Introduction of Discontinuous Galerkin Methods

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14-5-3

Conservation laws

- Hyperbolic Conservation Laws:

$$\begin{cases} u_t + \nabla \cdot f(u) = 0 & x \in R^d, t > 0, \\ u(x,0) = u_0(x) & x \in R^d, \end{cases}$$

- Hyperbolic conservation laws and convection dominated PDEs play an important role arise in applications
  - gas dynamics
  - modelling of shallow waters
  - weather-forecasting
  - magneto-hydrodynamics
  - semiconductor device simulation

Conservation laws

- Gas dynamics

$$u = (\rho, \rho v, E)^T,\ f(u) = (\rho v, \rho v^2 + p, v(E+p))^T$$

- Modelling of shallow waters

$$u = (h, hv)^T,\ f(u) = (hv, hv^2 + gh^2/2)^T$$

- Burgers' equation: $\begin{cases} u_t + (u^2/2)_x = 0 & x \in R, t > 0, \\ u(x,0) = 0.5 + \sin(\pi x) & x \in R, \end{cases}$

Exact solution

Conservation laws

- Monotone Schemes, the first order schemes
  - Godunov scheme
  - Lax-Friedrichs scheme
  - EO(Engquist-Osher) scheme
  - Up-Wind scheme
- TVD (Total-Variation-Diminishingschemes
- ENO (Essentially Non-Oscillatory) scheme
- WENO (Weighted Essentially Non-Oscillatory) scheme
- Discontinuous Galerkin finite element methods

What is DG Method?

Discontinuous Galerkin (DG) method is a high order finite element method. DG schemes are very important numerical methods for convection dominated PDEs.

We consider conservation law: $u_t + f(u)_x = 0, \quad u(x,0) = u_0(x).$

Multiplying with a test function v, integrate over a cell $I_i = [x_{i-1/2}, x_{i+1/2}]$, and integrate by parts:

$$\int_{I_i} u_t v dx - \int_{I_i} f(u)v' dx + f(u_{i+1/2})v_{i+1/2} - f(u_{i-1/2})v_{i-1/2} = 0$$

What is DG Method?

Now assume both the solution u and the test function v come from a finite dimensional approximation space V_h, which is usually taken as the space of piecewise polynomials of degree up to k:

$$V_h^k = \{p : p|_{I_i} \in P^k(I_i), i = 1, \cdots, N\}$$

However, the boundary terms $f(u_{i+1/2})$ and $v_{i+1/2}$ etc. are not well defined when u and v are in this space, as they are discontinuous at the cell interfaces.

What is DG Method?

From the conservation and stability (upwinding) considerations, we take

- A single valued monotone numerical flux to replace $f(u_{i+1/2})$:

$$\hat{f}_{i+1/2} = \hat{f}(u^-_{i+1/2}, u^+_{i+1/2})$$

where $\hat{f}(u;u) = f(u)$ (consistency); $\hat{f}(\uparrow,\downarrow)$ (monotonicity) and $\hat{f}$ is Lipschitz continuous with respect to both arguments.

- Values from inside I_i for the test function v

$$v^-_{i+1/2},\ \ v^+_{i-1/2}$$

What is DG Method?

Hence the DG scheme is: find $u \in V_h$ such that

$$\int_{I_i} u_t v dx - \int_{I_i} f(u) v' dx + \hat{f}_{i+1/2} v^-_{i+1/2} - \hat{f}_{i-1/2} v^+_{i-1/2} = 0 \qquad (1)$$

for all $v \in V_h$.

What is DG Method?

- A local orthogonal basis over I_i:

$$v_0^{(i)}(x) = 1, \qquad v_1^{(i)}(x) = \frac{x - x_i}{\Delta x_i/2}, \qquad v_2^{(i)}(x) = \left(\frac{x - x_i}{\Delta x_i/2}\right)^2 - \frac{1}{3}, \cdots$$

- The numerical solution $u^h(x,t)$:

$$u^h(x,t) = \sum_{l=0}^{k} u_i^{(l)}(t) v_l^{(i)}(x), \qquad \text{for } x \in I_i$$

- The degrees of freedom $u_i^{(l)}(t)$ are the moments defined by

$$u_i^{(l)}(t) = \frac{1}{a_l} \int_{I_i} u^h(x,t) v_l^{(i)}(x) dx, \qquad l = 0, 1, \cdots, k$$

where $a_l = \int_{I_i} (v_l^{(i)}(x))^2 dx$.

What is DG Method?

- Evolve the moments $u_i^{(l)}$, resulting in a semi-discretization scheme as:

$$\frac{d}{dt}u_i^{(l)} + \frac{1}{a_l}\left(-\int_{I_i} f(u^h(x,t))\frac{d}{dx}v_l^{(i)}(x)dx + \hat{f}(u^-_{i+1/2}, u^+_{i+1/2})v_l^{(i)}(x_{i+1/2})\right.$$
$$\left.-\hat{f}(u^-_{i-1/2}, u^+_{i-1/2})v_l^{(i)}(x_{i-1/2})\right) = 0, \qquad l = 0, 1, \cdots, k. \tag{2}$$

Time discretization

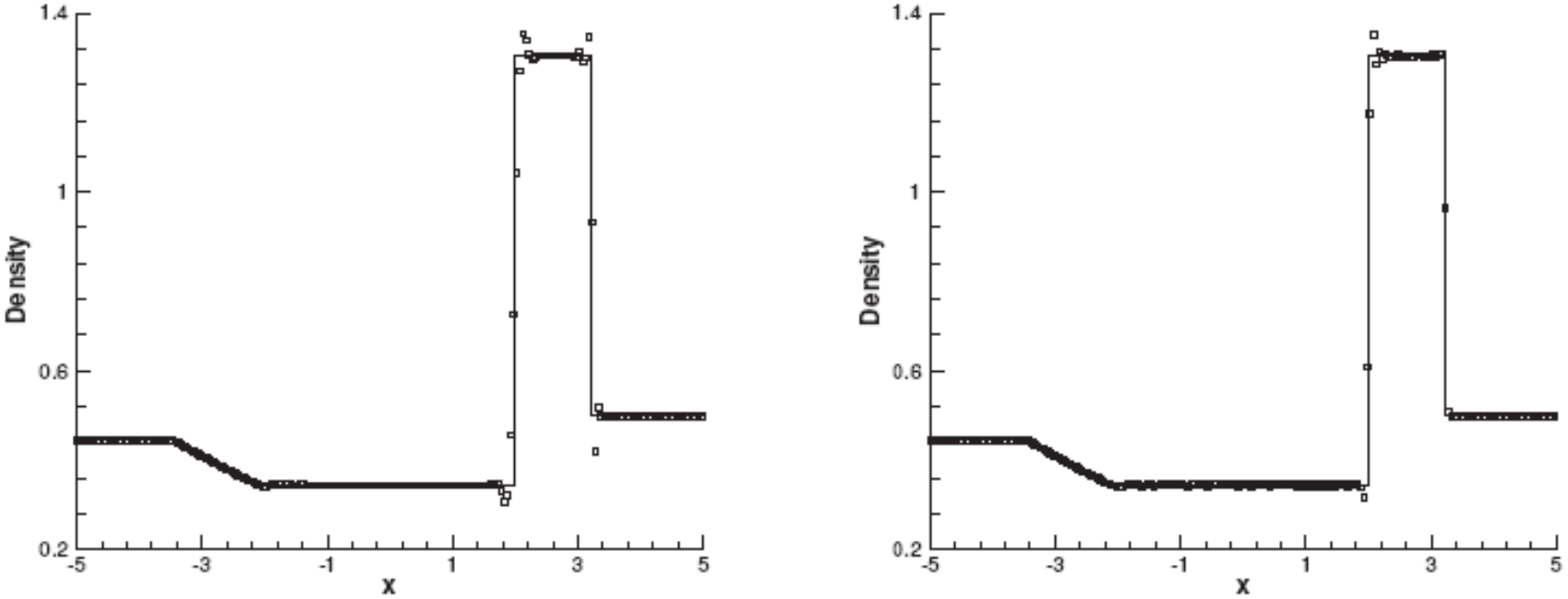
Using explicit, nonlinearly stable high order Runge-Kutta time discretizations .[Shu and Osher,JCP,1988]

The semidiscrete scheme (2) is written as:

$$u_t = L(u)$$

is discretized in time by a nonlinearly stable Runge-Kutta time discretization, e.g. the third order version.

