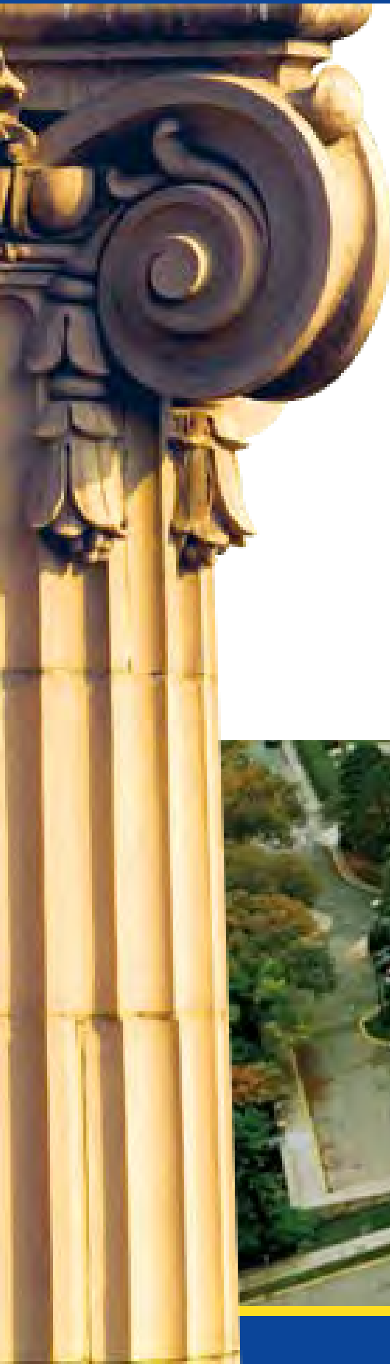

Adjoint Based hp-Adaptations for the CPR method

L. Shi and Z.J. Wang (zjwang.com)
Department of Aerospace Engineering
University of Kansas

Outline

- Introduction and motivation
- High-order CPR method
- Adjoint-based error estimate and hp-adaptation
  - Naive formulation
  - Dual-consistent formulation
- Numerical results
- Conclusions

Where is KU?

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The KUAE Legacy

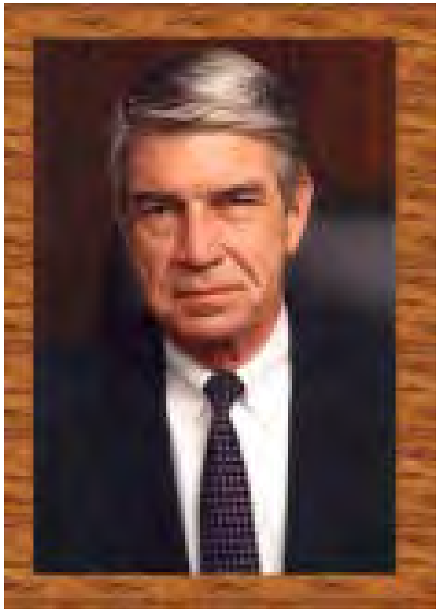
- First graduates in 1944
- KUAE is among of top universities in the world in AIAA student design competitions
  - No. 1 in AIAA Graduate Team Aircraft Design Competition
  - Swept top 3 in the AIAA Individual Aircraft Design Competition
  - No. 2 in AIAA’s team engine design competition
  - No. 2 in AIAA Undergraduate Team Space Transport Design
- Faculty with industry/lab experience
- Lots of faculty contact
- Focus on design/build/fly and hands on experience
- Very strong graduates

The Most Famous Alumni

Joe Engle
Astronaut

KUAE BS 1948

John Brizendine
CEO of
Douglas Aircraft



KUAE BS 1948

Alan Mulally
CEO of Ford

KUAE BS 1969

Who Are We -- By the Numbers?

- Undergraduate Students: ~170
- Graduate Students: ~40 with 50:50 split between MS and PhD
- Research Volume: ~$1.5 - 2.0M (FY09-FY12)
  - Top 10 unit on campus for faculty and student engagement in research, per capita expenditures
  - Several faculty are key leaders and active in CReSIS (Center for Remote Sensing of Ice Sheets), the largest research center at KU, which designs, builds and flies the Meridian UAS.
  - Strengths:
    - Unmanned Aerial Vehicles
    - Composites and adaptive structures
    - Computational fluid dynamics

Our People

- 11 tenured/tenure-track faculty members
  - Most serve on national committees in aerospace engineering
  - Many serve as editors of professional journals
  - Many have lab/industry experience
- All faculty supported by external research grants
  - Many are national leaders in structures, aerodynamics, control/robotics and propulsion
- 4 Staff
  - 2 administrative assistants
  - 2 technicians
- 2 adjunct faculty
  - One is an astronaut

Popular High-Order Methods

- Compact difference method
- Optimized difference method
- ENO/WENO methods
- MUSCL, PPM and K-exact FV
- Spectral element/hp finite element
- Residual distribution methods
- Discontinuous Galerkin
- Spectral volume/spectral difference
- Correction procedure via reconstruction

■ Structured grid ■ Unstructured grid

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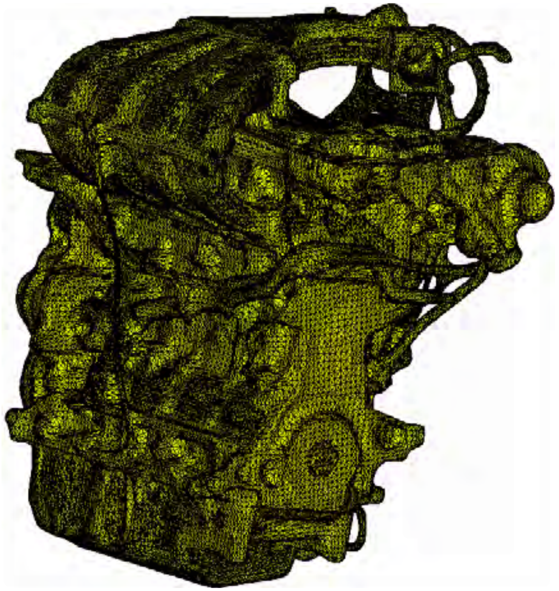
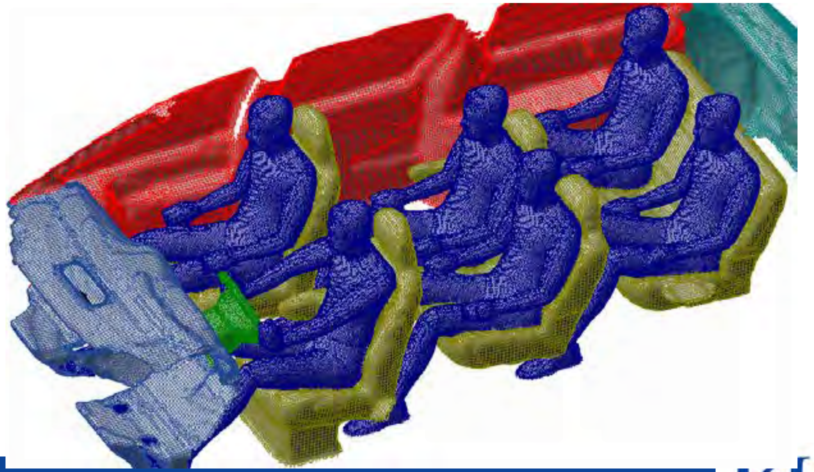
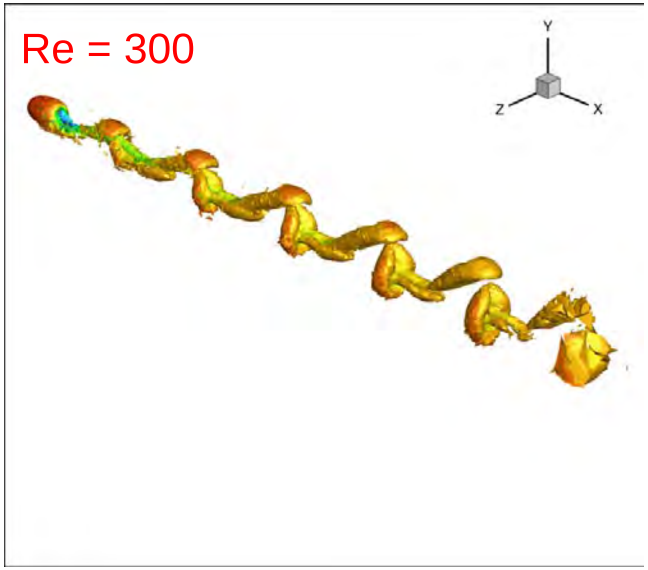
Structured vs. Unstructured Grid Methods

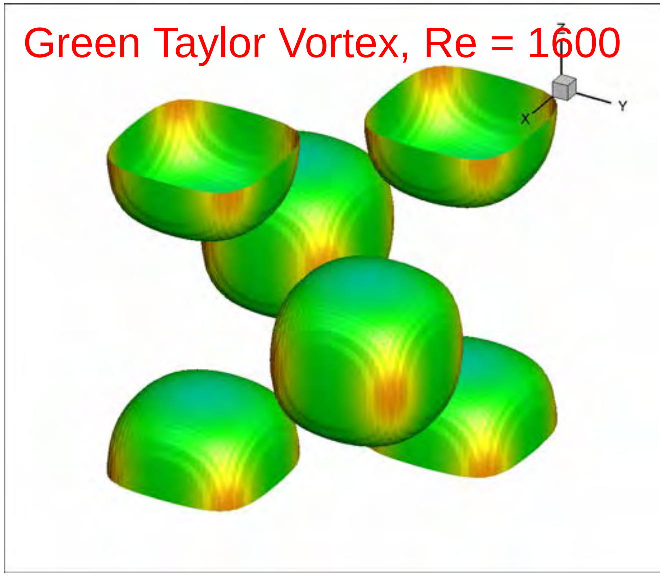
- Efficiency (S.)
- Ease of implementation (S.)
- Geometric flexibility (U.)
- Boundary condition (U.)
- Load balancing (U.)
- K-exactness on arbitrary grids (U.)





Demonstration of CPR Methods



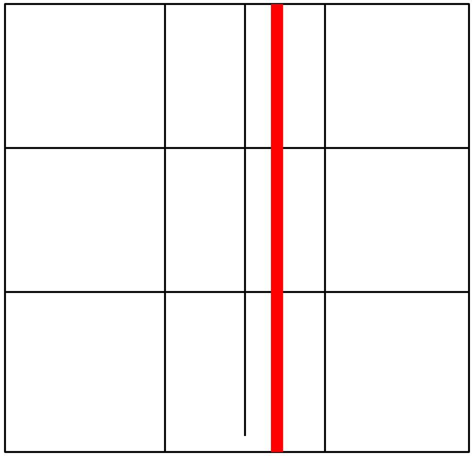


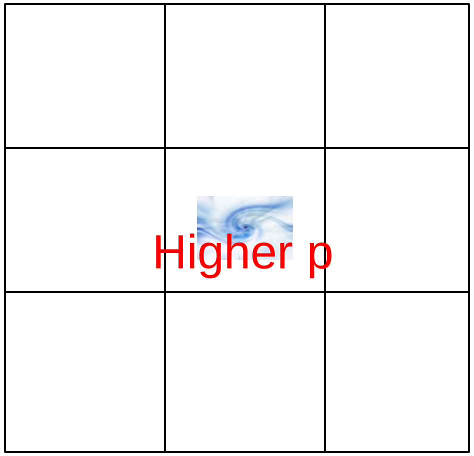
Introduction

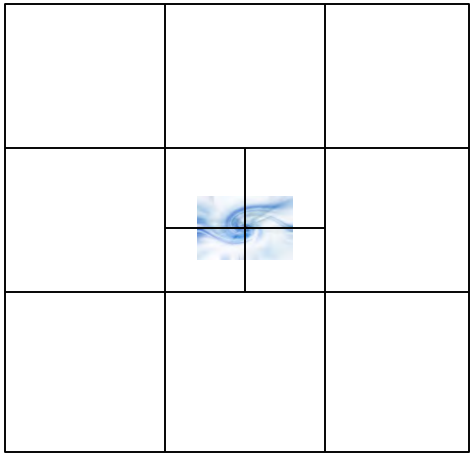
- According to the 1st and 2nd International Workshops
  - High-order methods outperform low order ones for smooth problems with smooth geometries
  - For non-smooth problems, no conclusion can be drawn
  - Solution-based hp-adaptations dramatically increase the efficiency, accuracy and robustness
- Pacing items
  - High-order mesh generation, robust and efficient solver, error estimate and hp-adaptations, and discontinuity capturing
- Objective: develop an adjoint method for the CPR method to obtain engineering outputs of desired accuracy with minimal cost

