Infinitely many sign-changing and semi-nodal solutions for a nonlinear Schrödinger system

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(joint work with Prof. Chang-Shou Lin and Wenming Zou)

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Outline

1. Introduction of the problem
2. Main results
3. Main ideas of the proof

The problem

Consider the following coupled Gross-Pitaevskii equations:

$$\begin{cases} -i\frac{\partial}{\partial t}\Phi_1 = \Delta\Phi_1 + \mu_1|\Phi_1|^2\Phi_1 + \beta|\Phi_2|^2\Phi_1, & x \in \Omega,\ t > 0, \\ -i\frac{\partial}{\partial t}\Phi_2 = \Delta\Phi_2 + \mu_2|\Phi_2|^2\Phi_2 + \beta|\Phi_1|^2\Phi_2, & x \in \Omega,\ t > 0, \\ \Phi_j = \Phi_j(x,t) \in \mathbb{C}, \quad j = 1,2, \\ \Phi_j(x,t) = 0, \quad x \in \partial\Omega,\ t > 0,\ j = 1,2, \end{cases} \tag{1}$$

where $\Omega = \mathbb{R}^N$ or $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \leq 3$, $i = \sqrt{-1}$ is the imaginary unit, $\mu_1, \mu_2 > 0$ and $\beta \neq 0$ is a coupling constant.

Motivation from Physics

- System (1) arises as mathematical models from several physical problems, such as nonlinear optics and the Hartree-Fock theory for a double condensate, i.e., a binary mixture of Bose-Einstein condensates.
- In physics, the sign of μ_j represents the self-interactions of the single j-th component. If $\mu_j > 0$ as considered here, it is called the focusing case, in opposition to the defocusing case where $\mu_j < 0$.
- The sign of β determines whether the interactions between the two components are repulsive or attractive, i.e., the interaction is attractive if $\beta > 0$, and the interaction is repulsive if $\beta < 0$.

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Solitary wave solutions

- To obtain solitary wave solutions of the system (1), we set $\Phi_j(x,t) = e^{i\lambda_j t}u_j(x)$ for $j = 1, 2$. Then system (1) is reduced to the following elliptic system:

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2, & x \in \Omega, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2, & x \in \Omega, \\ u_1|_{\partial\Omega} = u_2|_{\partial\Omega} = 0. \end{cases} \tag{2}$$

- When $\Omega = \mathbb{R}^N$, the Dirichlet boundary condition $u_1|_{\partial\Omega} = u_2|_{\partial\Omega} = 0$ means

$$u_1(x) \to 0 \text{ and } u_2(x) \to 0, \quad \text{as } |x| \to \infty,$$

and we assume $\lambda_1, \lambda_2 > 0$; When Ω is a smooth bounded domain, we let $\lambda_1(\Omega)$ be the first eigenvalue of $-\Delta$ in Ω with Dirichlet boundary condition, and assume $\lambda_1, \lambda_2 > -\lambda_1(\Omega)$. Then operators $-\Delta + \lambda_j$ are positive definite.

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Different kinds of solutions

Definition

We call a solution (u_1, u_2) *nontrivial* if $u_j \not\equiv 0$ for $j = 1, 2$, a solution (u_1, u_2) *semi-trivial* if (u_1, u_2) is type of $(u_1, 0)$ or $(0, u_2)$. A solution (u_1, u_2) is called *positive* if $u_j > 0$ in Ω for $j = 1, 2$, a solution (u_1, u_2) *sign-changing* if both u_1 and u_2 change sign, a solution *semi-nodal* if one component is positive and the other one changes sign.

Remark

Problem (2) has two kinds of semi-trivial solutions $(\omega_1, 0)$ and $(0, \omega_2)$, where ω_i are nontrivial solutions of the scalar equation

$$-\Delta u + \lambda_i u = \mu_i u^3, \quad u \in H_0^1(\Omega). \tag{3}$$

We are only concerned with nontrivial solutions of problem (2). The existence of infinitely many semi-trivial solutions makes the study of nontrivial solutions rather complicated and delicate.

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A more general system

There is a more general k-coupled system ($k \geq 3$), namely

$$\begin{cases} -\Delta u_j + \lambda_j u_j = \mu_j u_j^3 + \sum_{i \neq j} \beta_{ij} u_i^2 u_j, & \text{in } \Omega, \\ u_j|_{\partial\Omega} = 0, \quad j = 1, \cdots, k. \end{cases} \tag{4}$$

Here $\beta_{ij} = \beta_{ji}$. In general, system (4) is more delicate than system (2).