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n) \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L(u^{(1)}) \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L(u^{(2)}). \end{aligned}$$



Lax problem. t=1.3. 200 cells. Density.
Left: k=1. Right: k=2.
For k=3 and k=4, the codes blow up

The Limiters

- The minmod based TVB limiter.(Cockburn and Shu, Math. Comp. 1989)
- Moment limiter. (Biswas, Devine and Flaherty, Appl. Numer. Math, 1994)
- A modification of moment limiter.(Burbean, Sagaut and Brunean, JCP, 2001)
- The monotonicity preserving (MP) limiter.(Suresh and Huynh,JCP,1997)
- A modification of the MP limiter. (Rider and Margolin, JCP, 2001)
- WENO limiter (Qiu and Shu, SIAM. J. Sci. Comput. 2005)
- HWENO limiter (Qiu and Shu, JCP, 2004, Computers & Fluids 2005)

Advantages of the DG method:

- Easy handling of complicated geometry and boundary conditions (common to all finite element methods). Allowing hanging nodes in the mesh;
- Compact. Communication only with immediate neighbors, regardless of the order of the scheme;
- Explicit. Because of the discontinuous basis, the mass matrix is local to the cell, resulting in explicit time stepping (no systems to solve);
- Parallel efficiency. Achieves 99% parallel efficiency for static mesh and over 80% parallel efficiency for dynamic load balancing with adaptive meshes (Flaherty et al.);

Advantages of the DG method:

- Provable cell entropy inequality and L_2 stability, for arbitrary scalar equations in any spatial dimension and any triangulation, for any order of accuracy, without limiters;

- At least $(k+1/2)$-th order accurate, and often $(k+1)$-th order accurate for smooth solutions when piecewise polynomials of degree k are used, regardless of the structure of the meshes.

- Easy $h-p$ adaptivity.

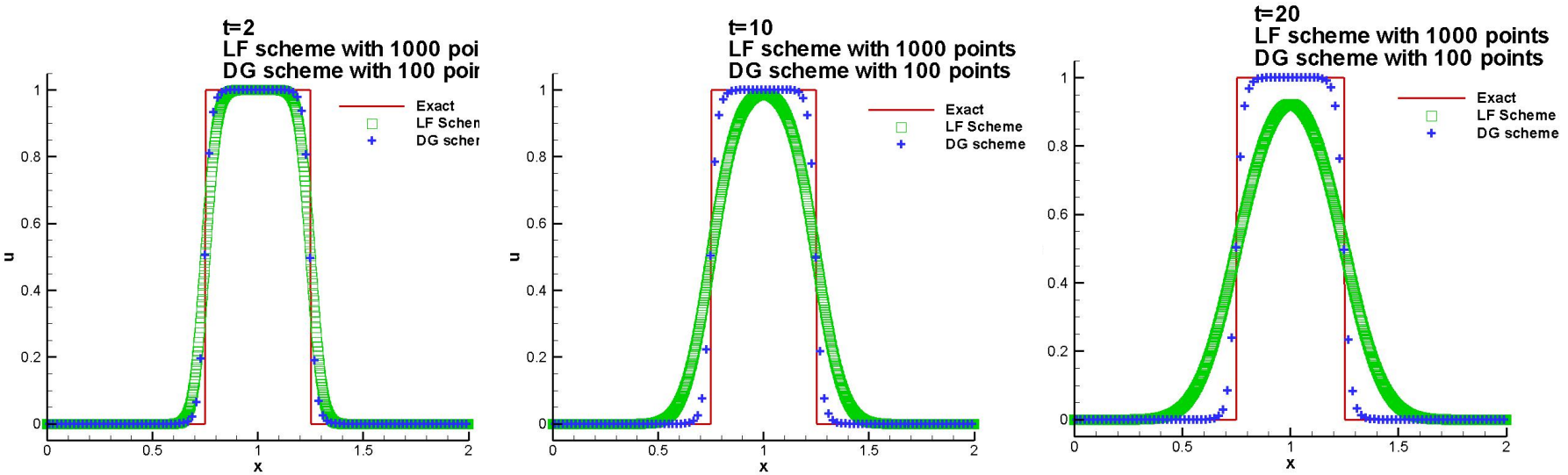
- Stable and convergent DG methods are now available for many nonlinear PDEs containing higher derivatives: convection diffusion equations, KdV equations, ...

A example of DG

Linear equation $u_t + u_x = 0$ with initial condition:

$$u(x,0) = \begin{cases} 1 & \frac{3}{4} < x \leq \frac{5}{4} \\ 0 & 0 < x \leq \frac{3}{4}, \text{ or } \frac{5}{4} < x \leq 2 \end{cases}$$

with periodic boundary condition. The exact solution is: $u(x,t) = u_0(x-t)$.



History of the DG method:

Here is a (very incomplete) history of the study of DG methods:

- 1973: First discontinuous Galerkin method for steady state linear scalar conservation laws (Reed and Hill).
- 1974: First error estimate (for tensor product mesh) of the discontinuous Galerkin method of Reed and Hill (LeSaint and Raviart).
- 1986: Error estimates for discontinuous Galerkin method of Reed and Hill (Johnson and Pitkaranta).
- 1989-1998: Runge-Kutta discontinuous Galerkin method for ***nonlinear*** conservation laws (Cockburn, Shu, ...).

History of the DG method:

- 1994: Proof of cell entropy inequality for discontinuous Galerkin method for nonlinear conservation laws in general multidimensional triangulations (Jiang and Shu).

- 1997-1998: Discontinuous Galerkin method for convection diffusion problems (Bassi and Rebay, Cockburn and Shu, Baumann and Oden, ...).

- 2002: Discontinuous Galerkin method for partial differential equations with third or higher order spatial derivatives (KdV, biharmonic, ...) (Yan and Shu, Xu and Shu, ...)

- 2007: Discontinuous Galerkin method for two-medium flow (Qiu, Zhu, Liu and Khoo)

Collected works on the DG methods:

- Discontinuous Galerkin Methods: Theory, Computation and Applications, B. Cockburn, G. Karniadakis and C.-W. Shu, editors, Lecture Notes in Computational Science and Engineering, volume 11, Springer, 2000. (Proceedings of the first DG Conference)

- Journal of Scientific Computing, special issue on DG methods, 2005，2009.

- Computer Methods in Applied Mechanics and Engineering, special issue on DG methods, 2006.

Cell Entropy inequality of DG

It is well known that weak solutions of

$$u_t + f(u)_x = 0 \tag{3}$$

may not be unique and the unique, physically relevant weak solution (the so-called entropy solution) satisfies the following entropy inequality:

$$U(u)_t + F(u)_x \leq 0 \tag{4}$$

in distribution sense, for any convex entropy $U(u)$ satisfying $U''(u) \geq 0$ and the corresponding entropy flux $F(u) = \int^u U'(u)f'(u)du$.

It will be nice if a numerical approximation to (3) also shares a similar entropy inequality as (4).

Cell Entropy inequality of DG

We recall the DG scheme is: find $u \in V_h$ such that

$$\int_{I_i} u_t v dx - \int_{I_i} f(u) v' dx + \hat{f}_{i+1/2} v^-_{i+1/2} - \hat{f}_{i-1/2} v^+_{i-1/2} = 0 \qquad (1)$$

for all $v \in V_h$. Let $v = u$ and $U(u) = u^2/2$, then we have:

$$\int_{I_i} U(u)_t dx - \int_{I_i} f(u) u' dx + \hat{f}_{i+1/2} u^-_{i+1/2} - \hat{f}_{i-1/2} u^+_{i-1/2} = 0 \qquad (5)$$

We denote $\tilde{F}(u) = \int^u f(u) du$, then the (5) can be written as:

$$\int_{I_i} U(u)_t dx - \tilde{F}(u^-_{i+1/2}) + \tilde{F}(u^+_{i-1/2}) + \hat{f}_{i+1/2} u^-_{i+1/2} - \hat{f}_{i-1/2} u^+_{i-1/2} = 0 \quad (6)$$

Cell Entropy inequality of DG

Let the numerical entropy flux $\hat{F}_{i+1/2} = -\tilde{F}(u^-_{i+1/2}) + \hat{f}_{i+1/2}u^-_{i+1/2}$, then

$$\int_{I_i} U(u)_t dx - \hat{F}_{i+1/2} + \hat{F}_{i-1/2} + \Theta_{i-1/2} = 0 \tag{7}$$

with

$$\Theta_{i-1/2} = -\tilde{F}(u^-_{i-1/2}) + \hat{f}_{i-1/2}u^-_{i-1/2} + \tilde{F}(u^+_{i-1/2}) + \hat{f}_{i-1/2}u^+_{i-1/2}$$

It is easy to verify $\hat{F}$ is consistent with the entropy flux $F(u) = \int^u U'(u)f'(u)du$ for $U(u) = u^2/2$.