Solution Adaptive Approaches

- Discretization error reduction
  - P-enrichment: smooth flow regions (Weierstrass theorem)
  - H-refinement: geometry or flow singularities
  - Anisotropic adaptation: shear layers, shocks,…







Solution Adaptive Approaches

- Adaptation criteria/error indicators
  - Feature-based: simple, ad hoc, less rigorous
  - Residual-based: may lead to false refinements
  - Adjoint-based: adapt the mesh in regions affecting the output, and estimate the error in the output

Main Idea of CPR (Huynh)

We solve

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0$$

using a differential formulation

$$\frac{\partial U_i(x)}{\partial t} + \frac{\partial F_i(x)}{\partial x} = 0, \quad U_i(x) \in P^k, \quad F_i(x) \in P^{k+1}$$

The DOFs are solutions at a set of "solution points"

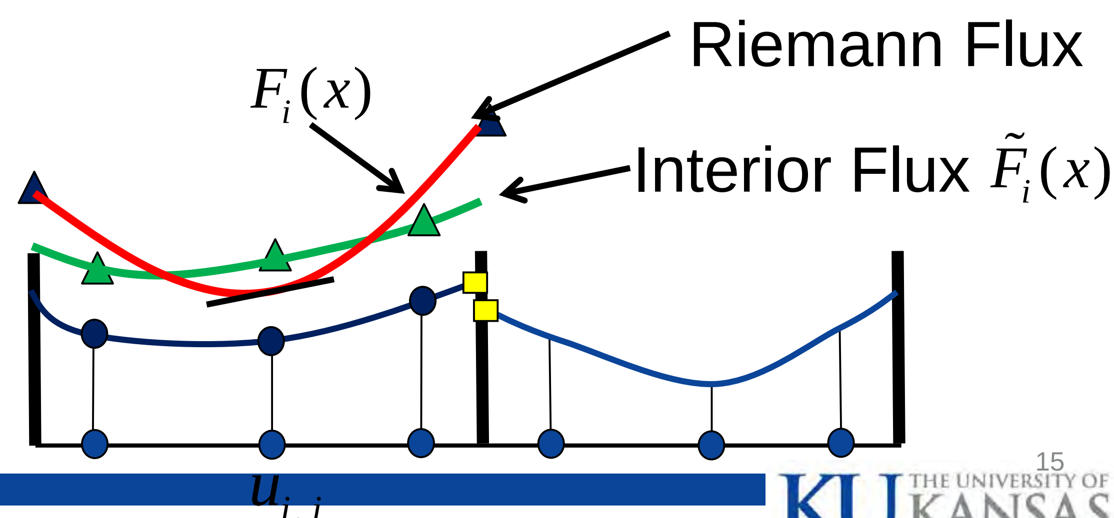
Main Idea of CPR (FR)

Find a flux polynomial $F_i(x)$ one degree higher than the solution, which minimizes

$$\left\| \tilde{F}_i(x) - F_i(x) \right\|$$

The use the following to update the DOFs

$$\frac{du_{i,j}}{dt} + \frac{dF_i(x_{i,j})}{dx} = 0$$



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The 2D CPR Method on Simplex

- Consider a hyperbolic conservation law in 2D

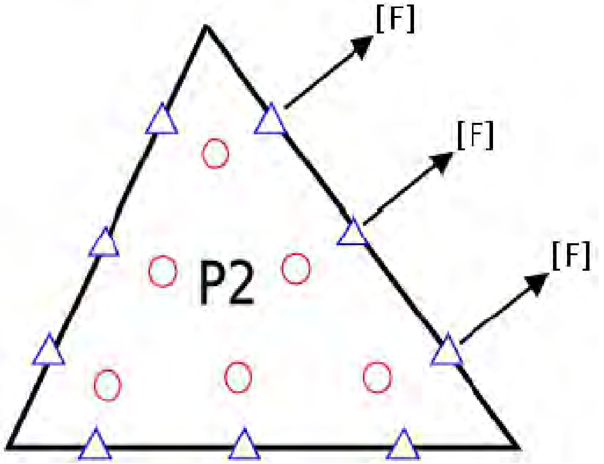
$$\frac{\partial Q}{\partial t} + \nabla \cdot \vec{f}(Q) = 0$$

- The 2D CPR formulation for a linear triangle is

$$\frac{\partial Q_{i,j}}{\partial t} + \nabla \cdot \vec{f}(Q_{i,j}) + \frac{1}{|V_i|} \sum_{f \in \partial V_i} \sum_{l} \alpha_{j,f,l} [\mathrm{F^n}]_{\mathrm{f,l}} S_f = 0$$

The correction coefficient $\alpha_{j,f,l}$

- Constants for any linear triangle
- $\int_{V_i} W \delta_i dV = \int_{\partial V_i} W [\mathrm{F^n}] dV$



Review of Adjoint-Based Adaptive Methods

Adjoint-based adaptive methods

Dynamically distribute computer resources to regions which are important for predicting engineering outputs

Current status of the output-based adaptation methods

- 2D/3D complex geometry
- Steady/unsteady
- Euler/NS/RANS
- Anisotropic hp-adaptations

[Giles and Pierce,1997; Becker and Rannacher,2001; Venditti and Darmofal, 2002; Hartmann and ouston,2002; Nielsen et al, 2004; Fidkowski and Darmofal,2007; Hartmann,2007; Mani and Mavriplis, 2007; Nemec et al, 2008; Park, 2008; Wang and Mavriplis,2009; Oliver and Darmofal, 2008; Fdikoswski and Roe, 2009; Ceze and Fdikoswski,2012;...]

Fully Discrete Adjoint

Let $R_h(\mathrm{Q}_h)$ be the residual, $J_h(\mathrm{Q}_h)$ be the output. The adjoint $\tilde{\psi}_h$ is defined in

$$\delta J_h = J_h(\mathrm{Q}_h + \delta \mathrm{Q}_h) - J_h(\mathrm{Q}_h) = \tilde{\psi}_h^T \delta R_h$$

Since

$$\frac{\partial R_h}{\partial \mathrm{Q}_h}\delta Q_h + \delta R_h = 0$$

Then

$$\delta J_h = \frac{\partial J_h}{\partial Q_h}\delta Q_h = \tilde{\psi}_h^T \delta R_h = -\tilde{\psi}_h^T \frac{\partial R_h}{\partial Q_h}\delta Q$$

Finally

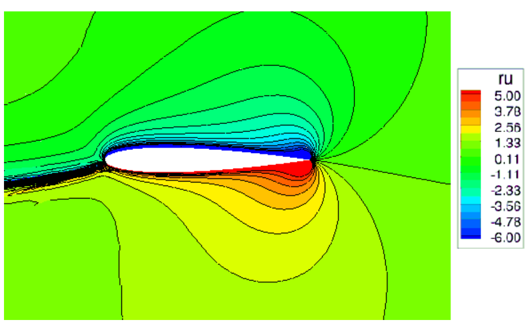
$$-\frac{\partial J_h}{\partial Q_h} = \tilde{\psi}_h^T \frac{\partial R_h}{\partial Q_h} \quad \text{or} \quad -\frac{\partial R_h}{\partial Q_h}^T \tilde{\psi}_h = \frac{\partial J_h}{\partial Q_h}^T$$

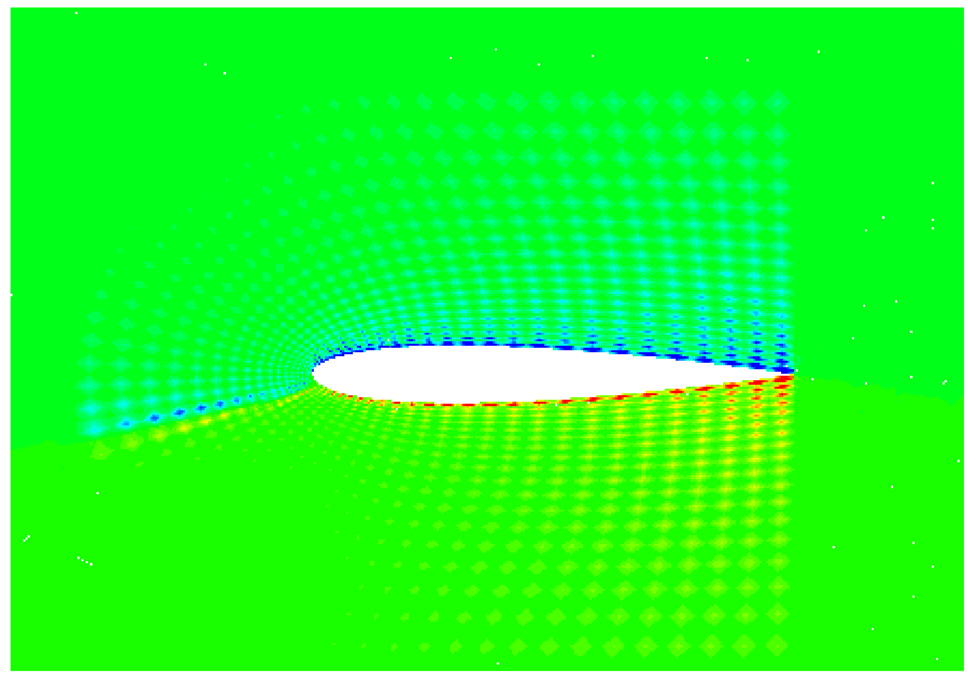
The Fully Discrete Adjoint for the CPR Method

NACA 0012 at $M_\infty = 0.4$, $\alpha = 5°$

- The x-mom of the lift adjoint
- Fully discrete adjoint
- Highly-oscillating adjoint solution

$$-\frac{\partial R_h}{\partial Q_h}^T \tilde{\psi}_h = \frac{\partial J_h}{\partial Q_h}^T$$





Dual Consistency

A residual from a differential schemes at SP j of cell i

$$R(Q)_{i,j} = \nabla \cdot \vec{f}(Q_i)_j + \frac{1}{|V_i|} \sum_{f \in \partial V_i} \sum_{l} \alpha_{j,f,l} [F^n]_{f,l} S_f$$

With fully discrete approach

$$-\sum_i \sum_j \frac{\partial R_{i,j}}{\partial Q_k} \tilde{\psi}_{i,j} = \frac{\partial J}{\partial Q_k}, \quad k = 1, \ldots, N_{DOF}$$

To be dual consistent,

$$-\int_\Omega \frac{\partial N(Q)}{\partial Q}^T \psi \, dV = \frac{\partial J}{\partial Q}^T$$

Discrete Adjoint for the CPR in the Integral Form

$$-\int_{\Omega} \frac{\partial N(Q)}{\partial Q}^{T} \psi \, dV = \frac{\partial J}{\partial Q}^{T}$$

- Approximate ψ_i using the basis L_j from the primal solution space

$$\psi_i = \sum_j L_j \hat{\psi}_{i,j}$$

- Directly discretizing the continuous adjoint eqn

$$-\sum_i \sum_j \frac{\partial R_{i,j}}{\partial Q_k} \omega_j \left| J_{i,j} \right| \hat{\psi}_{i,j} = \frac{\partial J}{\partial Q_k}$$

