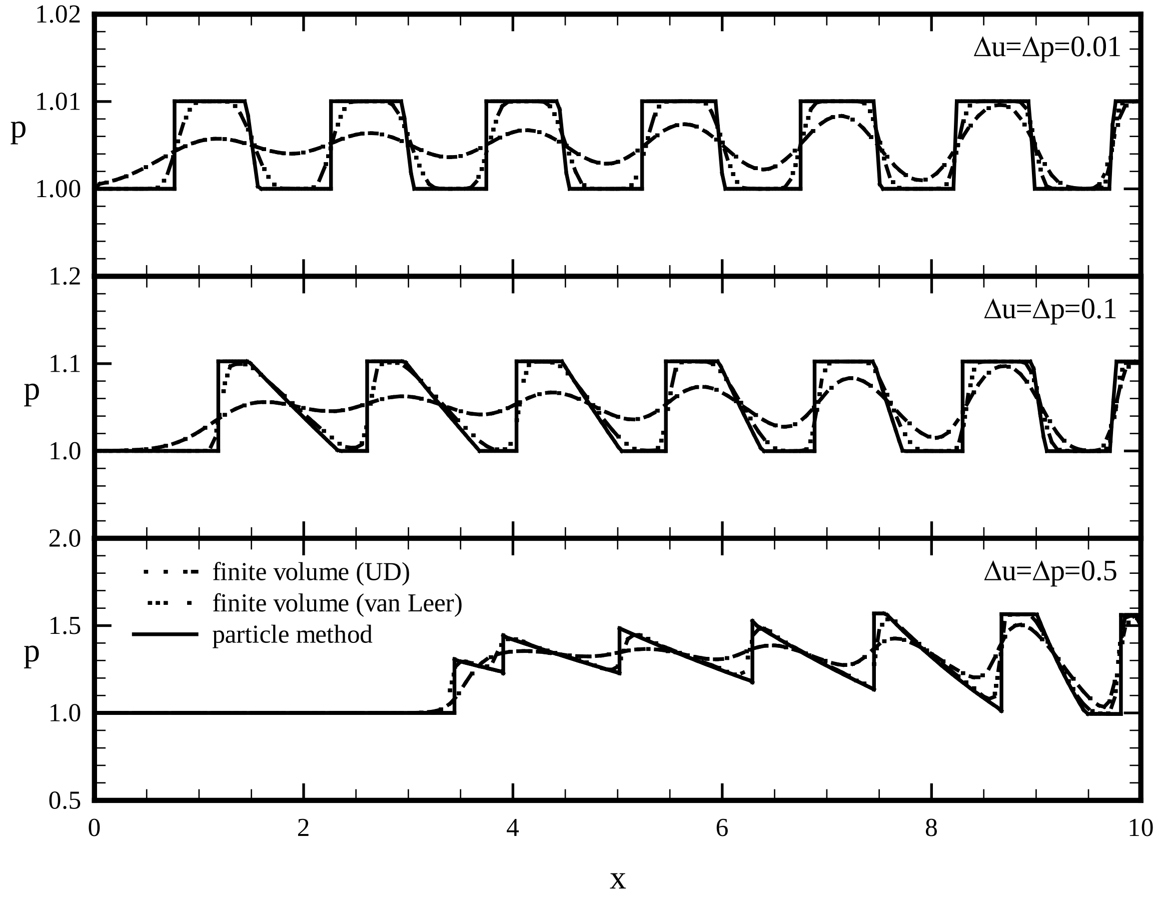
$$(h_L, v_L) = (1, 0) \qquad (h_R, v_R) = (1, -1)$$

Shock Tube Problem (3)