For any monotone flux $\hat{f}_{i+1/2} = \hat{f}(u^-_{i+1/2}, u^+_{i+1/2})$, we have

$$\Theta_{i-1/2} = (\tilde{F}'(\xi) - \hat{f}_{i-1/2})(u^+_{i-1/2} - u^-_{i-1/2}) \geq 0$$

from a mean value theorem, ξ is a value between $u^+_{i-1/2}$ and $u^-_{i-1/2}$.

Cell Entropy inequality of DG

Proposition 1. The solution u to the semi-discrete DG scheme (1) satisfies the following cell entropy inequality

$$\int_{I_i} U(u)_t dx - \hat{F}_{i+1/2} + \hat{F}_{i-1/2} \leq 0 \tag{8}$$

for the square entropy $U(u) = u^2/2$, and some consistent entropy flux $\hat{F}_{i+1/2} = \hat{F}(u^-_{i+1/2}, u^+_{i+1/2})$ satisfying, $\hat{F}(u,u) = F(u)$.

Proposition 2. For periodic or compactly supported boundary conditions, the solution u to the semi-discrete DG scheme (1) satisfies the following L^2-stability:

$$\frac{d}{dt}\int_R (u)^2 dx \leq 0$$

or

$$||u(\cdot,t)||_2 \leq ||u(\cdot,0)||_2.$$

Local DG for convection-diffusion problem

We consider the one-dimensional convection-diffusion equation:

$$u_t + f(u)_x = (a(u)u_x)_x \tag{9}$$

with $a(u) \geq 0$. We rewrite this equation as the system:

$$u_t + f(u)_x = (b(u)q)_x, \;\; q - B(u)_x = 0 \tag{10}$$

where

$$b(u) = \sqrt{a(u)}, \;\; B(u) = \int^u b(u)du \tag{11}$$

Local DG for convection-diffusion problem

Hence the LDG scheme is: find $u, q \in V_h$ such that

$$\int_{I_i} u_t v dx - \int_{I_i} (f(u) - b(u)q)v' dx \tag{12}$$
$$+(\hat{f} - \hat{b}\hat{q})_{i+1/2} v^-_{i+1/2} - (\hat{f} - \hat{b}\hat{q})_{i-1/2} v^+_{i-1/2} = 0$$
$$\int_{I_i} qp dx + \int_{I_i} B(u)p' dx - \hat{B}_{i+1/2} p^-_{i+1/2} + \hat{B}_{i-1/2} p^+_{i-1/2} = 0 \tag{13}$$

for all $v, p \in V_h$.

Here, all the "hat" terms are the numerical fluxes, namely single-valued functions defined at the cell interfaces which typically depend on the discontinuous numerical solution from both sides of the interface.

Local DG for convection-diffusion problem

The convection flux $\hat{f}$ should be chosen as a monotone flux. However, the upwinding principle is no longer a valid guiding principle for the design of the diffusion fluxes $\hat{b}, \hat{q}$ and $\hat{B}$.

To guarantee the stability of the scheme (12)-(13), we will discuss a particularly attractive choice, called alternating fluxes, defined as:

$$\hat{b} = \frac{B(u^+) - B(u^-)}{u^+ - u^-}, \quad \hat{q} = q^+, \quad \hat{B} = B(u^-) \tag{14}$$

or

$$\hat{b} = \frac{B(u^+) - B(u^-)}{u^+ - u^-}, \quad \hat{q} = q^-, \quad \hat{B} = B(u^+) \tag{15}$$

The important point is that, $\hat{q}$ and $\hat{B}$ should be chosen from different directions.

Local DG for convection-diffusion problem

Similar to the case for hyperbolic conservation laws, we have the following "cell entropy inequality" for the LDG method (12)-(13).

Proposition 3. The solution u, q to the semi-discrete DG scheme (12)-(13) satisfies the following "cell entropy inequality":

$$\int_{I_i} ((u^2/2)_t + q^2)dx - \hat{F}_{i+1/2} + \hat{F}_{i-1/2} \leq 0 \tag{16}$$

for some consistent entropy flux

$$\hat{F}_{i+1/2} = \hat{F}(u^-_{i+1/2}, q^-_{i+1/2}, u^+_{i+1/2}, q^+_{i+1/2})$$

satisfying, $\hat{F}(u, q, u, q) = F(u) - ub(u)q$, where as before $F(u) = \int^u uf'(u)du$.

Local DG for convection-diffusion problem

Proposition 4. For periodic or compactly supported boundary conditions, the solution u, q to the semi-discrete DG scheme (12)-(13) satisfies the following L^2-stability:

$$\frac{d}{dt}\int_R (u)^2 dx + 2\int_R q^2 dx \leq 0$$

or

$$||u(\cdot,t)||_2 + 2\int_0^t ||q(\cdot,\tau)||_2 d\tau \leq ||u(\cdot,0)||_2.$$

DG with Non-polynomial basis

- The main objective: to study DG method when the approximation space V_h consists of non-polynomial functions.
- This is made possible because of the DG framework, which does not require any continuity at element interfaces.
- The motivation to use non-polynomial finite element spaces is to obtain better approximations for specific solutions of PDEs, such as the boundary layer solutions and oscillatory solutions, for which exponential / trigonometric functions instead of polynomials as basis functions for the new approximation spaces are expected to yield better numerical results.

DG with Non-polynomial basis

We consider the trigonometric polynomial space: $V_h^k = \{p : p|_{I_i} \in T^k(I_i)\}$, the trigonometric polynomial spaces of degree at most k on the cell I_i. We adopt the following local orthogonal basis over I_i, $\{v_l^{(i)}(x), l = 0, ..., k\}$:

$$v_0^{(i)}(x) = 1,$$

$$v_1^{(i)}(x) = \sin(\alpha(x - x_i)),$$

$$v_2^{(i)}(x) = \cos(\alpha(x - x_i)) - \frac{\sin(h\alpha/2)}{h\alpha/2},$$

$$\sin((\alpha + 1)(x - x_i)),$$

$$\cos((\alpha + 1)(x - x_i)) - \frac{\sin(h(\alpha + 1)/2)}{h(\alpha + 1)/2}$$

DG with Non-polynomial basis

Then, the numerical solution $u^h(x,t)$ in the space V_h^k can be written as:

$$u^h(x,t) = \sum_{l=0}^{k} u_i^{(l)}(t) v_l^{(i)}(x), \text{ for } x \in I_i$$

and due to the local orthogonal basis the degrees of freedom (the moments) $u_i^{(l)}(t)$ are defined by

$$u_i^{(l)}(t) = \frac{1}{\int_{I_i} (v_l^{(i)}(x))^2 dx} \int_{I_i} u^h(x,t) v_l^{(i)}(x) dx, \; l = 0, ..., k.$$

DG with Non-polynomial basis

In order to determine the approximate solution, we evolve the degrees of freedom $u_i^{(l)}(t)$ by using the following relation:

$$\begin{aligned}\frac{d}{dt}u_i^{(l)}(t) &= (\int_{I_i} f(u^h(x,t))\frac{d}{dx}v_l^{(i)}(x)dx - \hat{f}(u^-_{i+1/2},u^+_{i+1/2})v_l^{(i)}(x_{i+1/2}) \\ &\quad +\hat{f}(u^-_{i-1/2},u^+_{i-1/2})v_l^{(i)}(x_{i-1/2}))/a_l,\ \ l=0,...,k, \qquad (17)\end{aligned}$$

where $a_l = \int_{I_i}(v_l^{(i)}(x))^2dx$.

The semi-discrete scheme (17) is discretized in time by a nonlinearly stable Runge-Kutta time discretization, e.g. the third order TVD Runge-Kutta method.