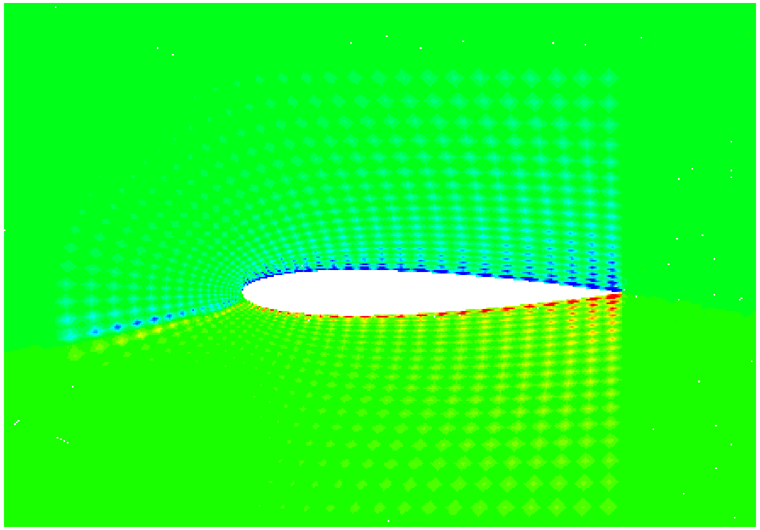
- The difference between $\hat{\psi}_{i,j}$ and $\tilde{\psi}_{i,j}$

$$\tilde{\psi}_{i,j} = \omega_j \left| J_{i,j} \right| \hat{\psi}_{i,j}$$

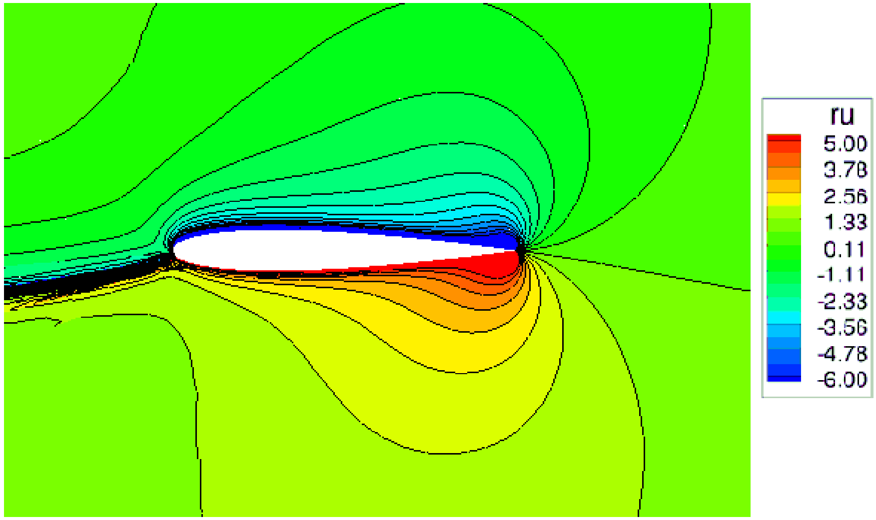
Comparison of the Adjoints with the CPR Method

NACA 0012 at $M_\infty = 0.4$, $\alpha = 5°$

- The x-mom of the lift adjoint
- Fully discrete adjoint
- Discrete adjoint in the integral form



The inconsistent adjoint



Dual consistent adjoint

The Adjoint-based Error Estimation

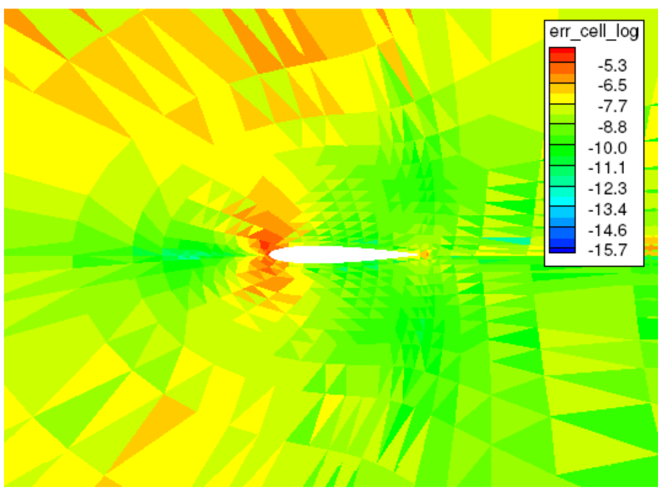
- Output error est.: adjoint solution weighted primal residual

$$\delta J_h(\mathrm{Q_h}) = J_h(\mathrm{Q}_H) - J_h(\mathrm{Q}_h) \approx -(\hat{\psi}_h)^T R_h(Q_h^H)$$

- Adjoint-based local error indicator

$$\eta = \left| (\hat{\psi}_h)^T R_h(Q_h^H) \right|$$

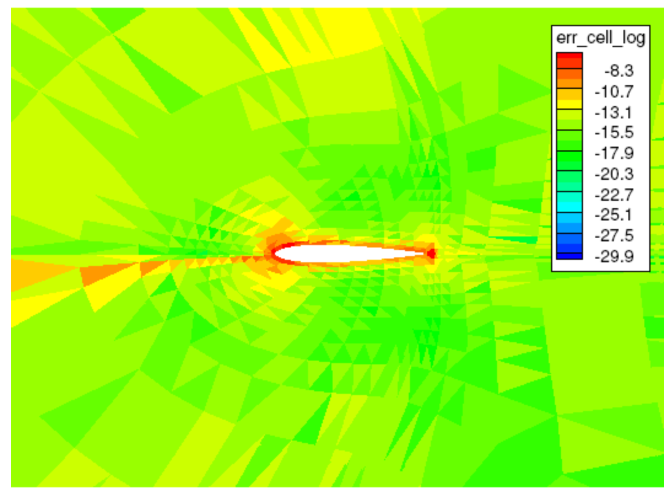
- Multi-p residual-based error indicator $\eta = \left| R_h(Q_h^H) \right|$



Local residual distribution

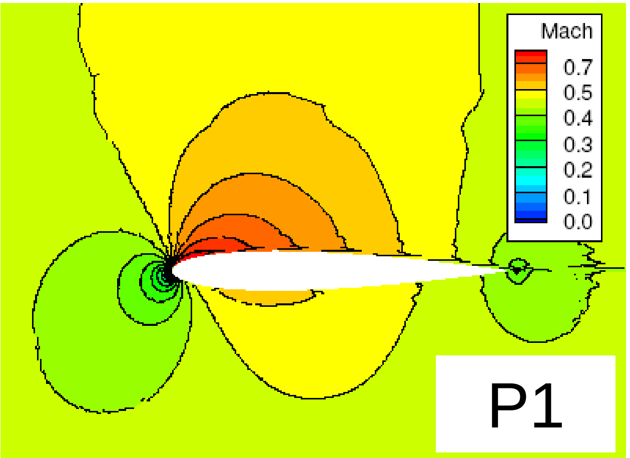
ψ

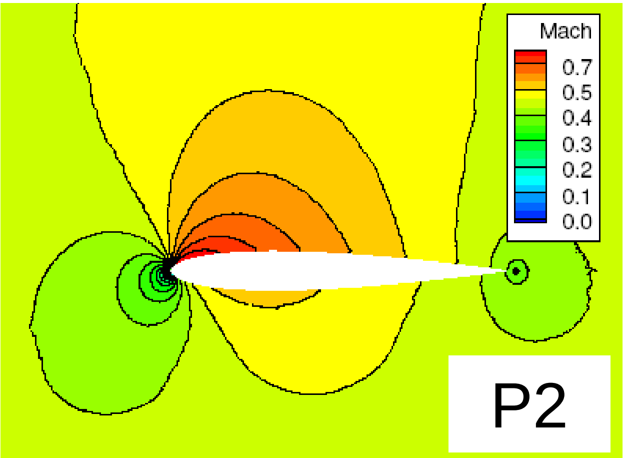
$\Rightarrow$



Adjoint-based error indicator

Accuracy Test of the Adjoint-based Error Est.





NACA 0012 at $M_\infty = 0.5$, $\alpha = 2°$

- The lift as the output J
- The effectivity of the error est.

$$\eta_H^e = \frac{-(\psi_h)^T R_h(Q_h^H)}{J_H(Q_H) - J_h(Q_h)}$$

Cells	$\mathcal{J}_H(Q_H) - \mathcal{J}_h(Q_h)$	$-(\psi_h)^T \mathcal{R}_h(Q_h^H)$	η_H^e
280	-5.859e-3	-1.103e-3	1.88
1120	-2.638e-3	-4.002e-3	1.52
4480	-8.736e-4	-9.995e-4	1.14
17920	-1.933e-4	-1.988e-4	1.03

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HP-adaptation Procedure

- Solve the flow field until a convergence criterion is reached
- Compute the error indicator η_i for each cell
- Mark cells with the largest error indicator with a fixed fraction f
- Perform the h-refine on the marked cells with large jumps
- Apply the p-enrichment on the rest of the marked cells
- Inject the current solution into the new one
- Repeat step 1-6

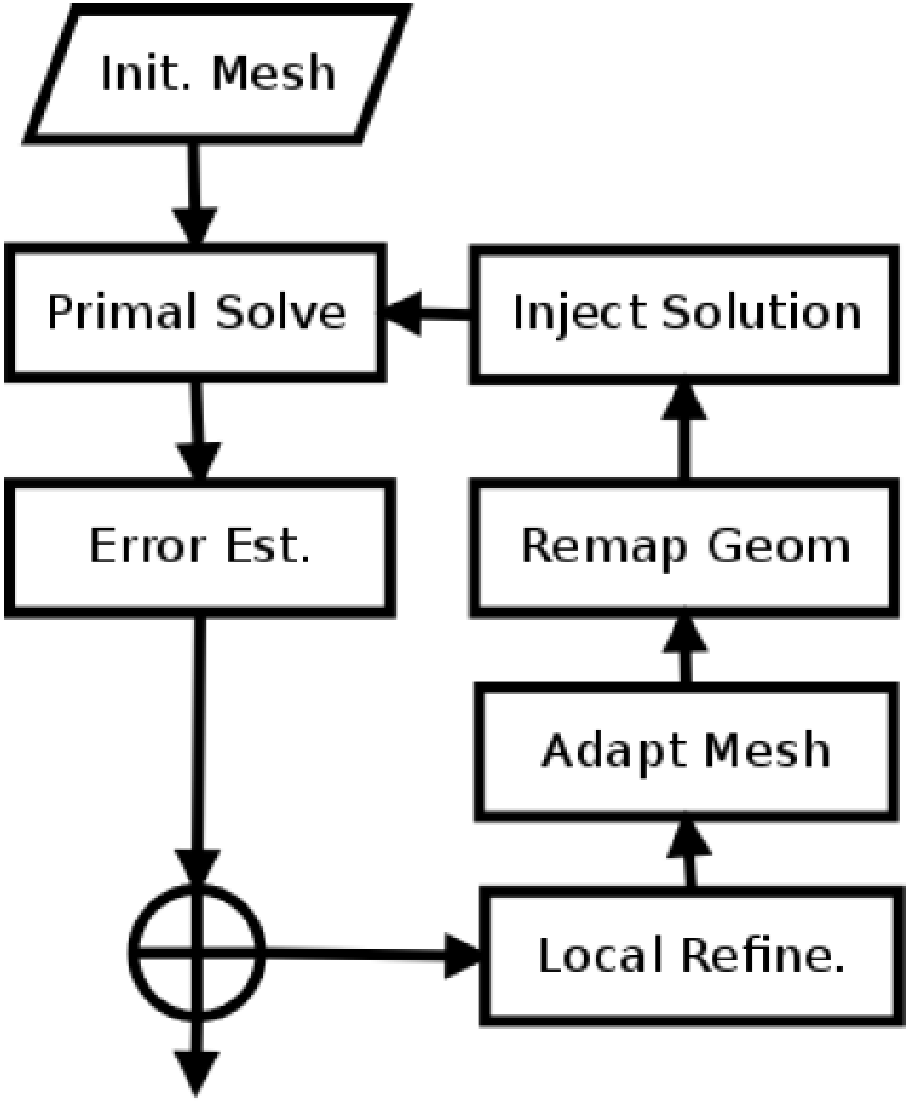
Smoothness indicator

$$\phi_i = \frac{1}{|\partial\Omega_i|}\int_{\partial\Omega_i}\left|\frac{[\![q]\!]\cdot\vec{n}}{\{q\}}\right| dS$$

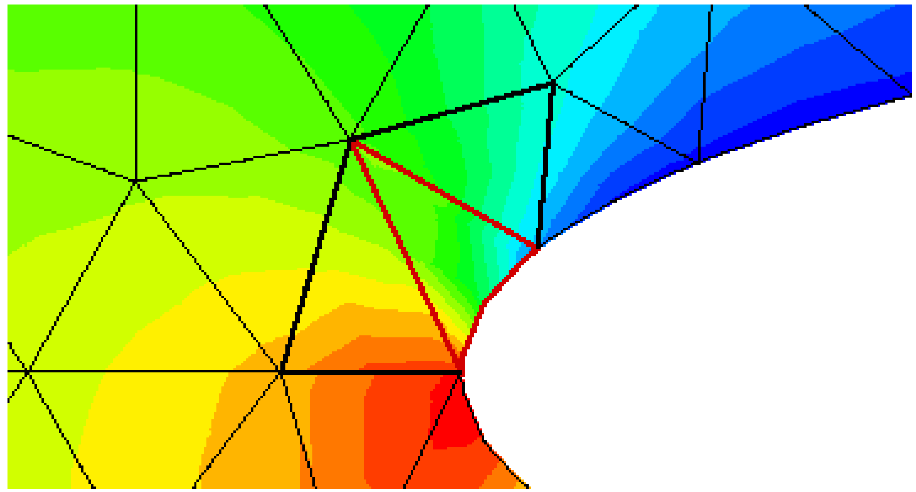
$$\begin{cases} \phi_i > \dfrac{1}{K} & h-refinements \\ \phi_i < \dfrac{1}{K} & p-enrichments \end{cases}$$