In the last decades, systems (2) and (4) (both in the focusing case and defocusing case) have received ever-increasing interest and have been studied intensively from many aspects, such as ground state (or least energy) solutions, multiple solutions, a priori estimates, asymptotic behaviors and phase seperations and so on. See papers such as

Conti-Terracini-Verzini(ANIHPC2002, JFA2003), Chang-Lin-Lin-Lin(PhysD2004), Lin-Wei(CMP2005, ANIHPC2005, JDE2006), Bartsch-Wang(JPDE2006), Ambrosetti-Colorado(JLMS2007), Sirakov(CMP2007), Bartsch-Wang-Wei (JFPTA-2007), Liu-Wang(CMP2008, ANS2010), Wei-Weth(ARMA2008, Nonlinearity2008), Terracini-Verzini(ARMA2009), Dancer-Wei(TAMS2009), Dancer-Wei-Weth(ANIHPC2010), Noris-Ramos(PAMS2010), Bartsch-Dancer-Wang(CVPDE2010), Noris-Tavares-Terracini-Verzini(CPAM2010, JEMS2012), Tavares-Terracini-Verzini-Weth(CPDE2011), Ikoma-Tanaka(CVPDE2011), Tavares-Terracini(CVPDE2012, ANIHPC2012), Chen-Zou(ARMA2012, CVPDE2012), Wei-Yao(CPAA2012), Dancer-Wang-Zhang(JFA2012), Quittner-Souplet(CMP2012), Sato-Wang(ANIHPC2013),

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Multiple positive solutions under symmetric conditions

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- Dancer, Wei and Weth (ANIHPC2010) proved the existence of infinitely many positive solutions of (2), provided $\beta \le -\mu$. The assumption $\beta \le -\mu$ is sharp due to a priori estimates for $\beta > -\sqrt{\mu_1\mu_2}$:

$$\|u_1\|_{L^\infty(\Omega)} + \|u_2\|_{L^\infty(\Omega)} \le C(\beta), \text{ for all positive solutions } (u_1, u_2).$$

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- If Ω is a ball or $\mathbb{R}^N$ (i.e., radially symmetric), infinitely many positive radial solutions, whose profiles are closely related to sign-changing solutions of the scalar equation $-\Delta u + \lambda u = \mu u^3$, were also obtained. See Wei-Weth (ARMA2008) for sysem (2), where they proved: Let Ω be a ball and give $k \in \mathbb{N}$, then for each $\beta \leq -\mu$, (2) has a positive radial solution (u_β, v_β) such that $u_\beta - v_\beta$ changes sign precisely k times. Moreover, letting $\beta \to -\infty$, $u_\beta \to W^+$ and $v_\beta \to W^-$ in $H_r(\Omega) \cap C(\overline{\Omega})$, where $W^\pm = \max\{0, \pm W\}$ and W is a radial sign-changing solution of $-\Delta u + \lambda u = \mu u^3$ which changes sign precisely k times.
- See also Terracini-Verzini(ARMA2009) for more general results for k-coupled system (4), and Bartsch-Dancer-Wang (CVPDE2010) for sysem (2) without assumption $\mu_1 = \mu_2$ (An idea of global bifurcation).

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An open question about Multiple positive solutions

- An open question: For $\beta \leq -\sqrt{\mu_1\mu_2}$, whether (2) has infinitely many positive solutions without symmetric assumptions $\lambda_1 = \lambda_2$ or $\mu_1 = \mu_2$?
- A recent progress: Guo and Wei(2013) give an positive answer in the case where $N = 2$, Ω is a bounded domain and $\mu_1 = \mu_2 = \mu$, but without assumption $\lambda_1 = \lambda_2$. An idea of perturbation:

$$\begin{cases} -\Delta u_1 + \lambda u_1 = \mu u_1^3 + \beta u_1 u_2^2 + \varepsilon u_1, & x \in \Omega, \\ -\Delta u_2 + \lambda u_2 = \mu u_2^3 + \beta u_1^2 u_2 - \varepsilon u_2, & x \in \Omega, \\ u_1|_{\partial\Omega} = u_2|_{\partial\Omega} = 0. \end{cases}$$

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Multiple nontrivial solutions without symmetric condition

If we do not assume symmetric conditions $\lambda_1 = \lambda_2$ or $\mu_1 = \mu_2$, multiple nontrivial solutions can also be obtained.

- Liu and Wang (CMP2008) obtained finite multiple nontrivial solutions provided $\beta > 0$ small or large, i.e., given $k \in \mathbb{N}$, there exists $0 < \beta_k < \beta'_k$ such that (2) has at least k nontrivial solutions for $0 < \beta < \beta_k$ or $\beta > \beta'_k$. Infinitely many nontrivial solutions for $\beta < 0$ were also obtained by Liu and Wang (ANS2010). *No information about the sign of these solutions*.
- Sato and Wang (ANIHPC2013) obtained infinitely many semi-positive solutions (i.e., at least one component is positive) for $\beta < 0$ and finite multiple semi-positive solutions for $\beta > 0$ small or large. *Whether the other component positive or sign-changing is unknown*.