DG with Non-polynomial basis

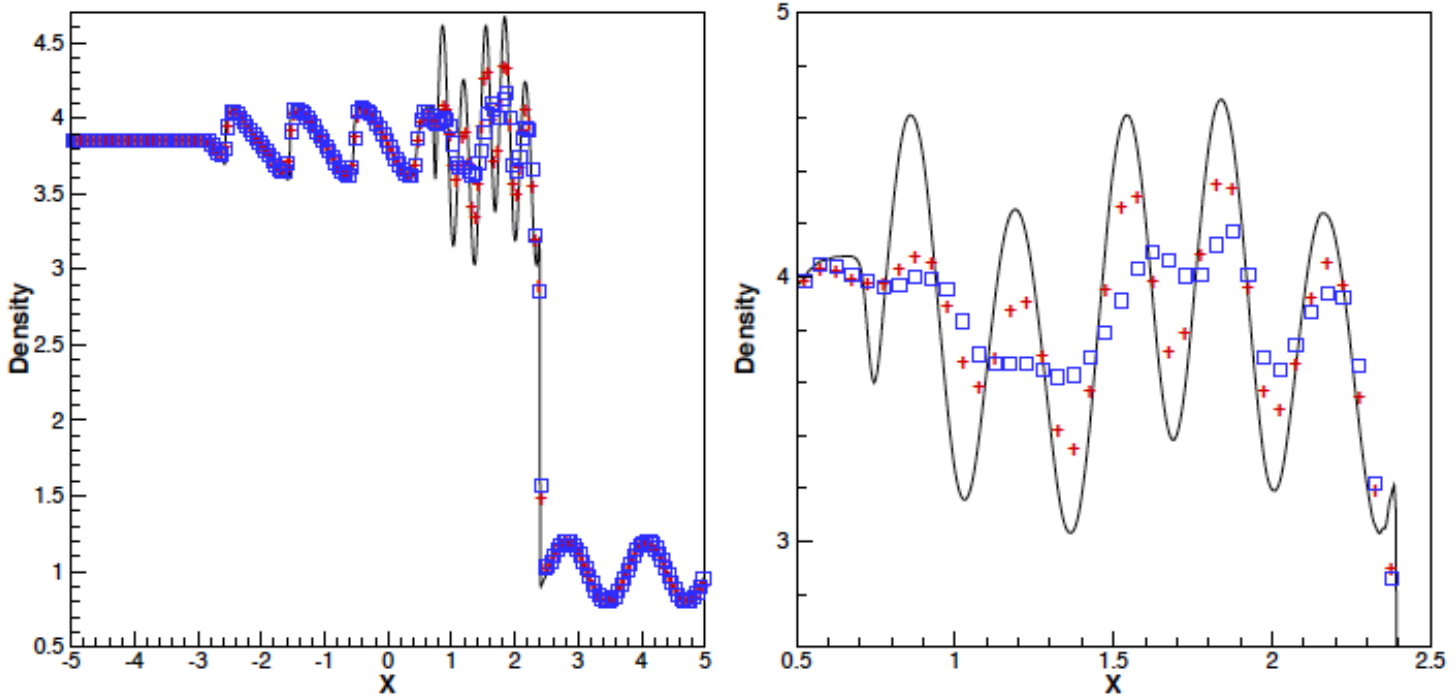
$u_t + (u^2/2)_x = 0$. $u(x,0) = 0.5 + \sin(\pi x)$. Periodic boundary conditions.

$t = 0.5/\pi$, L^1 and L^∞ errors and numerical orders of accuracy.

	RKDG-T, $\alpha = \pi$				RKDG-A			
cells	L^1 error	order	L^∞ error	order	L^1 error	order	L^∞ error	order
10	1.64E-3		2.49E-2		1.76E-3		2.92E-2	
20	1.94E-4	3.08	3.76E-3	2.73	2.07E-4	3.08	4.55E-3	2.68
40	2.51E-5	2.96	6.50E-4	2.53	2.67E-5	2.96	7.59E-4	2.58
80	3.21E-6	2.97	9.14E-5	2.83	3.38E-6	2.98	1.06E-4	2.84
160	4.08E-7	2.98	1.27E-5	2.85	4.27E-7	2.99	1.46E-5	2.86
320	5.15E-8	2.99	1.68E-6	2.92	5.37E-8	2.99	1.92E-6	2.93

2D Euler equations. $\rho(x,y,0) = 1 + 0.2(\sin(\pi x) + \sin(\pi y))$, $u(x,y,0) = 0.7$
$v(x,y,0) = 0.3$, $p(x,y,0) = 1$. Periodic boundary conditions.
$t = 1.0$. L^1 and L^∞ errors and numerical orders of accuracy.

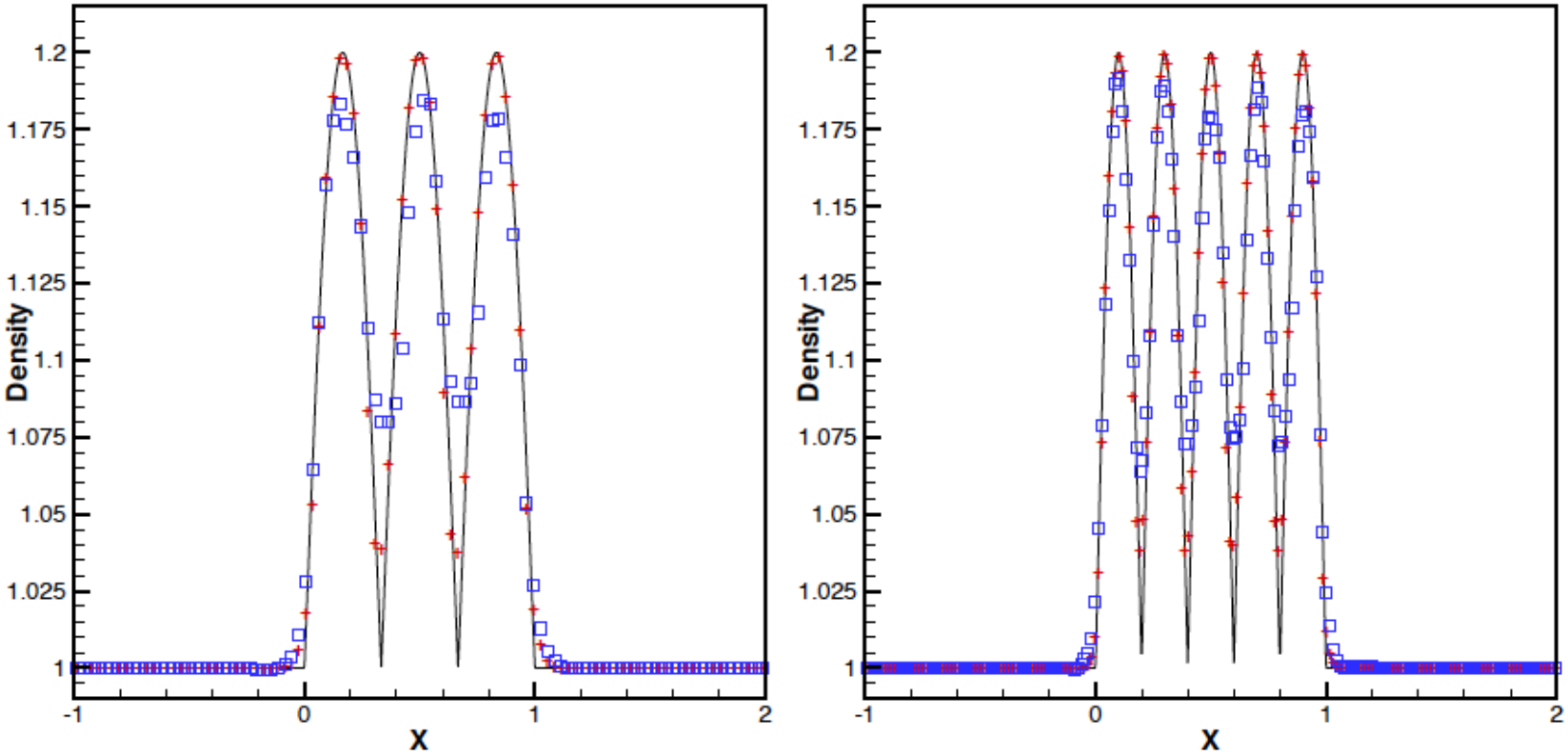
	RKDG-T, $\alpha = \pi$							
cells	L^1 error				L^∞ error			
10×10	7.89E-12				4.58E-11			
20×20	2.47E-14				1.62E-13			
40×40	7.75E-15				4.60E-14			
80×80	4.45E-14				2.12E-13			
160×160	1.69E-13				7.09E-13			
	RKDG-T, $\alpha = 1.0$				RKDG-A			
cells	L^1 error	order	L^∞ error	order	L^1 error	order	L^∞ error	order
10×10	3.33E-4		2.02E-3		3.71E-4		2.25E-3	
20×20	6.28E-5	2.41	3.84E-4	2.40	6.99E-5	2.41	4.27E-4	2.40
40×40	9.79E-6	2.68	5.91E-5	2.70	1.09E-5	2.68	6.57E-5	2.70
80×80	1.32E-6	2.88	7.95E-6	2.89	1.47E-6	2.88	8.84E-6	2.89
160×160	1.69E-7	2.97	1.01E-6	2.97	1.88E-7	2.97	1.12E-6	2.97



One dimensional Euler equations, with initial condtion:

$$(\rho, u, p)^T = \begin{cases} (3.857143, 2.629369, 10.333333)^T, & x < -4, \\ (1 + 0.2\sin(5x), 0, 1)^T, & x \geq -4. \end{cases}$$

Computed densities at $t = 1.8$ with 200 cells. RKDG-T with $\alpha = 1.0$ (plusses) and RKDG-A (squares) against the reference solution (solid line).