[Krivodonova et al, 2004]

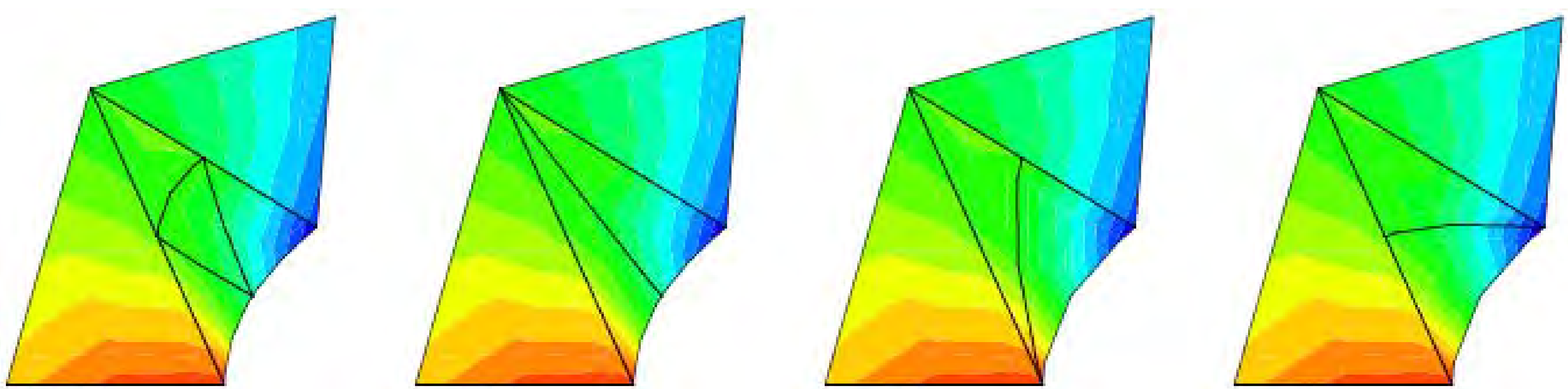
Anisotropic h-Adaptation



The local output-error sampling procedure for the anisotropic adaptations



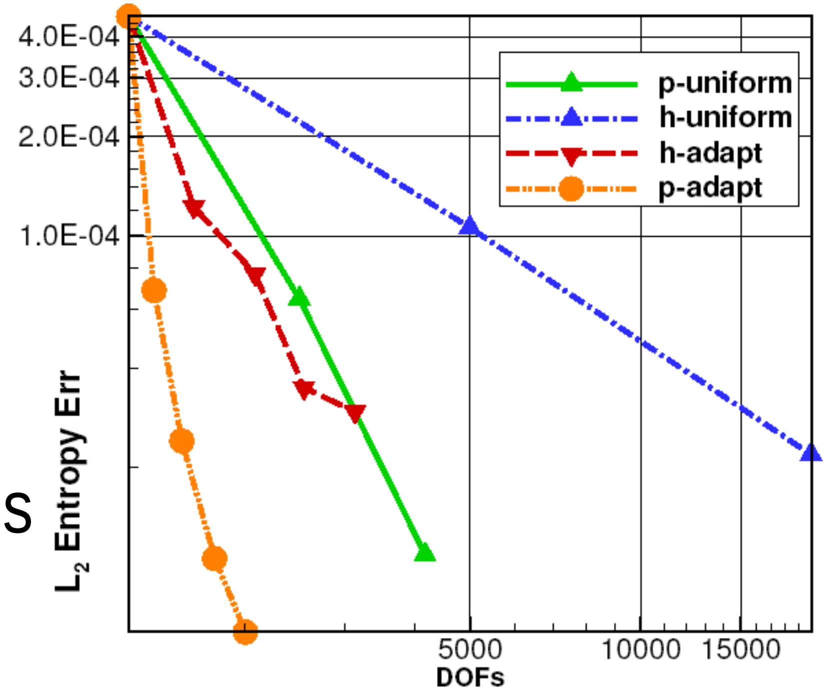
The candidate cell



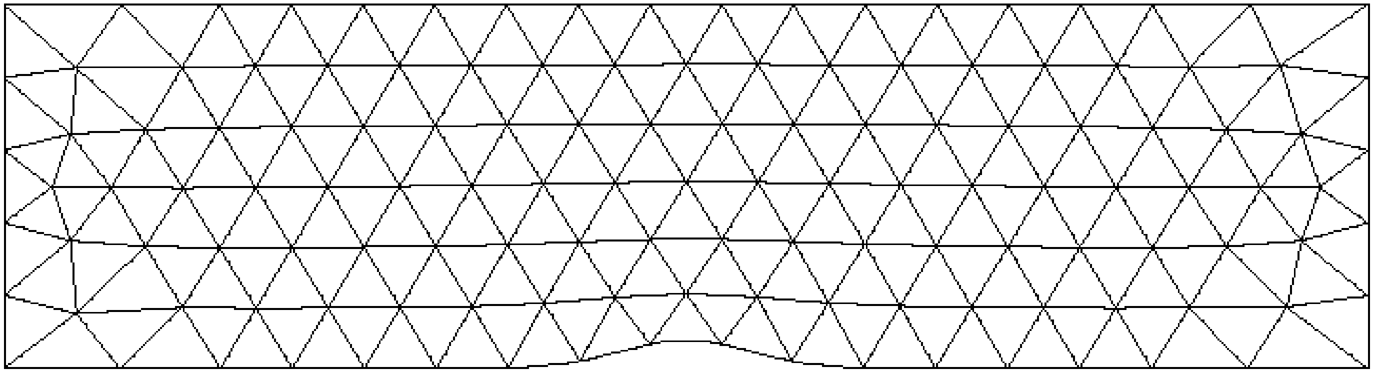
Iso Edge 1 Edge 2 Edge 3

Subsonic Flow over a Smooth Bump

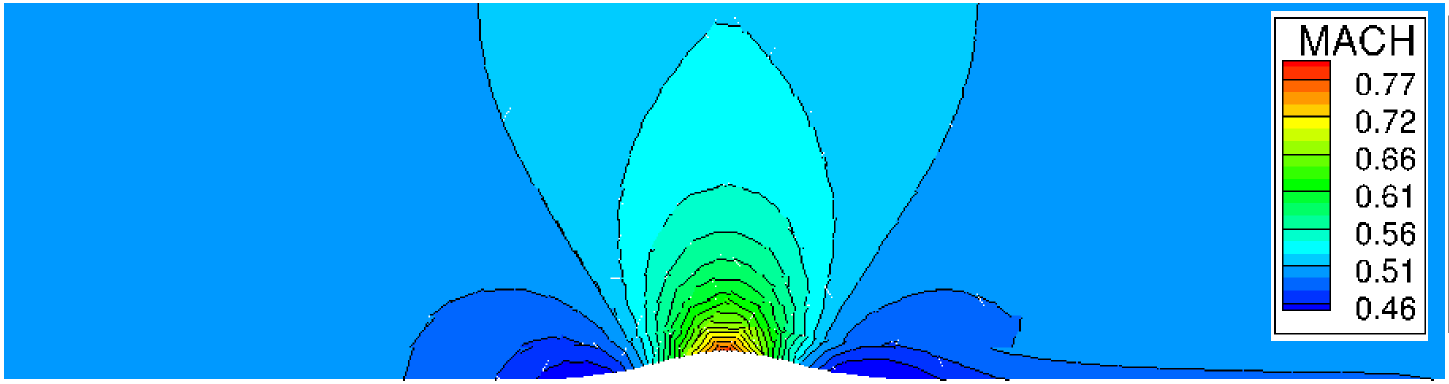
- Fixed fraction f = 0.1
- Inviscid flow, $M_\infty = 0.5$
- Start from uniform p = 1
- Residual based adaptations
- 5 h- and p-adaptation iterations