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Multiple sign-changing or semi-nodal solutions

There are few results about the existence of sign-changing or semi-nodal solutions to (2) in the literature.

- When $\Omega = \mathbb{R}^N$, radially symmetric sign-changing solutions with a prescribed number of zeros were proved either for $\beta > 0$ large (Maia-Montefusco-Pellacci (CCM2008)) or for $\beta > 0$ small (Kim-Kwon-Lee (NA2013)), i.e., given $h, k \in \mathbb{N} \cup \{0\}$, there exist $0 < \beta_{hk} < \beta'_{hk}$ such that for $0 < \beta < \beta_{hk}$ or $\beta > \beta'_{hk}$, (2) has a radially symmetric solution (u_1, u_2) with u_1 changing sign precisely h times and u_2 changing sign precisely k times. *Clearly their methods can not be applied to obtain multiple sign-changing solutions for the non-radial bounded domain case.*

Sign-changing solutions in the defocusing case $\mu_j < 0$

Tavares and Terracini (ANIHPC2012) studied the following general k-coupled system in the defocusing case

$$\begin{cases} -\Delta u_j - \mu_j u_j^3 - \beta u_j \sum_{i \neq j} u_i^2 = \lambda_j u_j, & \text{in } \Omega, \\ u_j \in H_0^1(\Omega), \quad \int_\Omega u_j^2 \, dx = 1, \quad j = 1, \cdots, k, \end{cases} \tag{5}$$

where Ω is a smooth bounded domain, and $\beta < 0$, $\mu_j \leq 0$ are all fixed constants. They proved that there exist infinitely many $\lambda = (\lambda_1, \cdots, \lambda_k) \in \mathbb{R}^m$ and $u = (u_1, \cdots, u_k) \in H_0^1(\Omega, \mathbb{R}^m)$ such that (u, λ) are sign-changing solutions of (5). That is, λ_j is not fixed *a priori* but appears as a Lagrange multiplier. Therefore, *problem (5) is completely different from problem (2) we consider here, for which, we deal with the focusing case $\mu_j > 0$, and λ_j, μ_j, β are all fixed constants*.

Natural questions

For problem (2), since the related functional is even both for u_1 and u_2, it is natural to suspect that (2) may have sign-changing solutions. So far, there are two natural questions about sign-changing solutions which seem to be still open.

1. When $\beta < 0$, whether (2) has infinitely many sign-changing solutions for both the entire space case and the bounded domain case?
2. When $\beta > 0$, whether (2) has multiple sign-changing solutions for the non-radial bounded domain case?

We can ask similar questions about semi-nodal solutions.

The repulsive case $\beta < 0$: Sign-changing solutions

Theorem 1(C-Lin-Zou2012)

Let $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and $\beta < 0$. Then (2) has infinitely many sign-changing solutions $(u_{n,1}, u_{n,2})$ such that

$$\|u_{n,1}\|_{L^\infty(\Omega)} + \|u_{n,2}\|_{L^\infty(\Omega)} \to +\infty \quad \text{as} \quad n \to +\infty.$$

The repulsive case $\beta < 0$: Sign-changing solutions

Definition

A nontrivial solution is called *a least energy solution*, if its functional energy is minimal among all nontrivial solutions. A sign-changing solution is called *a least energy sign-changing solution*, if its functional energy is minimal among all sign-changing solutions.

For $-\infty < \beta < \beta_0$ where β_0 is a positive constant, Lin and Wei (ANIHPC2005) proved that (2) has a least energy solution which turns out to be a positive solution. See Sirakov (CMP2007) for large β.

Theorem 2(C-Lin-Zou2012)

Under the same assumptions in Theorem 1, problem (2) has a least energy sign-changing solution (u_1, u_2). Moreover, both u_1 and u_2 have exactly two nodal domains.

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The repulsive case $\beta < 0$: Semi-nodal solutions

Theorem 3(C-Lin-Zou2012)

Under the same assumptions in Theorem 1, problem (2) has infinitely many semi-nodal solutions $\{(u_{n,1}, u_{n,2})\}_{n\geq 2}$ such that

1. $u_{n,1}$ changes sign and $u_{n,2}$ is positive;
2. $\|u_{n,1}\|_{L^\infty(\Omega)} + \|u_{n,2}\|_{L^\infty(\Omega)} \to +\infty$ as $n \to +\infty$;
3. $u_{n,1}$ has at most n nodal domains. In particular, $u_{2,1}$ has exactly two nodal domains, and $(u_{2,1}, u_{2,2})$ has the least functional energy among all nontrivial solutions whose first component changes sign.