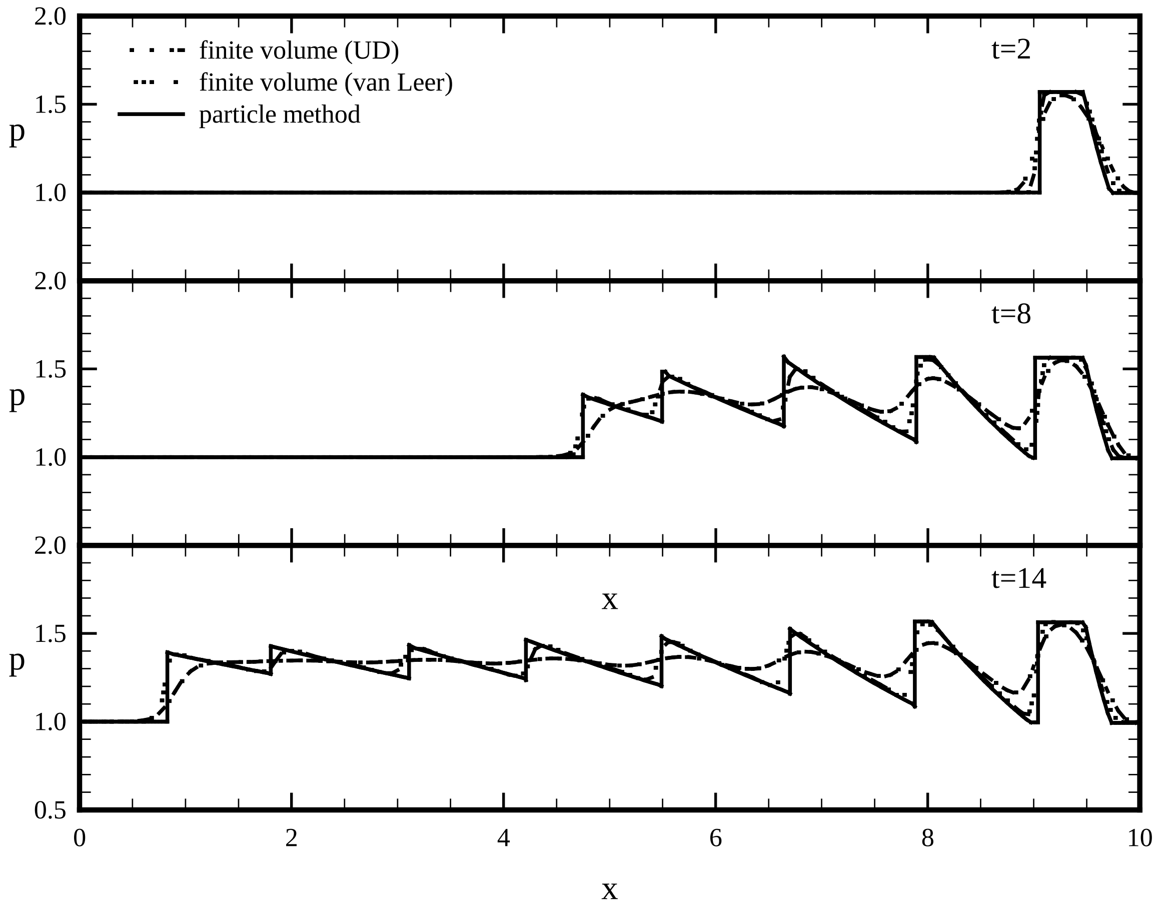


$$(h_L, v_L) = (1, -1) \qquad (h_R, v_R) = (1, 0)$$

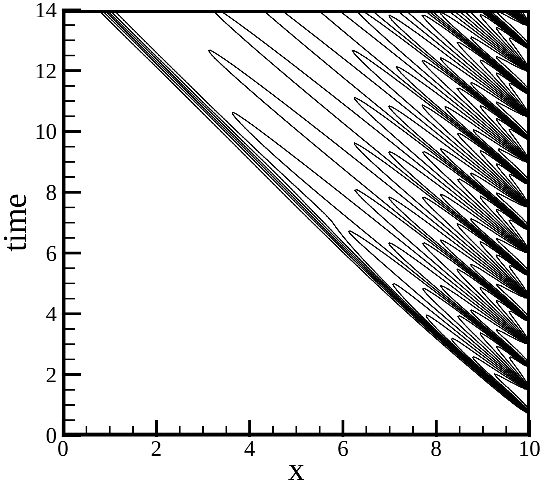
Valve Action (1)



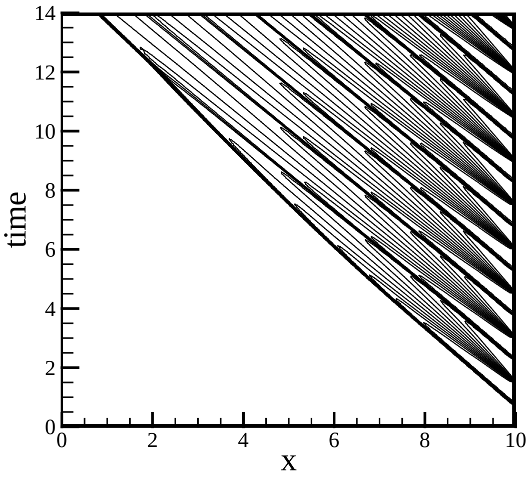
Valve Action (2)