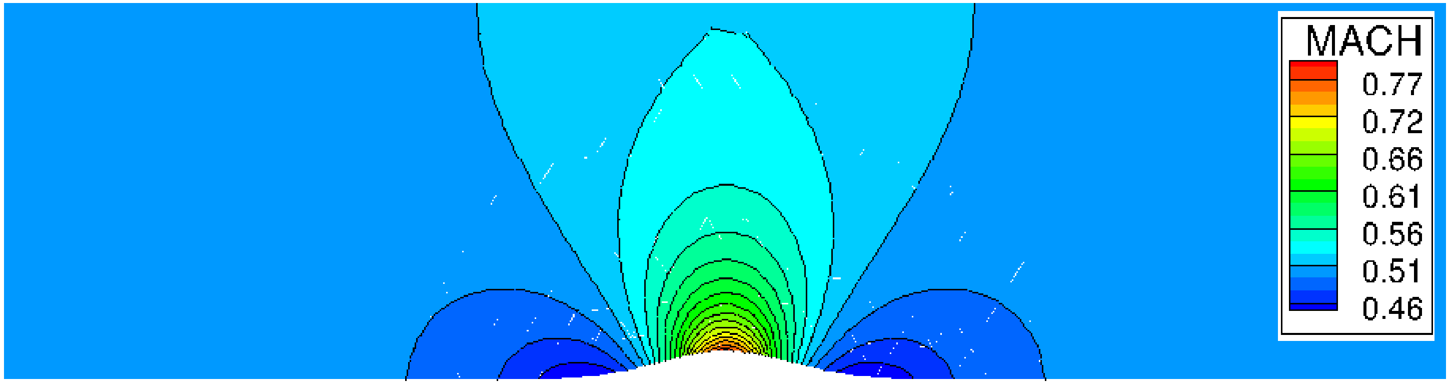
The initial mesh



Subsonic Flow over a Smooth Bump

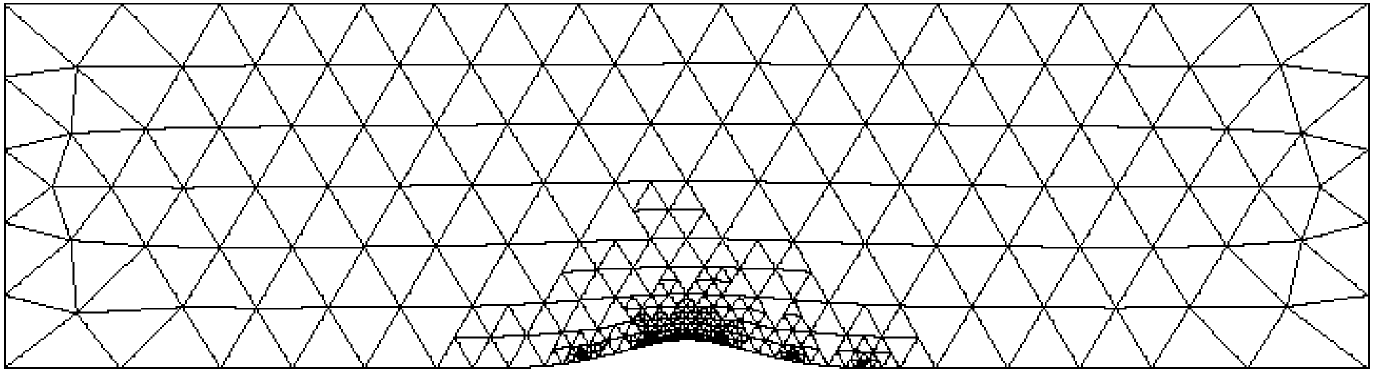


Mach contours on the initial mesh

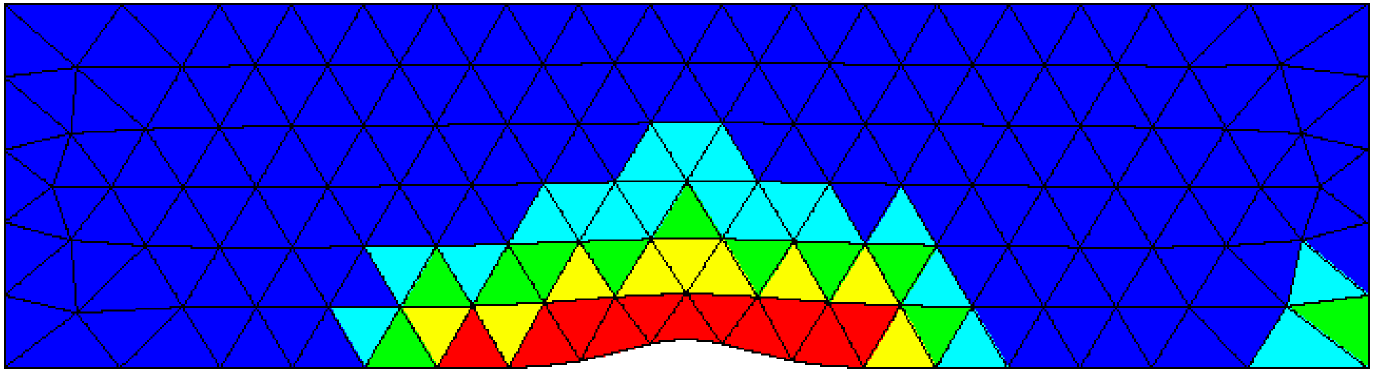


Mach contours on the adapted mesh

Subsonic Flow over a Smooth Bump



H-adapt level 5

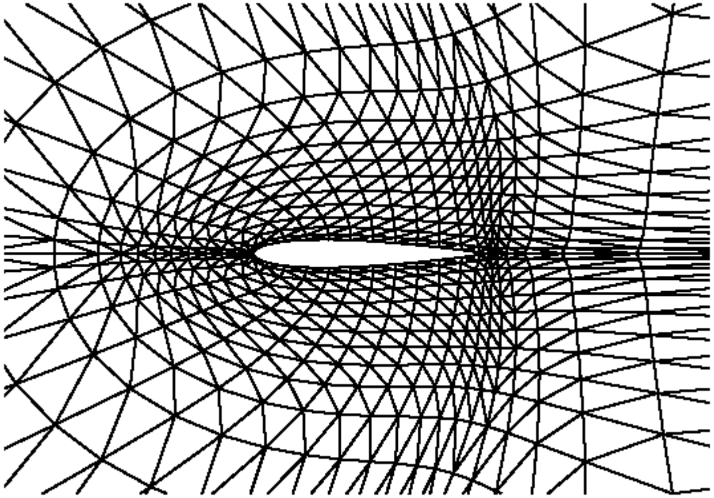


P-adapt level 5

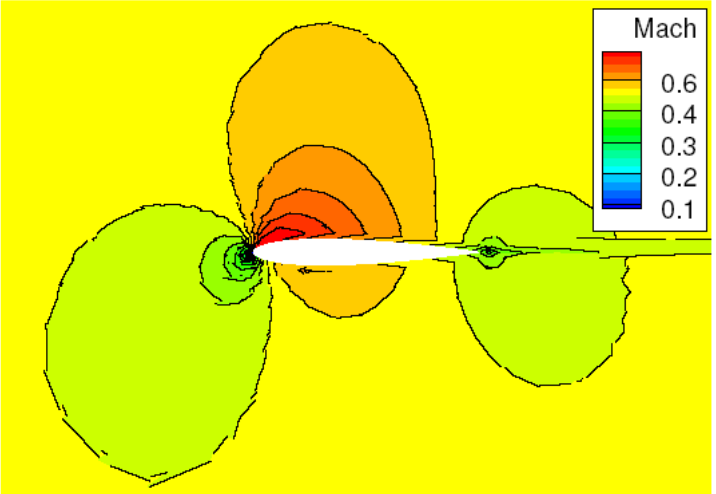
Subsonic Flow Over a NACA 0012 Airfoil

- Fixed fraction f = 0.1
- Inviscid, $M_\infty = 0.5$, $\alpha = 2°$
- Start from uniform p = 1
- H- and p-adaptation
- 'Truth' CL, CD from 1st HOW

- Error indicators
  - The lift adjoint
  - The drag adjoint
  - Unweighted residual