One dimensional Euler equations, with initial condtion:

$$(\rho, u, p)^T|_{x+0.5} = \begin{cases} (2 - x\sin(1.5\pi x^2), 1, 1)^T, & -1 < x < -\frac{1}{3}, \\ (2 + |\sin(2\pi x)|, 1, 1)^T, & |x| < \frac{1}{3}, \\ (2 + 2x - 1 - \sin(3\pi x)/6, 1, 1)^T, & \frac{1}{3} < x < 1. \end{cases}$$

Computed densities at $t = 6$ with 100 cells. RKDG-T with $\alpha = 1.0$ (plusses) and RKDG-A (squares) against the reference solution (solid line).

Lax-Wendroff DG

• The method relies on converting all the time derivatives in a temporal Taylor expansion into spatial derivatives by repeatedly using the PDE and its differentiated versions.

•The spatial derivatives are then discretized by the DG approximations.

• The nonlinear limiter for controlling spurious oscillations is performed once per time step. LWDG is more compact than RKDG.

• The Lax-Wendroff time discretization procedure is more cost effective than the Runge-Kutta time discretizationsfor certain problems including two dimensional Euler systems of compressible gas dynamics.

Lax-Wendroff DG

Consider conservation law:

$$u_t + f(u)_x = 0, \quad u(x,0) = u_0(x). \tag{18}$$

By a temporal Taylor expansion we obtain

$$u(x, t+\Delta t) = u(x,t) + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + \frac{\Delta t^3}{6} u_{ttt} + \dots. \tag{19}$$

in order to obtain $(k+1)$th order accuracy in time, the first $k+1$ time derivatives: $u_t, \dots, \frac{\partial^{(k+1)} u}{\partial t^{k+1}}$ shoulkd be approximated. We will proceed up to third order in time in here, although the procedure can be naturally extended to any higher orders.

Lax-Wendroff DG

The temporal derivative terms in (19) can be replaced with the spatial ones using the governing equation (18):

$$u_t = -f(u)_x = -f'(u)u_x,$$

$$u_{tt} = -(f'(u)u_t)_x = -f''(u)u_x u_t - f'(u)u_{xt},$$

$$u_{xt} = -f''(u)(u_x)^2 - f'(u)u_{xx},$$

$$u_{ttt} = -(f''(u)(u_t)^2 + f'(u)u_{tt})_x.$$

Then we can rewrite the approximation to (1) up to third order as:

$$u(x, t + \Delta t) = u(x, t) - \Delta t F_x. \tag{20}$$

with $F = f + \frac{\Delta t}{2} f'(u)u_t + \frac{\Delta t^2}{6}(f''(u)(u_t)^2 + f'(u)u_{tt})$. The standard discontinuous Galerkin method is then used to discretize F_x.

Lax-Wendroff DG

$$u_i^{(l)}(t^{n+1}) = u_i^{(l)}(t^n)$$
$$+\frac{1}{a_l}\left(-\int_{I_i} F\ \frac{d}{dx}v_l^{(i)}(x)dx + \hat{F}_{i+1/2}v_l^{(i)}(x_{i+1/2}) - \hat{F}_{i-1/2}v_l^{(i)}(x_{i-1/2})\right) = 0,$$
$$l = 0, 1, \cdots, k, \tag{21}$$

where $\hat{F}_{i+1/2}$ is a numerical flux which depends on the values of the numerical solution u^h and its spatial derivatives at the cell interface $x_{i+1/2}$, both from the left and from the right. We use the following simple Lax-Friedrichs flux

$$\hat{F}_{i+1/2} = \frac{1}{2}\left(F^-_{i+1/2} + F^+_{i+1/2} - \alpha(u^+_{i+1/2} - u^-_{i+1/2})\right),$$

where $u^\pm_{i+1/2}$ and $F^\pm_{i+1/2}$ are the left and right limits of the discontinuous solution u^h and the flux F at the cell interface $x_{i+1/2}$, and $\alpha = \max_u |f'(u)|$.

Lax-Wendroff DG

Consider the two dimensional conservation laws:

$$\begin{cases} u_t + f(u)_x + g(u)_y = 0, \\ u(x, y, 0) = u_0(x, y). \end{cases} \tag{22}$$

By a temporal Taylor expansion we obtain

$$u(x, y, t + \Delta t) = u(x, y, t) + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + \frac{\Delta t^3}{6} u_{ttt} + \dots.$$

Lax-Wendroff DG

For third order accuracy in time we would need to reconstruct three time derivatives: u_t, u_{tt}, u_{ttt}. We again use the PDE (22) to replace time derivatives by spatial derivatives.

$$u_t = -f(u)_x - g(u)_y = -f'(u)u_x - g'(u)u_y,$$

$$u_{tt} = -(f'(u)u_t)_x - (g'(u)u_t)_y = -(f''(u)u_x u_t + f'(u)u_{xt} + g''(u)u_y u_t + g'(u)u_{yt}),$$

$$u_{xt} = -(f''(u)(u_x)^2 + f'(u)u_{xx} + g''(u)u_x u_y + g'(u)u_{xy}),$$

$$u_{yt} = -(f''(u)u_y u_x + f'(u)u_{xy} + g''(u)(u_y)^2 + g'(u)u_{yy}),$$

$$u_{ttt} = -(f''(u)(u_t)^2 + f'(u)u_{tt})_x - (g''(u)(u_t)^2 + g'(u)u_{tt})_y.$$

Lax-Wendroff DG

Then we rewrite the approximation to (22) up to third order as:

$$u(x, t + \Delta t) = u(x, t) - \Delta t(F_x + G_y), \tag{23}$$

with

$$F = f + \frac{\Delta t}{2} f'(u)u_t + \frac{\Delta t^2}{6}(f''(u)(u_t)^2 + f'(u)u_{tt}),$$

$$G = g + \frac{\Delta t}{2} g'(u)u_t + \frac{\Delta t^2}{6}(g''(u)(u_t)^2 + g'(u)u_{tt}).$$

The standard discontinuous Galerkin method is then used to discretize F_x and G_y in (23).

Numerical Results

		LWDG				RKDG			
	N	L_1 error	order	L_∞ error	order	L_1 error	order	L_∞ error	order
P^1	10	2.72E-03		6.46E-03		2.32E-03		7.49E-03	
	20	6.25E-04	2.12	2.04E-03	1.67	4.90E-04	2.24	2.07E-03	1.85
	40	1.52E-04	2.04	5.58E-04	1.87	1.16E-04	2.08	5.46E-04	1.93
	80	3.75E-05	2.02	1.45E-04	1.94	2.85E-05	2.03	1.40E-04	1.97
	160	9.32E-06	2.01	3.69E-05	1.97	7.08E-06	2.01	3.53E-05	1.98
	320	2.33E-06	2.00	9.31E-06	1.99	1.76E-06	2.00	8.89E-06	1.99
P^2	10	6.11E-04		1.50E-03		2.78E-03		4.51E-03	
	20	5.48E-05	3.48	2.23E-04	2.75	1.05E-04	4.73	4.46E-04	3.34
	40	4.67E-06	3.55	2.06E-05	3.43	2.31E-05	2.18	4.51E-05	3.31
	80	4.69E-07	3.32	1.82E-06	3.51	3.27E-06	2.82	5.36E-06	3.07
	160	5.32E-08	3.14	2.28E-07	2.99	4.21E-07	2.96	6.97E-07	2.94
	320	6.45E-09	3.04	2.86E-08	3.00	5.30E-08	2.99	9.87E-08	2.82

Euler equations. $\rho(x,0) = 1+0.2\sin(\pi x)$, $v(x,0) = 1$, $p(x,0) = 1$. LWDG comparing with RKDG. Local Lax-Friedrichs flux, using N equally spaced cells. $t = 2$. L_1 and L_∞ errors of the density ρ.