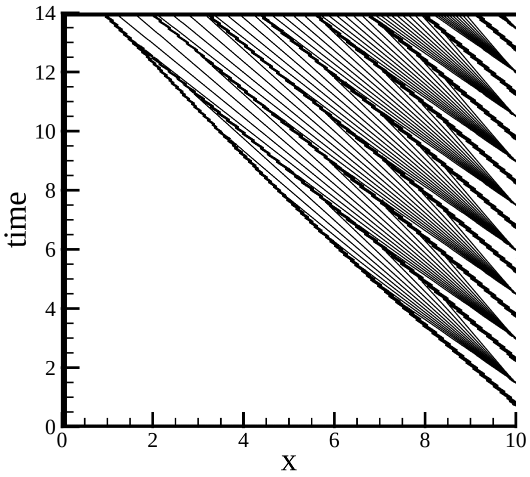
Valve Action (3)



UD

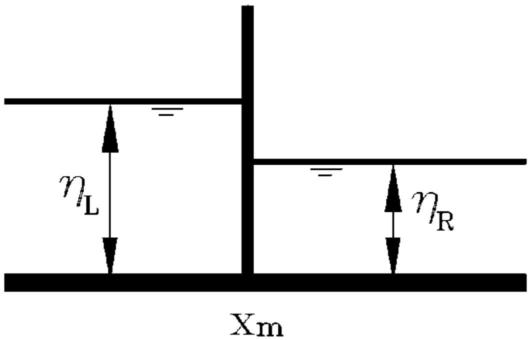


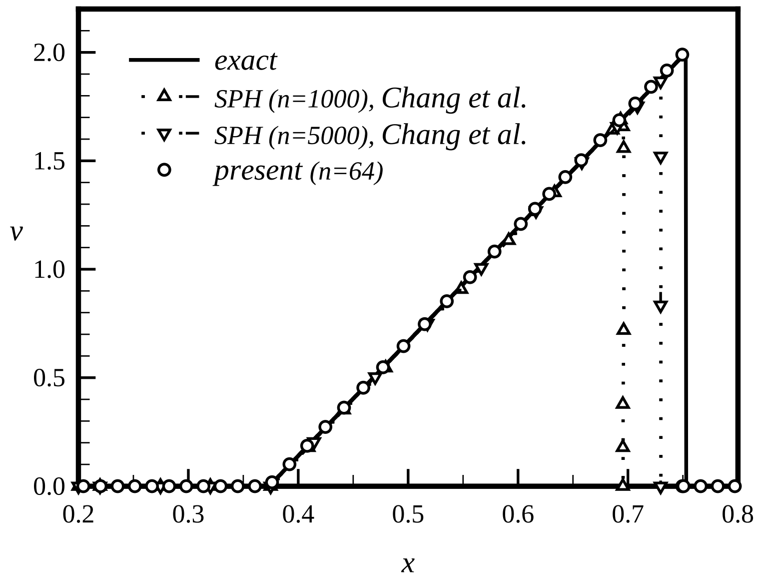
Van Leer

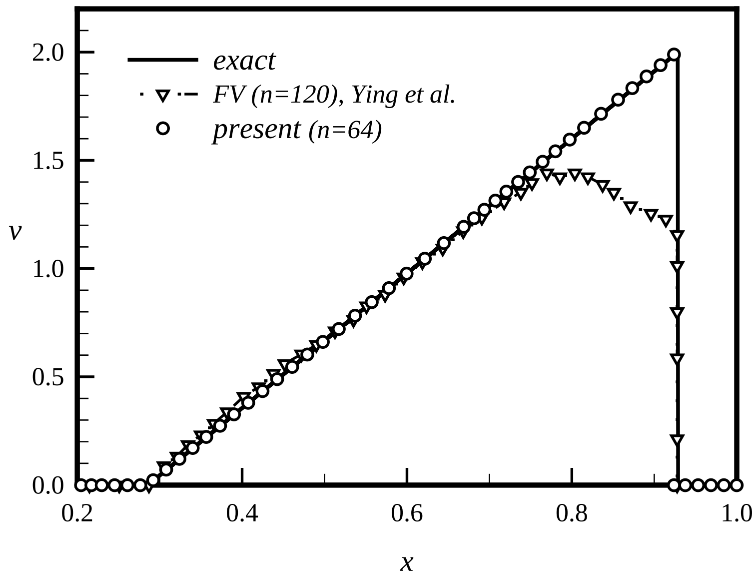


Present

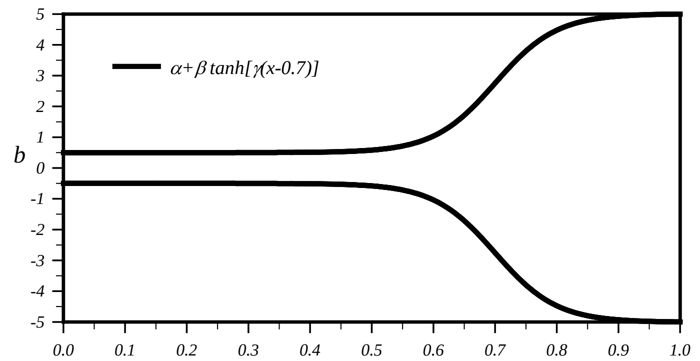
Open Channel