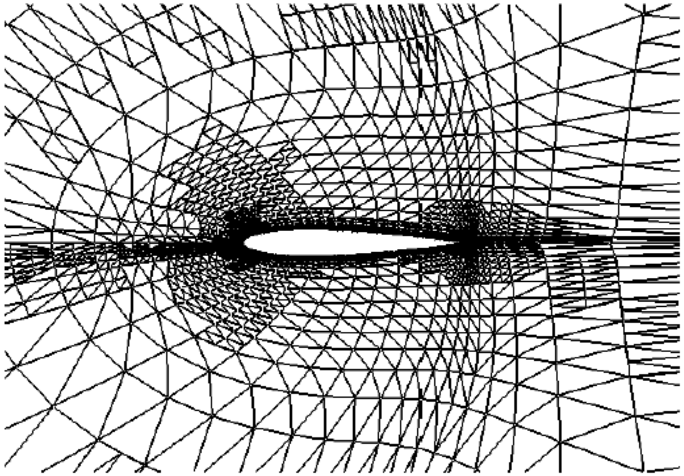


Initial mesh

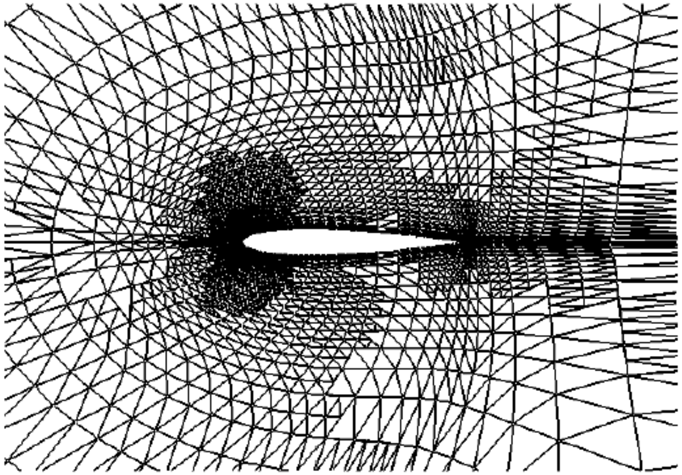


Initial solution

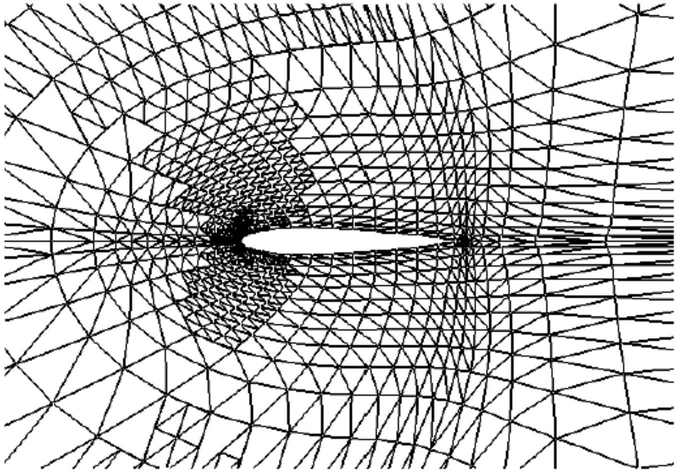
5 level h-adaptation



Lift adjoint

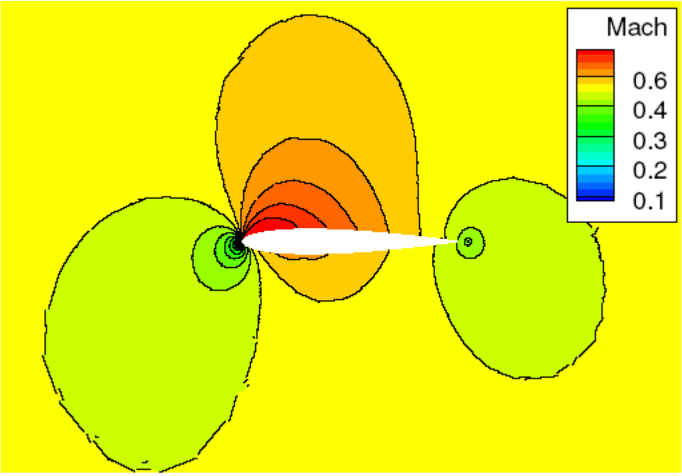


Drag adjoint

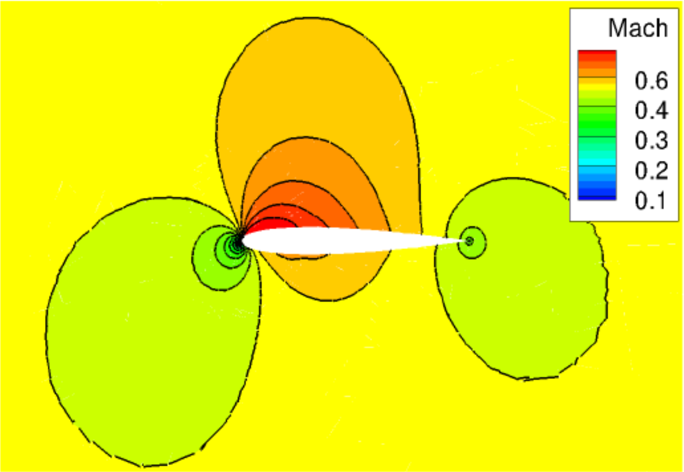


Residual-based

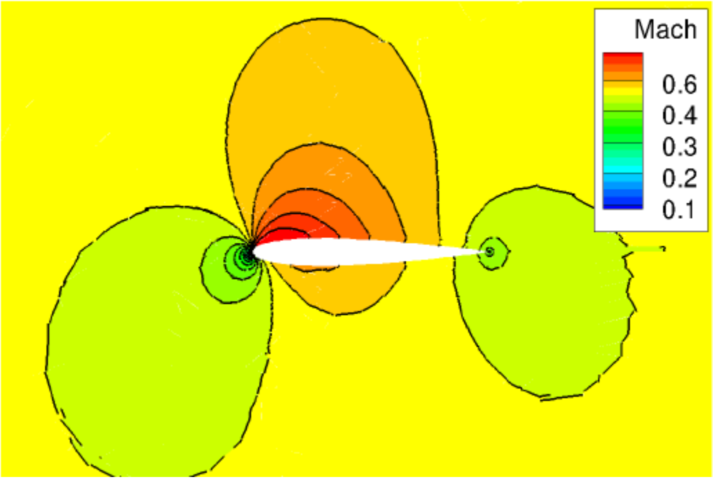
5 level h-adaptation



Lift adjoint

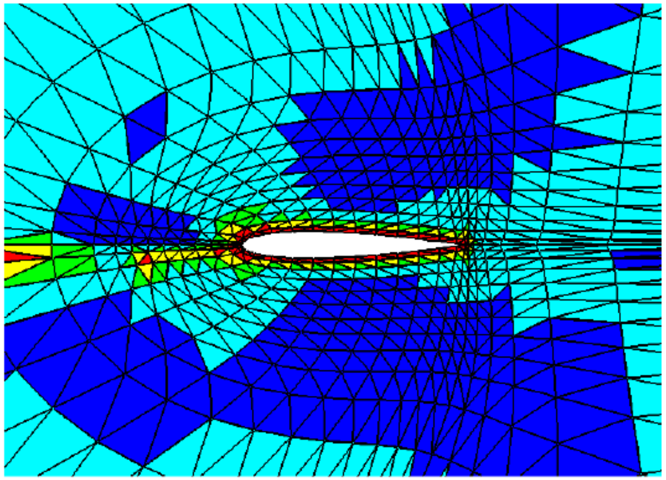


Drag adjoint

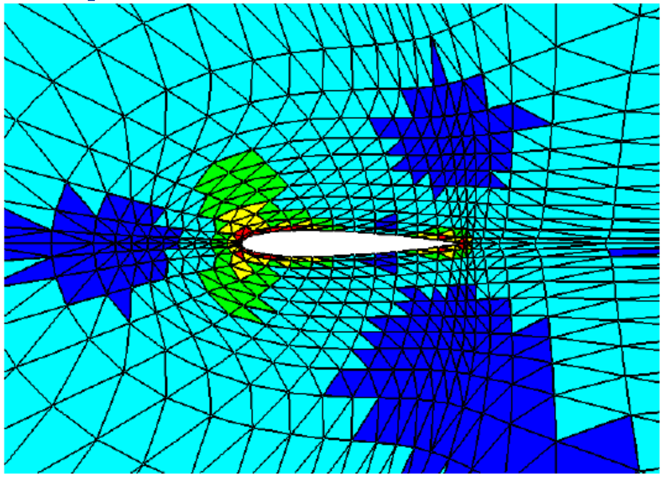


Residual-based

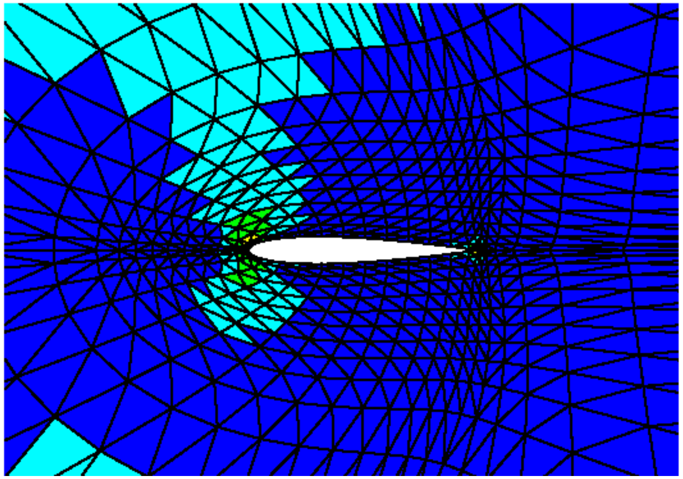
5 level p-adaptation



Lift adjoint

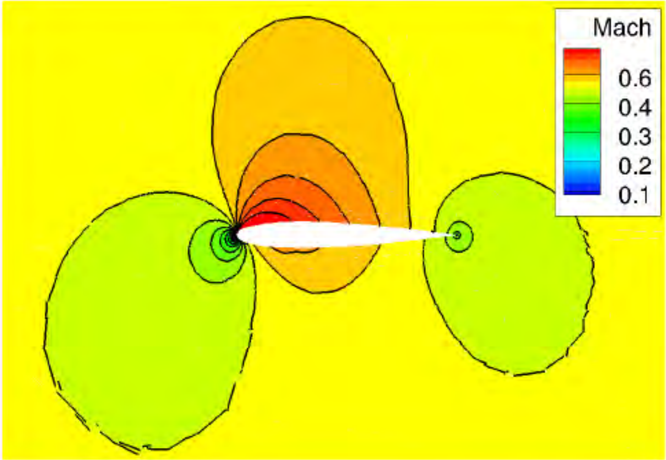


Drag adjoint

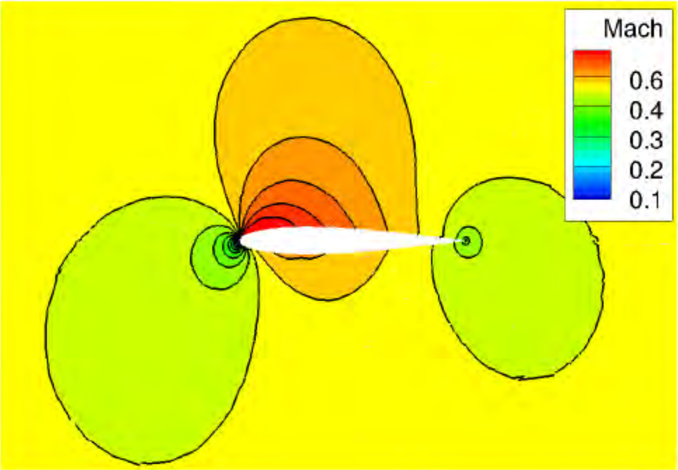


Residual-based

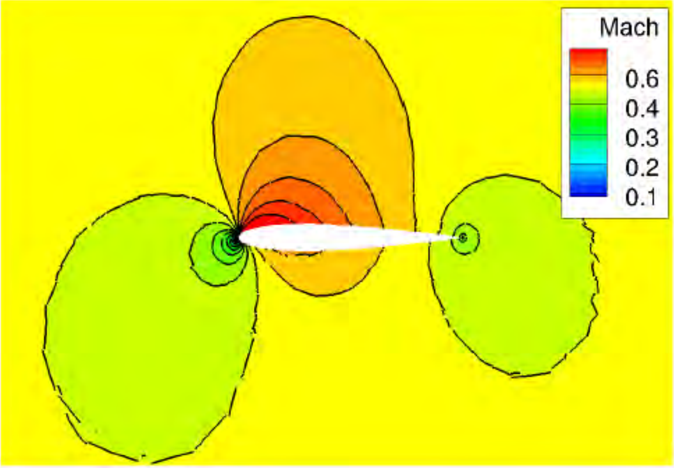
5 level p-adaptation



Lift adjoint

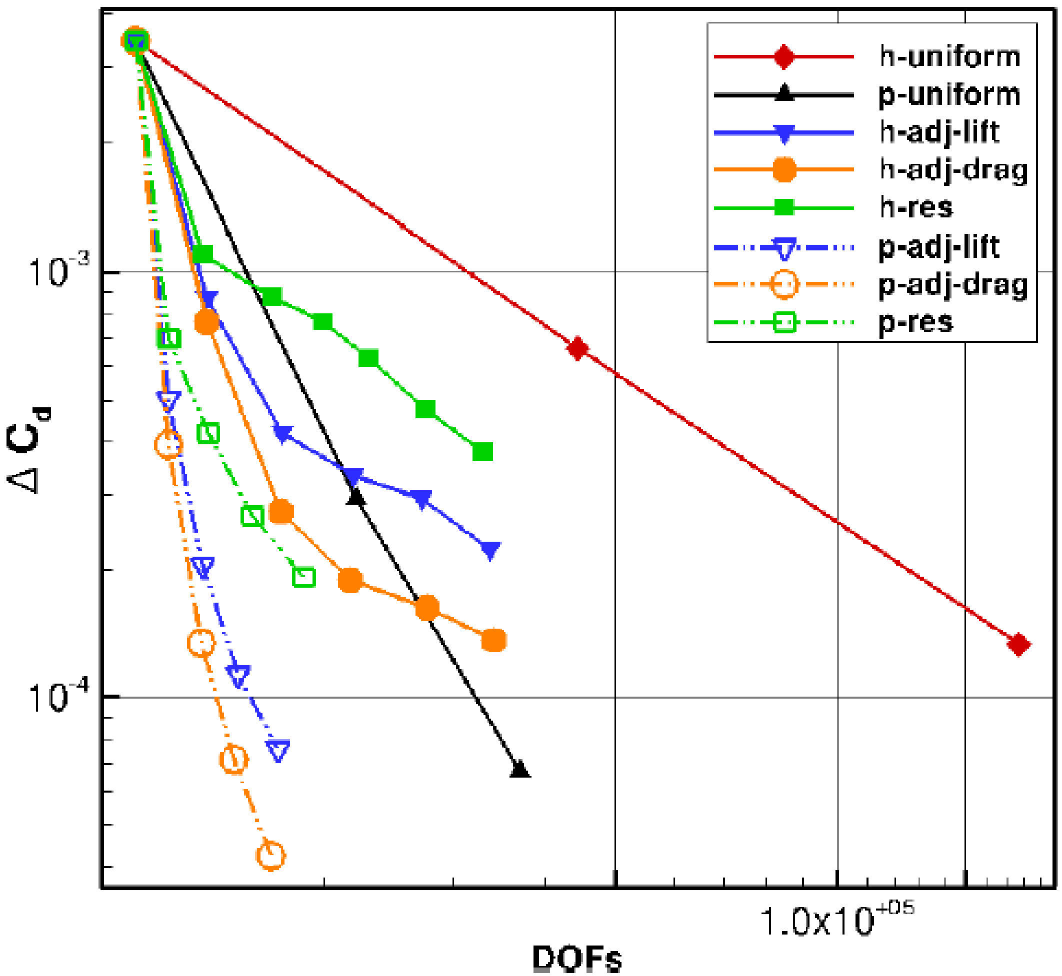


Drag adjoint



Residual-based

Subsonic Flow Over a NACA 0012 Airfoil

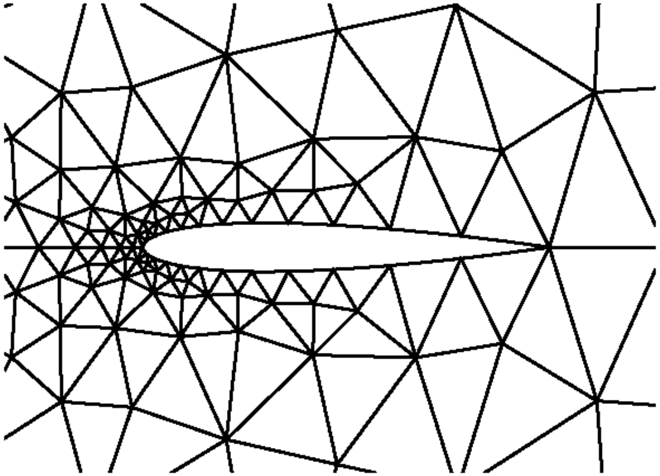


CD error

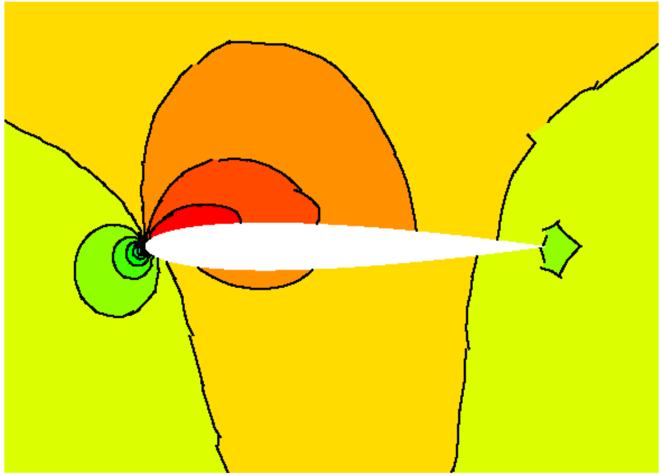
Subsonic Flow Over a NACA 0012 Airfoil

- Iso/aniso H-adaptation
- Fixed fraction f = 0.1
- Inviscid, $M_\infty = 0.5$, $\alpha = 2°$
- 3rd order scheme

Adaptation strategies
- Hanging nodes
- No-hanging nodes
- Error indicators
  - lift adjoint
  - drag adjoint



Initial mesh

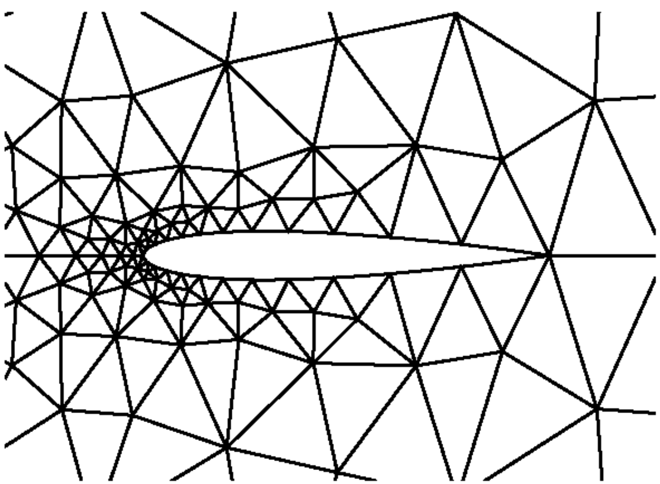


Initial solution

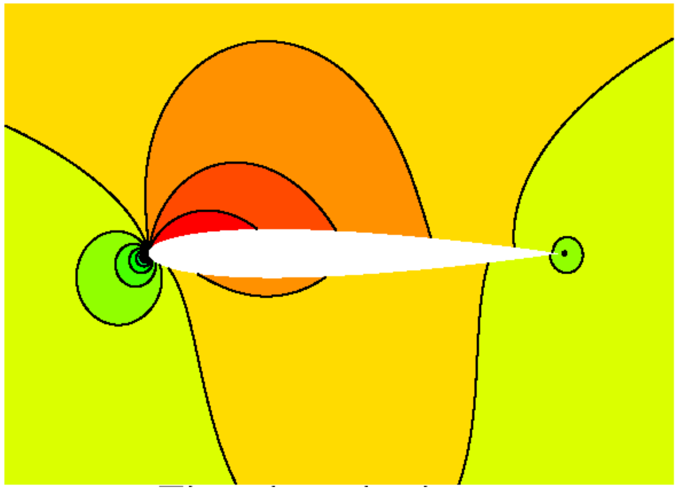
Subsonic Flow Over a NACA 0012 Airfoil

- Iso/aniso H-adaptation
- Fixed fraction f = 0.1
- Inviscid, $M_\infty = 0.5$, $\alpha = 2°$
- 3rd order scheme