Numerical Results

		LWDG				RKDG			
	$N \times N$	L_1 error	order	L_∞ error	order	L_1 error	order	L_∞ error	order
P^1	10 ×10	7.19E-02		5.37E-01		6.42E-02		6.66E-01	
	20×20	1.52E-02	2.25	1.99E-01	1.44	1.54E-02	2.06	2.47E-01	1.43
	40×40	3.66E-03	2.05	6.73E-02	1.56	3.04E-03	2.34	4.33E-02	2.51
	80×80	5.69E-04	2.69	9.13E-03	2.88	5.90E-04	2.37	9.16E-03	2.24
	160×160	1.32E-04	2.10	2.33E-03	1.97	1.42E-04	2.05	2.41E-03	1.93
P^2	10 ×10	2.99E-02		4.96E-01		2.98E-02		5.03E-01	
	20×20	1.66E-03	4.17	4.01E-02	3.63	1.81E-03	4.04	4.09E-02	3.62
	40×40	1.73E-04	3.26	5.82E-03	2.78	1.73E-04	3.38	6.04E-03	2.76
	80×80	2.09E-05	3.05	9.51E-04	2.61	2.07E-05	3.06	1.00E-03	2.59
	160×160	2.52E-06	3.05	1.30E-04	2.87	2.49E-06	3.06	1.38E-04	2.87

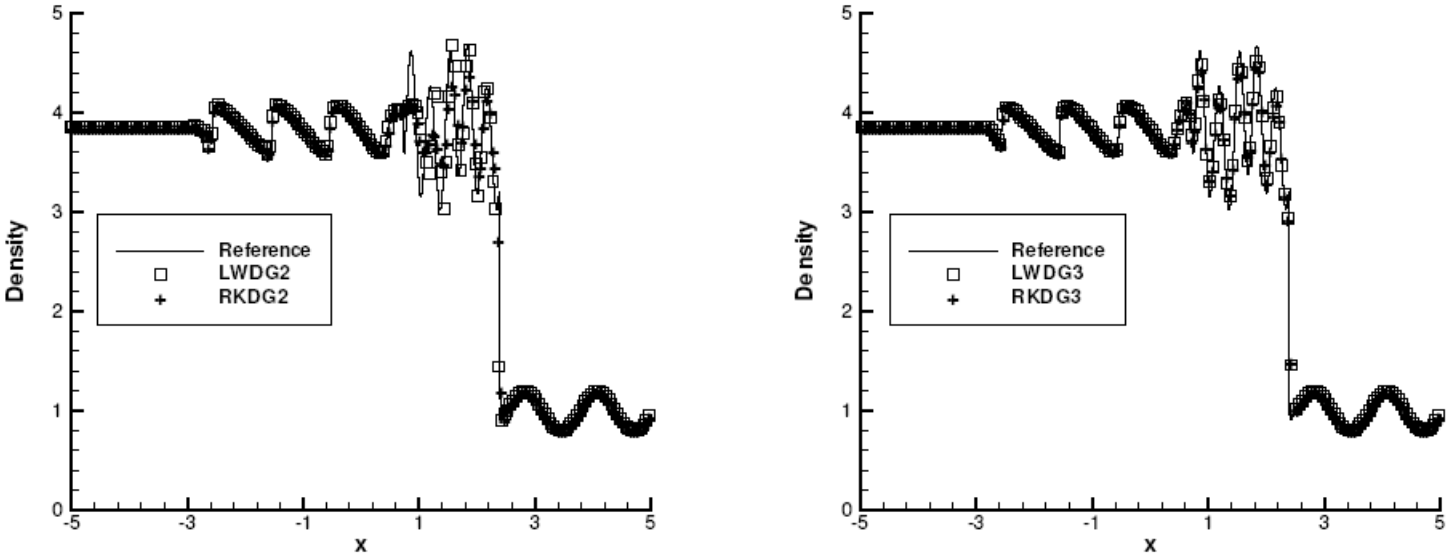
Burgers equation $u_t + (u^2/2)_x + (u^2/2)_y = 0$. Initial condition $u(x, y, 0) = 0.5 + \sin(\pi(x + y)/2)$ and periodic boundary conditions. LWDG comparing with RKDG. Local Lax-Friedrichs flux, $t = 0.5/\pi$. L_1 and L_∞ errors. Uniform meshes with $N \times N$ cells.

Numerical Results

		LWDG				RKDG			
	$N \times N$	L_1 error	order	L_∞ error	order	L_1 error	order	L_∞ error	order
P^1	10×10	2.59E-02		7.55E-02		3.48E-02		7.34E-02	
	20×20	8.76E-03	1.56	3.58E-02	1.08	6.89E-03	2.34	2.74E-02	1.42
	40×40	1.96E-03	2.16	1.06E-02	1.76	1.21E-03	2.51	7.36E-03	1.89
	80×80	1.65E-04	3.57	1.41E-03	2.91	2.33E-04	2.37	2.02E-03	1.87
	160×160	2.34E-05	2.82	2.87E-04	2.29	5.19E-05	2.17	6.45E-04	1.65
P^2	10×10	2.12E-03		7.42E-03		5.44E-03		1.39E-02	
	20×20	2.45E-04	3.12	9.98E-04	2.89	3.14E-04	4.11	1.22E-03	3.51
	40×40	2.44E-05	3.33	1.30E-04	2.94	2.66E-05	3.56	1.29E-04	3.24
	80×80	2.43E-06	3.33	1.72E-05	2.92	2.35E-06	3.50	1.71E-05	2.92
	160×160	3.16E-07	2.94	2.09E-06	3.04	2.19E-07	3.43	2.17E-06	2.97

Euler equations. Initial condition $\rho(x, y, 0) = 1 + 0.2\sin(\pi(x + y))$, $u(x, y, 0) = 0.7$, $v(x, y, 0) = 0.3$, $p(x, y, 0) = 1$ and periodic boundary conditions. LWDG comparing with RKDG. Local Lax-Friedrichs flux, $t = 2.0$. L_1 and L_∞ errors for the density ρ. Uniform meshes with $N \times N$ cells.

Numerical Results



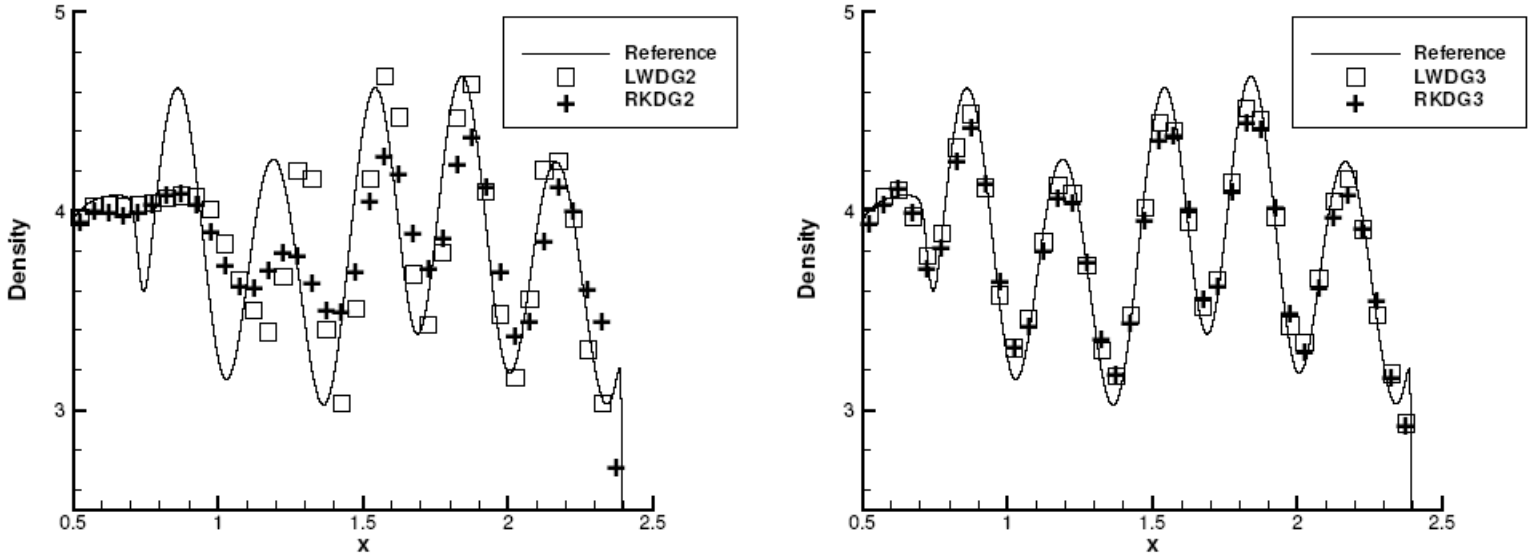
Euler equations. The problem of shock interaction with entropy waves, initial condition