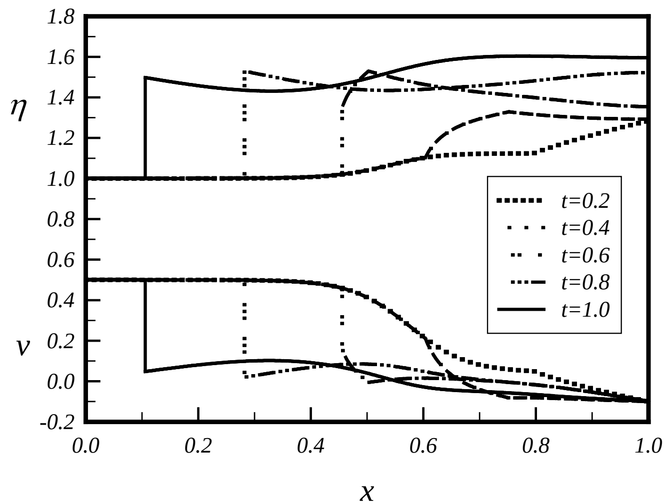


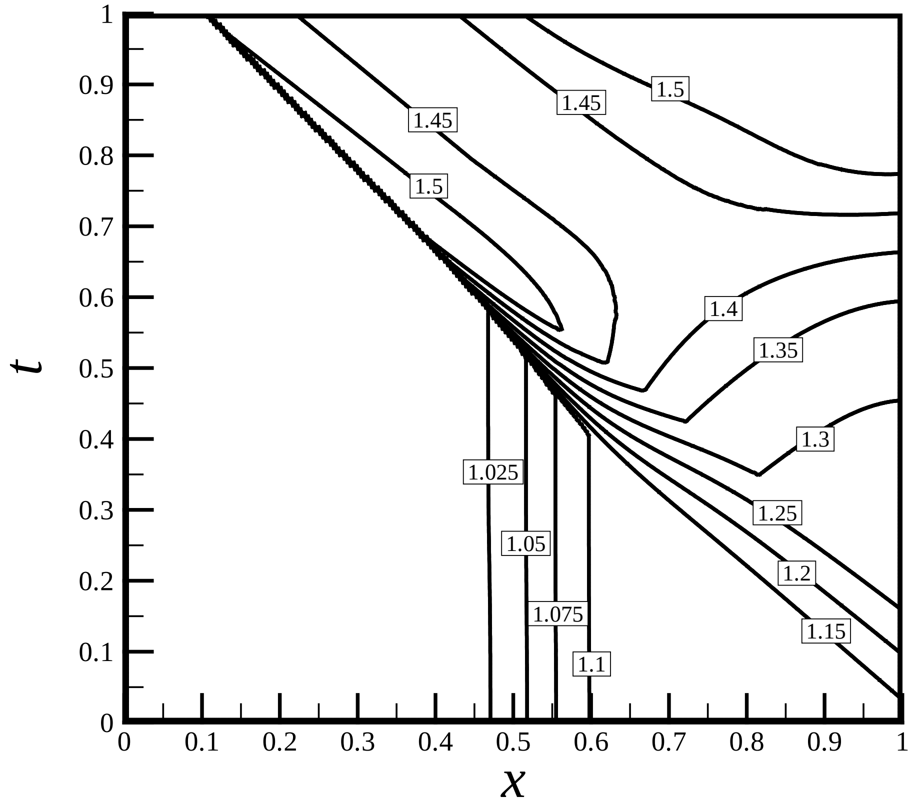




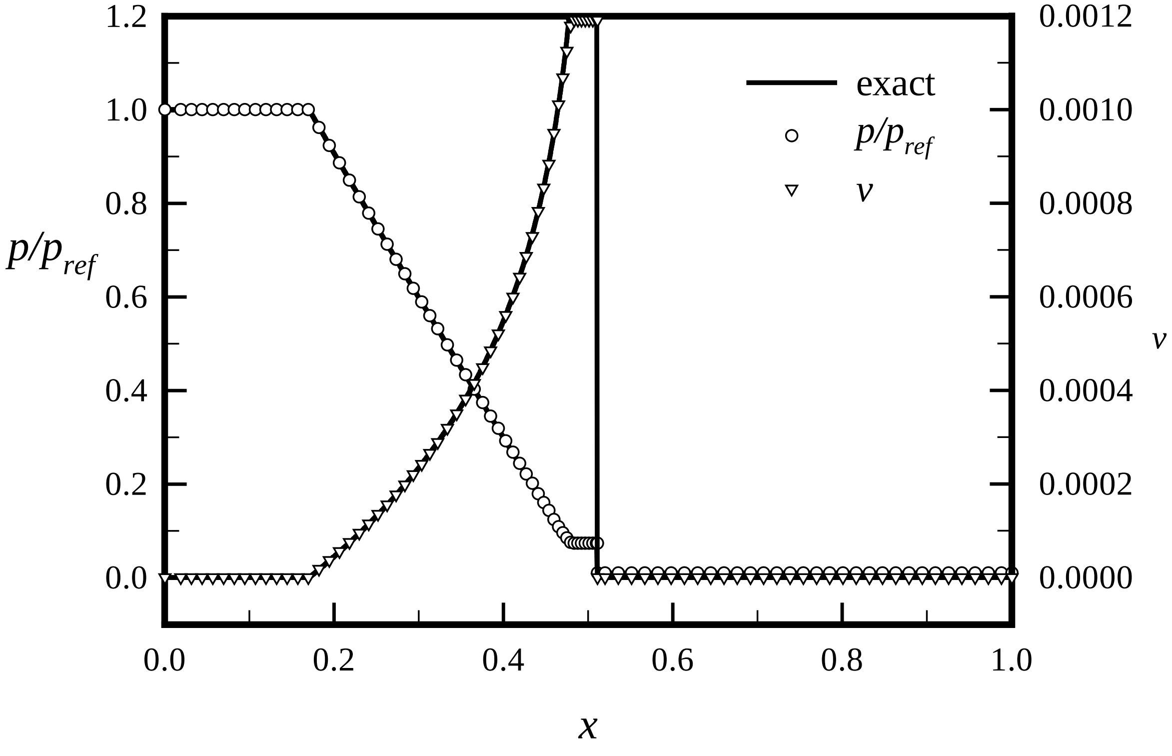
Tidal bore


α+β tanh[γ(x-0.7)]
b
5
4
3
2
1
0
-1
-2
-3
-4
-5
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0


η
v