Adaptation strategies
- Hanging nodes
- No-hanging nodes
- Error indicators
    - lift adjoint
    - drag adjoint

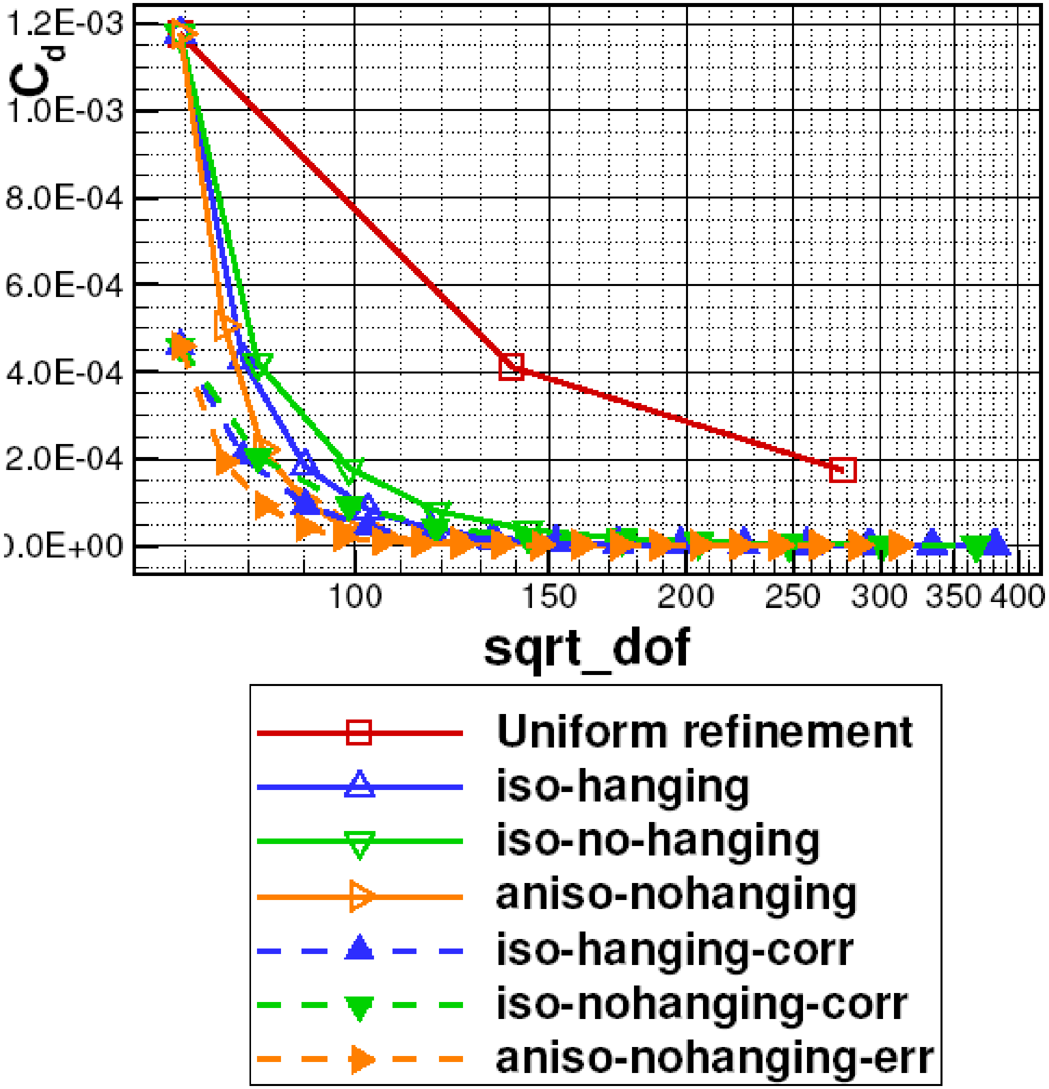


Initial mesh

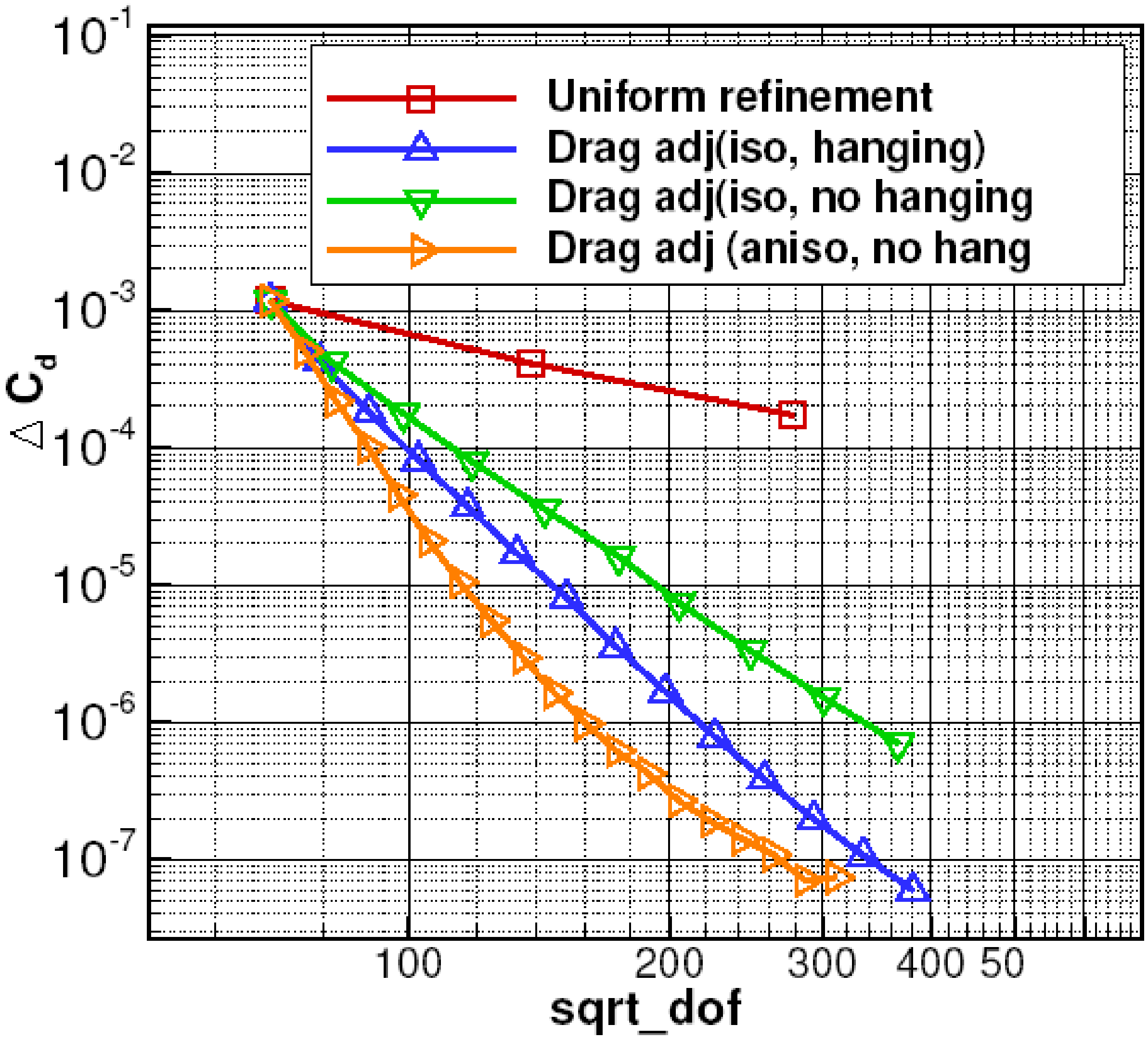


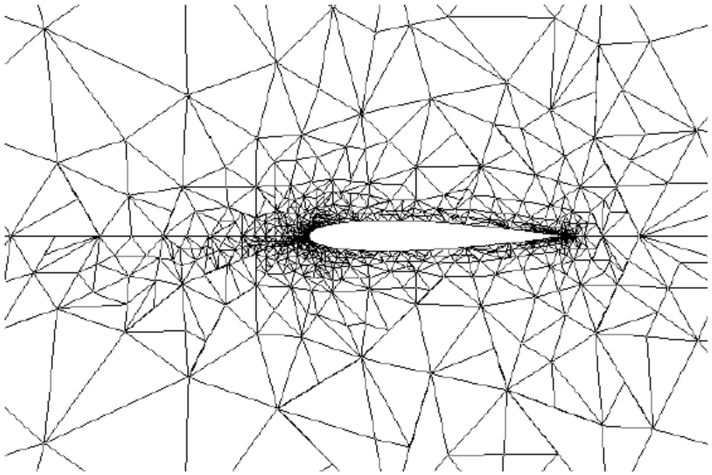
The adapted solution

Subsonic Flow Over a NACA 0012 Airfoil

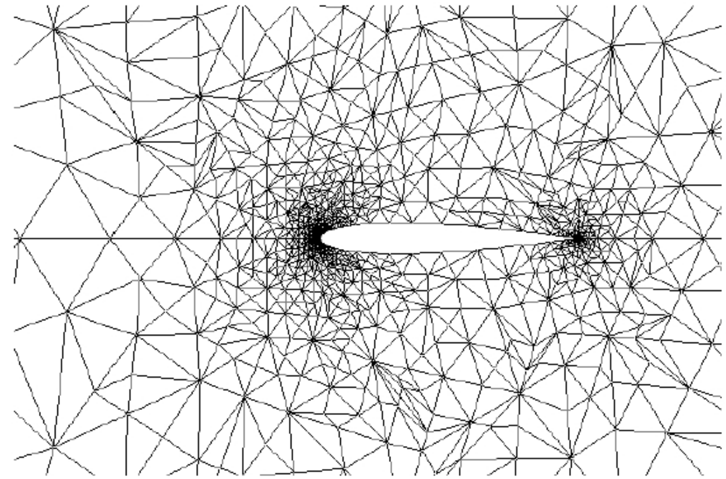


Subsonic Flow Over a NACA 0012 Airfoil

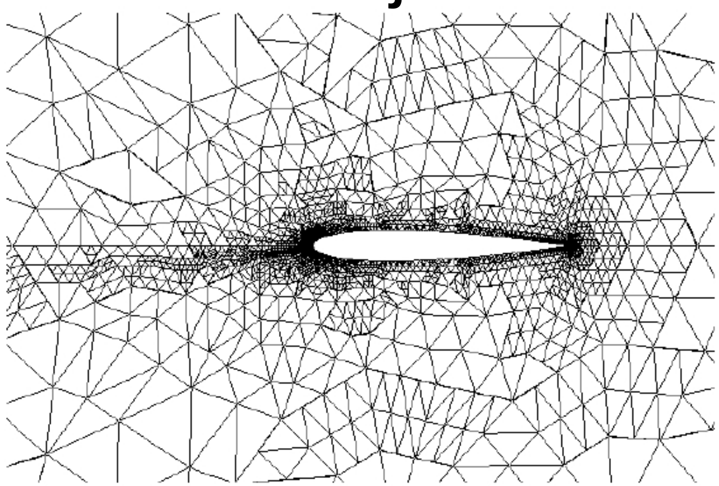




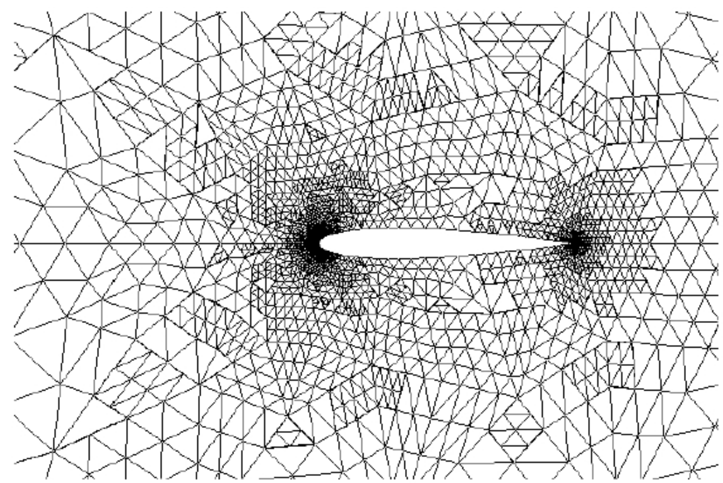
Aniso lift adjoint



Aniso drag adjoint



Iso lift adjoint

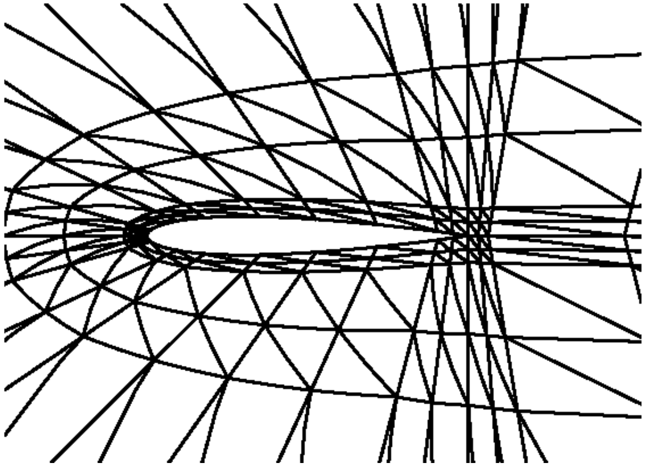


Iso drag adjoint

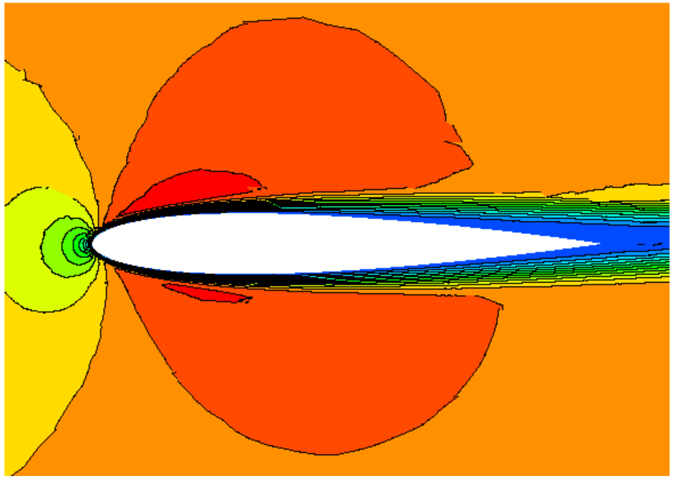
NACA 0012, $\mathbf{M_\infty = 0.5, \alpha = 1°}$, Re=5000

- Iso/aniso H-adaptation
- Fixed fraction f = 0.1
- Inviscid, $M_\infty = 0.5$, $\alpha = 2°$
- 4th order scheme
- 'Truth' CL, CD from 1st HOW

Adaptation strategies
- Hanging nodes
- Error indicators
  - drag adjoint



Initial mesh

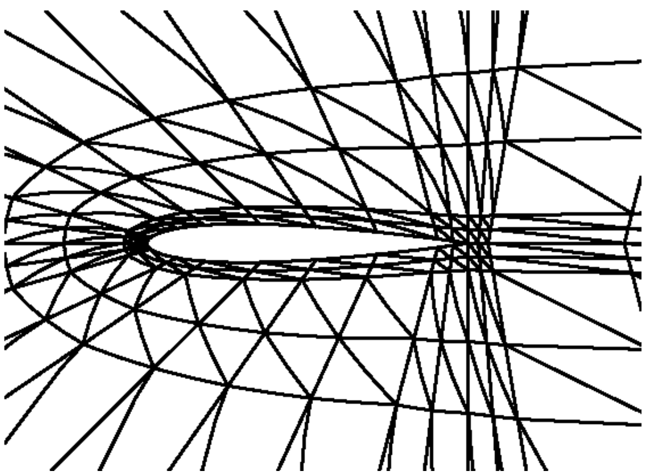


Initial solution

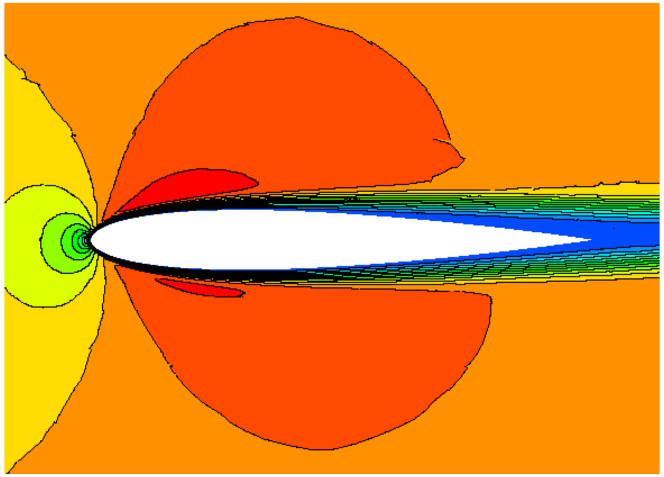
NACA 0012, $\mathbf{M}_\infty = \mathbf{0.5}, \boldsymbol{\alpha} = \mathbf{1°}$, Re=5000