$$(\rho, v, p) = (3.857143, 2.629369, 10.333333) \text{ for } x < -4;$$

$$(\rho, v, p) = (1 + \varepsilon \sin(5x), 0, 1) \text{ for } x \geq -4.$$

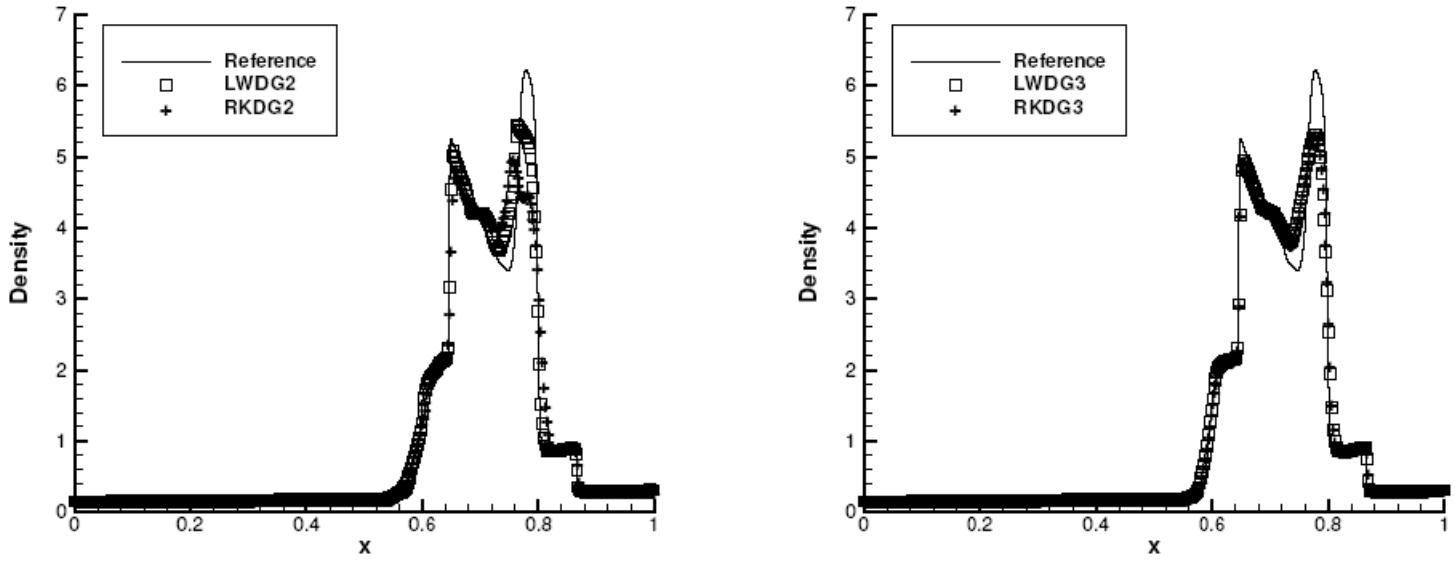
$\varepsilon = 0.2$. The computed density ρ is plotted at $t = 1.8$

Numerical Results



Details of the oscillatory portion of the solution between $x = 0.5$ and $x = 2.5$

Numerical Results



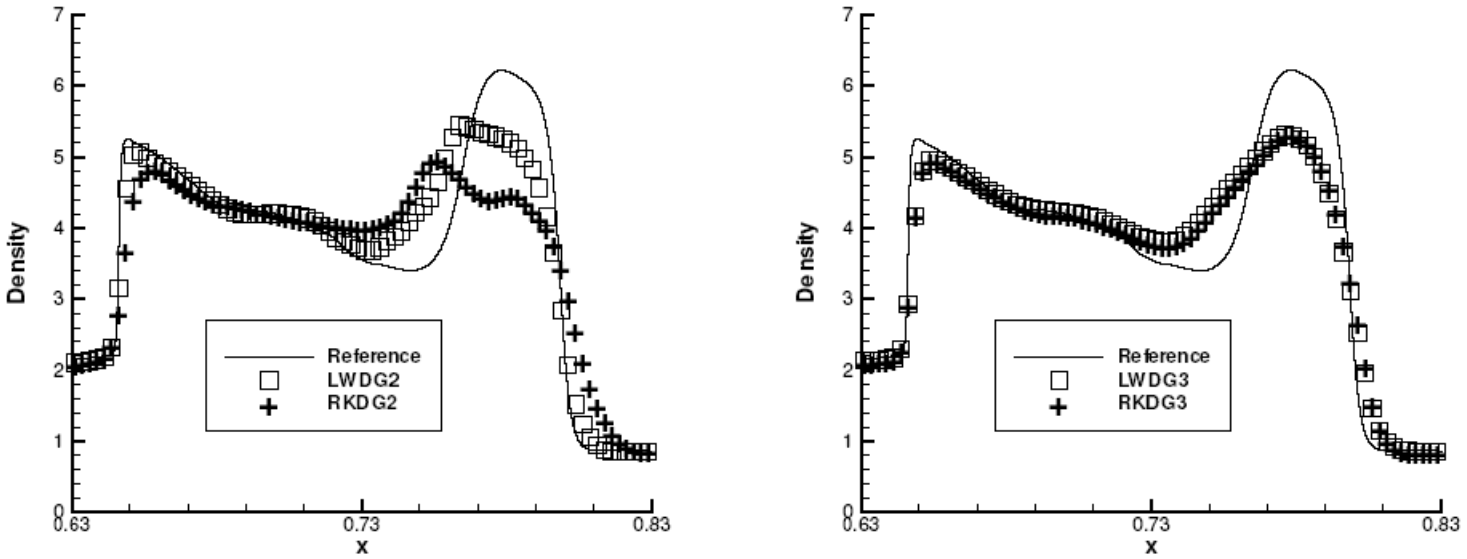
Blast waves of Euler equation with the initial condition:

$(\rho, v, p) = (1, 0, 1000)$ for $0 \leq x < 0.1$; $(\rho, v, p) = (1, 0, 0.01)$ for $0.1 \leq x < 0.9$;

$(\rho, v, p) = (1, 0, 100)$ for $0.9 \leq x$.

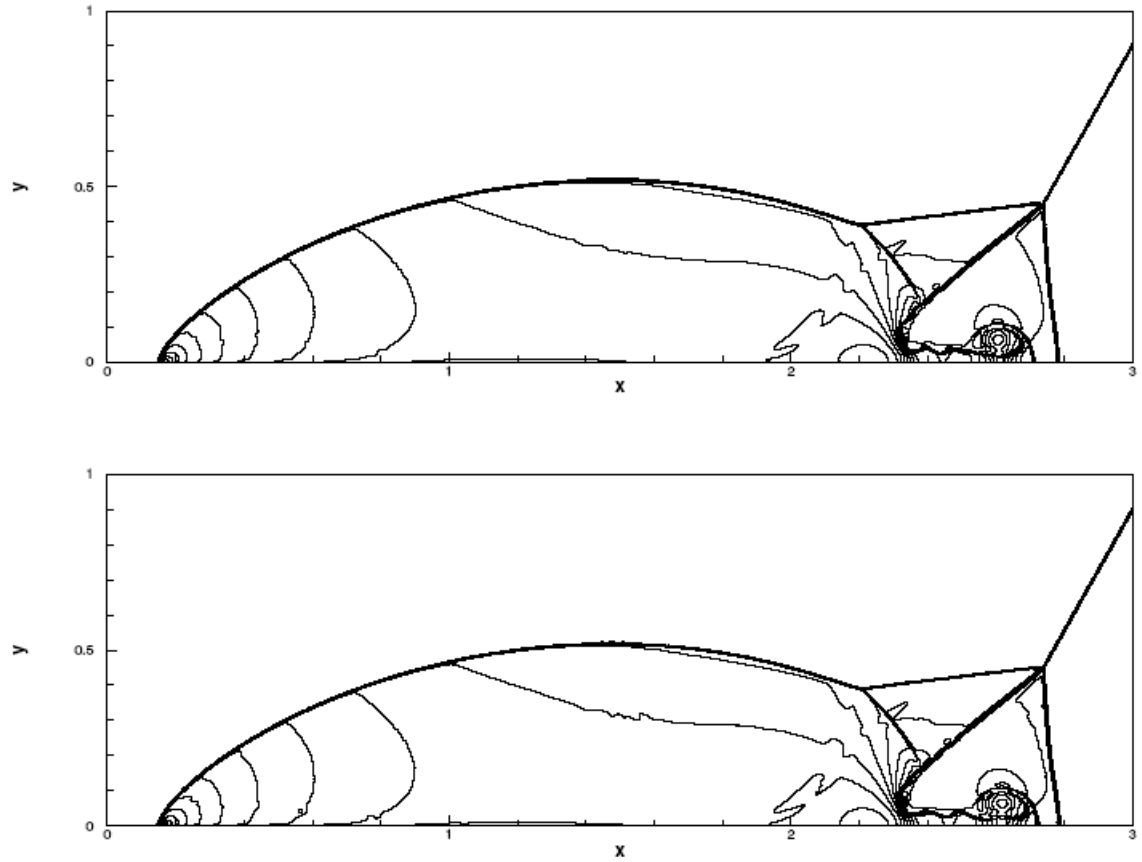
The computed density ρ is plotted at $t = 0.038$