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
0.0
0.2
0.4
0.6
0.8
1.0
x
t=0.2
t=0.4
t=0.6
t=0.8
t=1.0

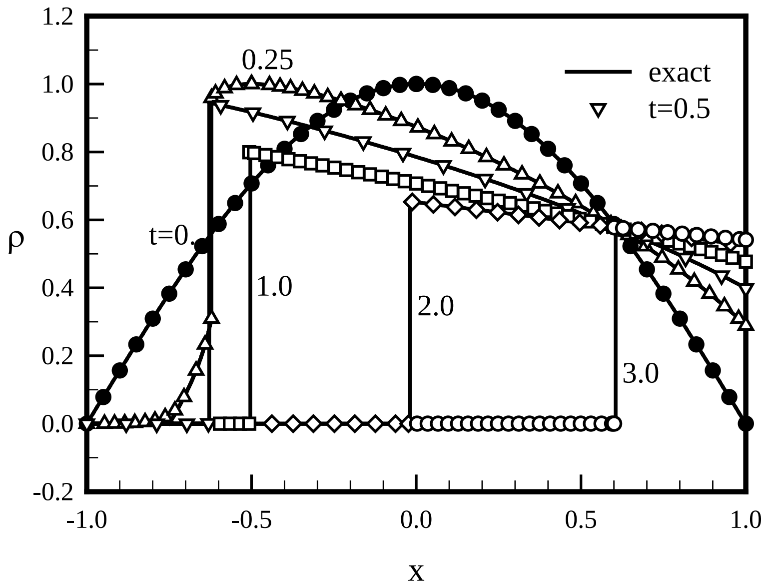

t
x
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.5
1.45
1.45
1.5
1.4
1.35
1.3
1.25
1.2
1.15
1.025
1.05
1.075
1.1