- Iso/aniso H-adaptation
- Fixed fraction f = 0.1
- Inviscid, $M_\infty = 0.5$, $\alpha = 2°$
- 4th order scheme
- 'Truth' CL, CD from 1st HOW

Adaptation strategies
- Hanging nodes
- Error indicators
  - drag adjoint



Initial mesh



Adapted solution

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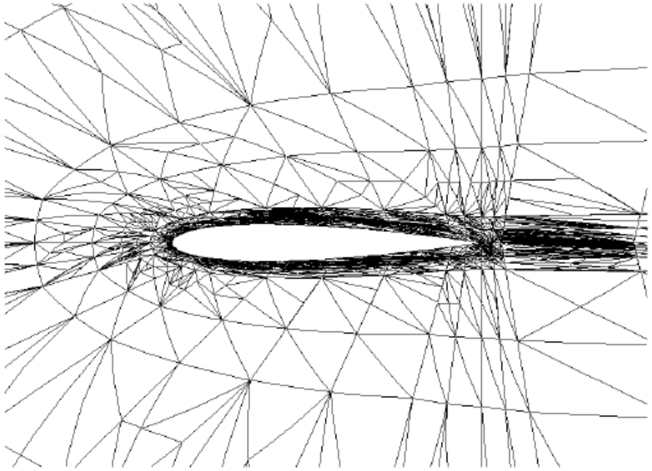
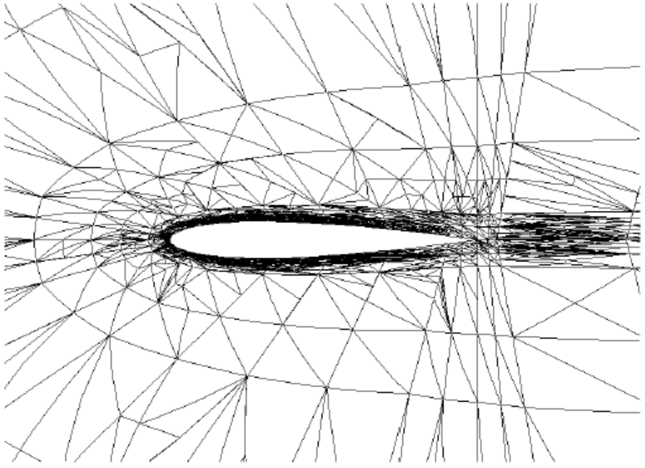
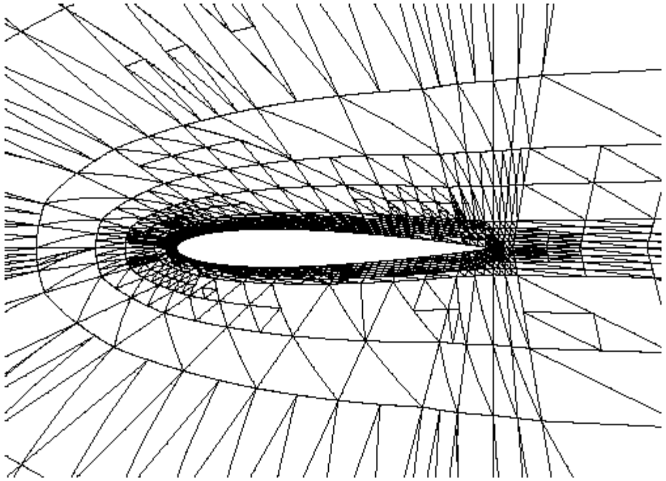


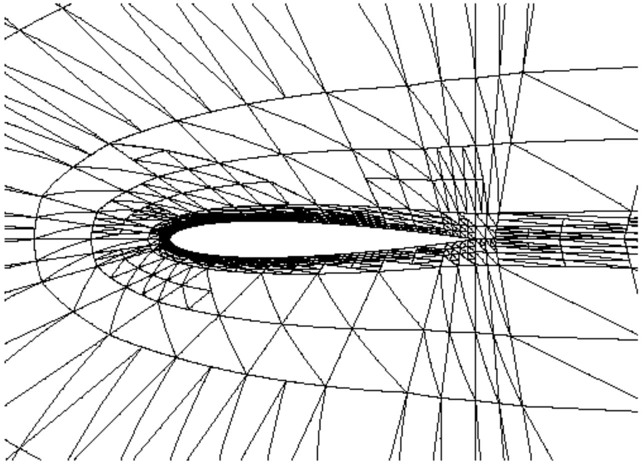
Aniso lift adjoint



Aniso drag adjoint

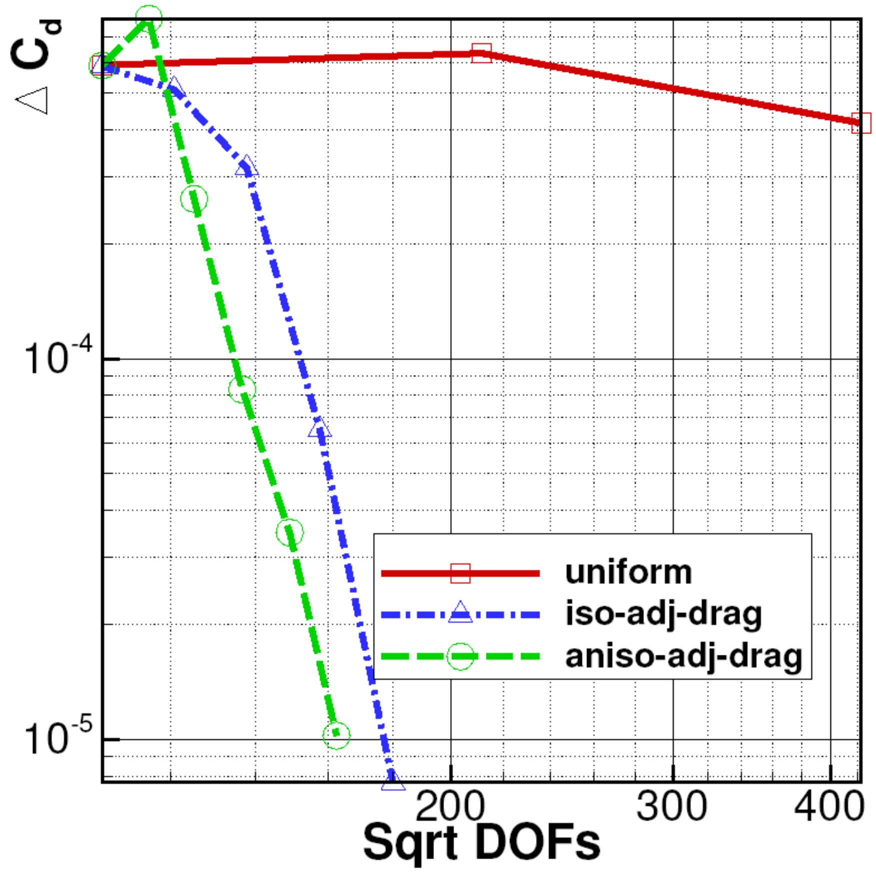


Iso lift adjoint



Iso drag adjoint

NACA 0012, $M_\infty = 0.5, \alpha = 1°$, Re=5000



Conclusions

- A dual consistent adjoint approach developed for the CPR method
- Isotropic/anisotropic h-adaptation method on simplex meshes using the CPR method is developed
- Results show significant savings of degrees of freedom when compared to the uniform h or p refinements
- Ongoing work:
  - Add the artificial viscosity to handle shocks
  - Further test the performance of the current method for viscous flow

Acknowledgements

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