Numerical Results



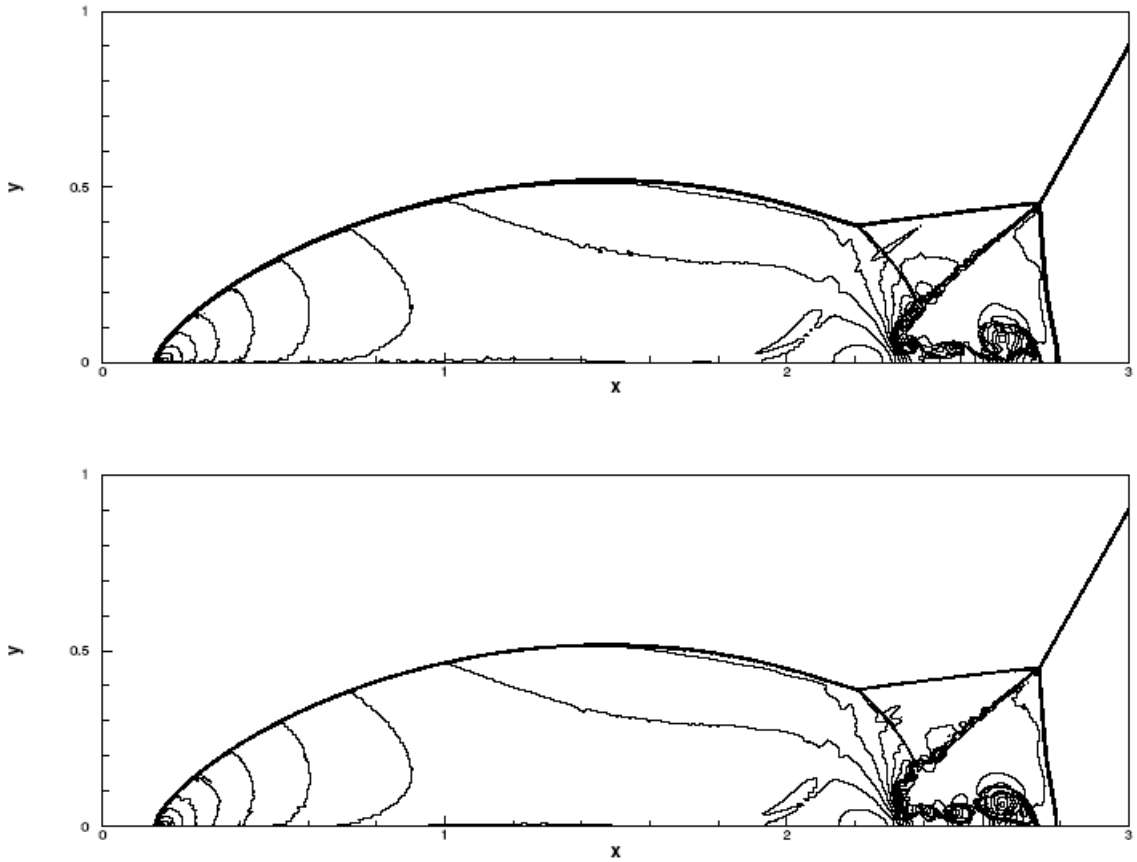
Details of the oscillatory portion of the solution between $x = 0.63$ and $x = 0.83$

Numerical Results

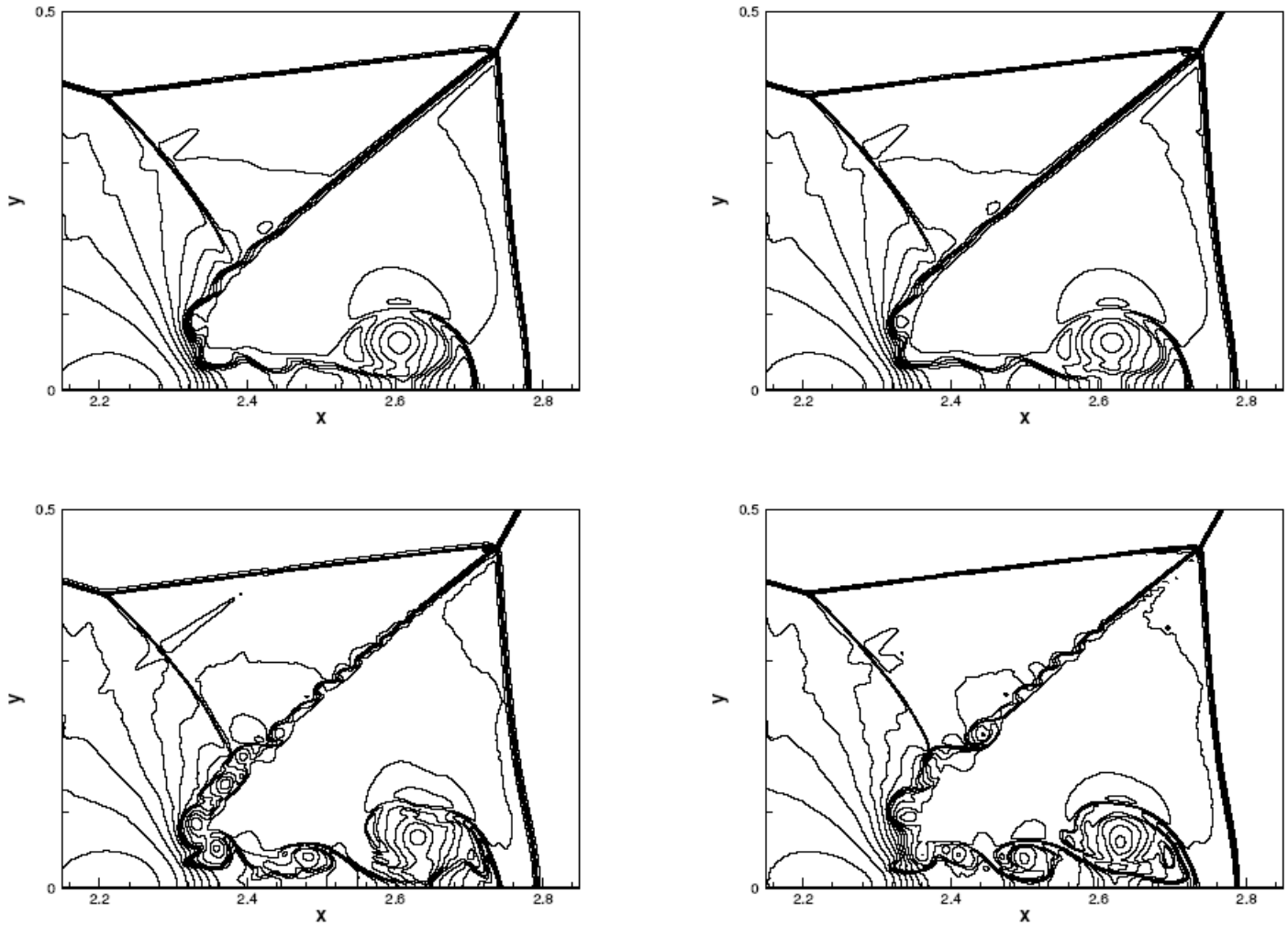


Double Mach reflection problem. Second order (k=1) LWDG (top) and RKDG (bottom).

Numerical Results



Double Mach reflection problem. Third order (k=2) LWDG (top) and RKDG (bottom).



Zoomed-in region to show more details. Top: second order (k=1); bottom: third order (k=2). Left: LWDG; Right: RKDG.

Numerical Results

Schemes	LWDG				RKDG			
	k=1		k=2		k=1		k=2	
M	0.01	100	0.01	100	0.01	100	0.01	100
120×30	95.59	77.44	740.96	569.54	131.18	86.20	1071.05	708.64
240×60	789.10	676.51	5925.01	4860.05	920.28	737.04	8361.42	6061.02

CPU time (in seconds) for the LWDG and RKDG methods to compute the double Mach reflection problem for the two meshes of 120×30 and 240×60 cells.
The computation is performed on a Compaq Digital personal workstation, 600au alpha-599MHZ with 256MB ram.

LWDG is more compact than RKDG and the Lax-Wendroff time discretization procedure is more cost effective than the Runge-Kutta time discretizations.

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Thank you !