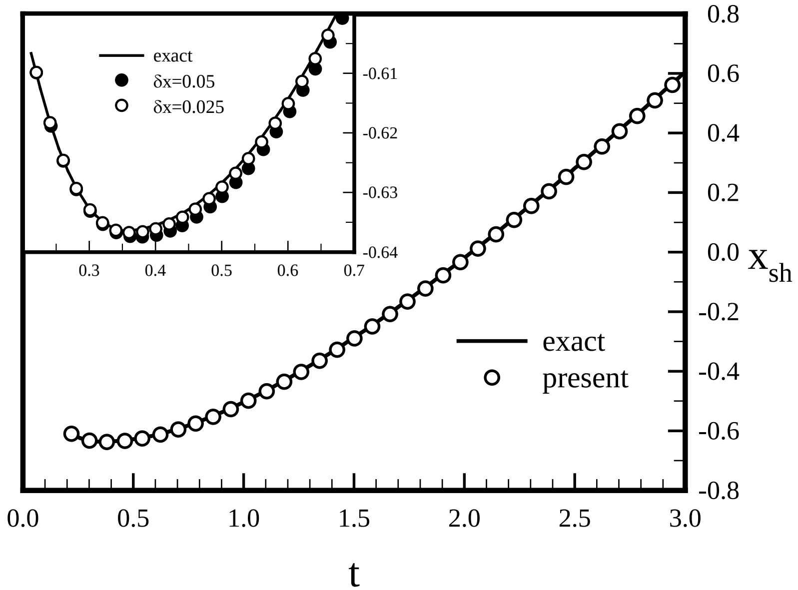
Two-phase flow



$$p_L / p_R = 100$$

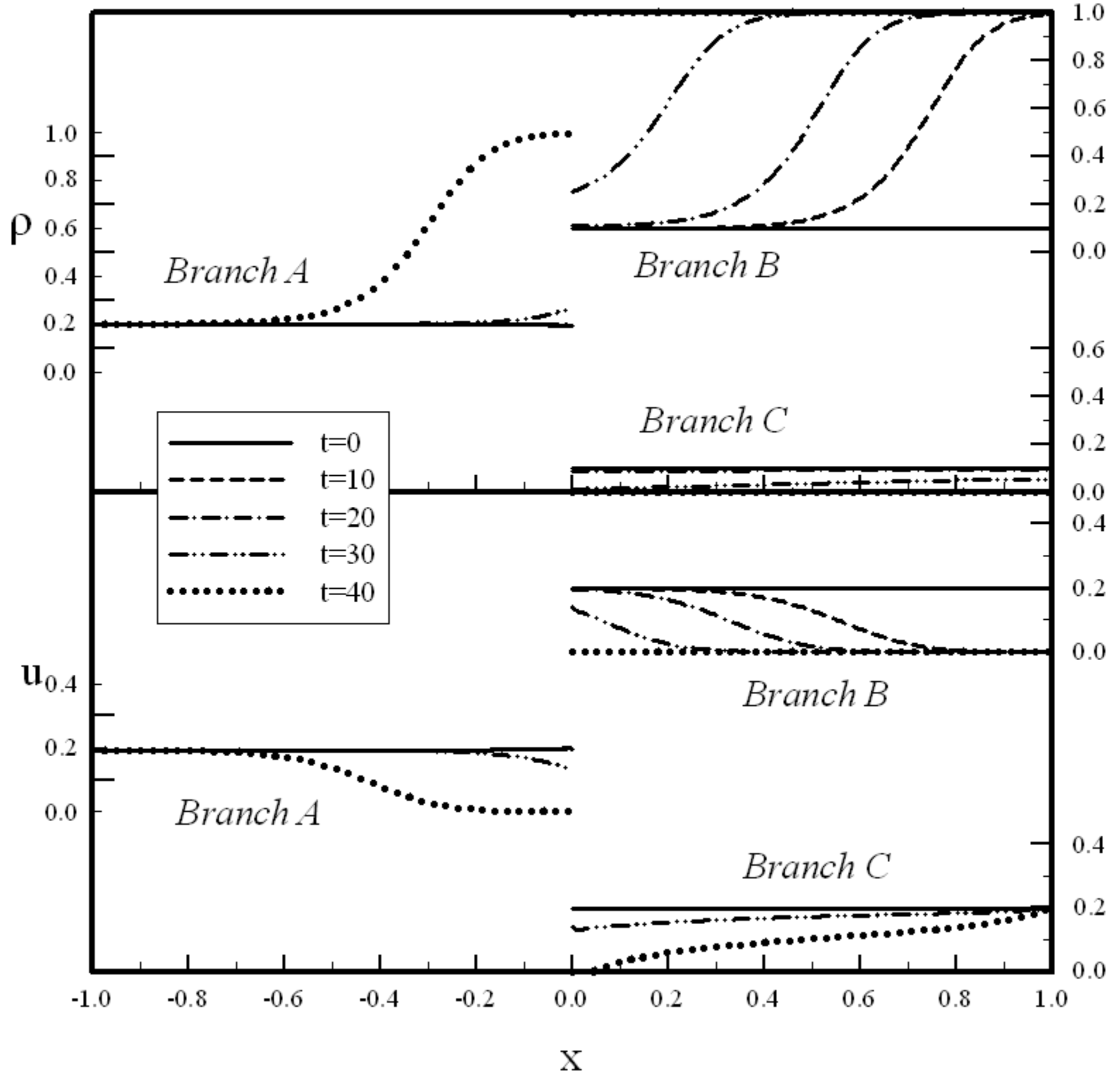
Traffic flow (LWR model)



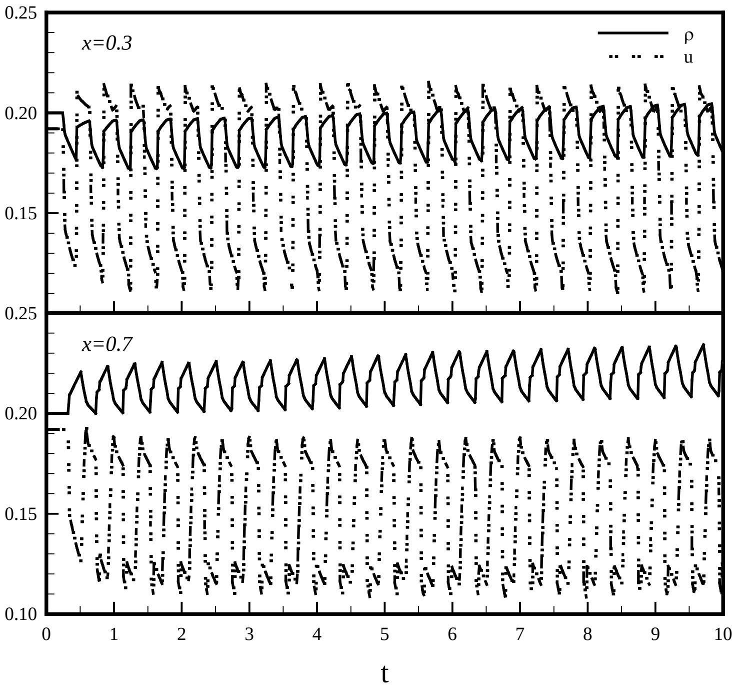


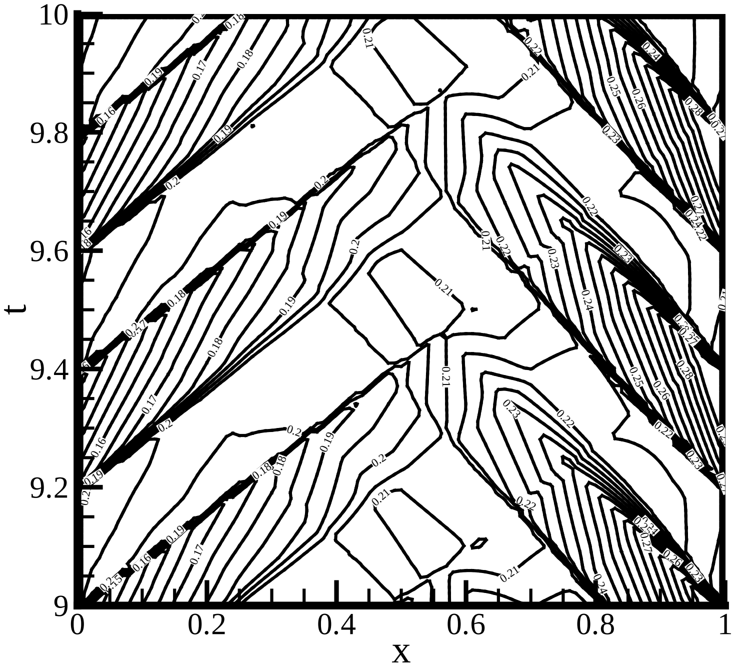
$$\rho(x,0) = \sin[(x - x_L)\pi / (x_R - x_L)]$$

Traffic flow (Payne model): Branch

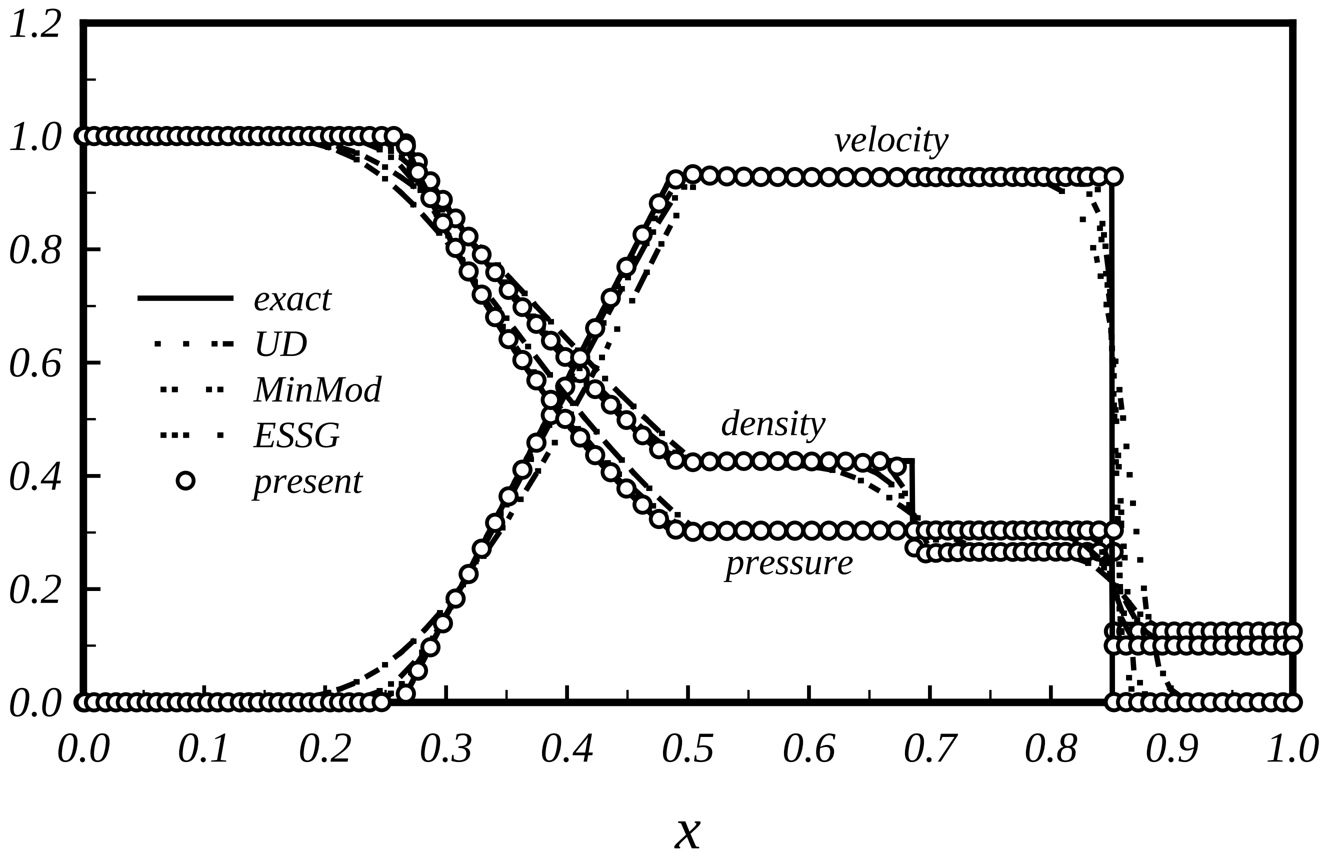


Traffic flow (Payne model): Red-Green





Gas dynamics



Gas dynamics