Remark

Similarly, we can prove that (2) has infinitely many semi-nodal solutions $\{(v_{n,1}, v_{n,2})\}_{n\geq 2}$ such that $v_{n,1}$ is positive, $v_{n,2}$ changes sign and has at most n nodal domains. In the symmetric case where $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$, $(u_{n,1}, u_{n,2})$ obtained in Theorem 3 and $(v_{n,1}, v_{n,2})$ may be the same solution in the sense of $u_{n,1} = v_{n,2}$ and $u_{n,2} = v_{n,1}$. However, if either $\lambda_1 \neq \lambda_2$ or $\mu_1 \neq \mu_2$, then $(u_{n,1}, u_{n,2})$ and $(v_{n,1}, v_{n,2})$ are really different solutions.

Z. Chen, C.-S. Lin and W. Zou, Infinitely many sign-changing and semi-nodal solutions for a nonlinear Schrödinger system, arXiv: 1212.3773v1 [math.AP].

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Consider the general k-coupled system (4)

$$\begin{cases} -\Delta u_j + \lambda_j u_j = \mu_j u_j^3 + \sum_{i \neq j} \beta_{ij} u_i^2 u_j, & \text{in } \Omega, \\ u_j|_{\partial\Omega} = 0, \quad j = 1, \cdots, k. \end{cases}$$

Remark

We did not consider this general k-coupled system in our paper. However, by using the same ideas, in the repulsive case where $\beta_{ij} < 0$ for any $i \neq j$, similar results as the three theorems above hold for the general k-system (4) with $k \geq 3$. That is, for any fixed $1 \leq m \leq k$, the general k-coupled system has infinitely many nontrivial solutions $(u_{1,n}, \cdots, u_{m,n}, u_{m+1,n}, \cdots, u_{k,n})$ with the first m components $u_{j,n}$, $1 \leq j \leq m$, sign-changing and the rest $k - m$ components positive.

The attractive case $\beta > 0$

The attractive case $\beta > 0$ is different from the repulsive case $\beta < 0$, and here we can only obtain finite multiple solutions.

Theorem 4(C-Lin-Zou, JDE2013)

Let $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. Then for any $k \in \mathbb{N}$ there exists $\beta_k > 0$ such that for each fixed $\beta \in (0, \beta_k)$, system (2) has at least k sign-changing solutions and k semi-nodal solutions with the first component sign-changing and the second component positive.

Theorem 5(C-Lin-Zou, JDE2013)

Let $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. Then there exists $\beta_1' \in (0, \beta_1]$ such that system (2) has a least energy sign-changing solution for each $\beta \in (0, \beta_1')$.

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The entire space case $\Omega = \mathbb{R}^N$

Remark

Theorems 1-5 are all stated in the bounded domain case. Let $\Omega = \mathbb{R}^N$ with $N = 2, 3$. Then by working in the radial function space $H^1_r(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) \ : \ u \text{ is radially symmetric}\}$ and recalling the compactness of $H^1_r(\mathbb{R}^N) \hookrightarrow L^4(\mathbb{R}^N)$, all the existence results above also hold via the same proof. The main difference is that, in the case $\Omega = \mathbb{R}^N$, all sign-changing and semi-nodal solutions are radially symmetric, and the least energy sign-changing solution is only in the sense of having the least energy among all radially symmetric sign-changing solutions.

Remarks

Open problem 1

From results of Dancer, Wei and Weth(ANIHPC2010), the sharp range is $(-\infty, -\sqrt{\mu_1\mu_2})$ when seeking infinitely many positive solutions. However, we do not know what is the sharp range of β when seeking infinitely many sign-changing or semi-nodal solutions. This seems to be a challenging open problem since, in general, we can not obtain a priori estimates for sign-changing and semi-nodal solutions.

Open problem 2

For the non-radial bounded domain case, we can only obtain multiple sign-changing and semi-nodal solutions for $\beta > 0$ small. Whether sign-changing or semi-nodal solutions exist or not for $\beta > 0$ large remains open. Different ideas are needed, since from the next result, we know that our method does not work for all $\beta > 0$.

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An nonexistence result of semi-nodal solutions

Theorem 6

Let $N \leq 3$ and (u_1, u_2) be a nontrivial solution of

$$\begin{cases} -\Delta u_1 + \lambda u_1 = \mu u_1^3 + \mu u_1 u_2^2, & x \in \mathbb{R}^N, \\ -\Delta u_2 + \lambda u_2 = \mu u_2^3 + \mu u_1^2 u_2, & x \in \mathbb{R}^N, \\ u_1, u_2 \in H^1(\mathbb{R}^N) \end{cases} \tag{6}$$

with $u_1 > 0$. Then $u_2 = Cu_1$ for some constant $C \neq 0$. In particular, (6) has no semi-nodal solutions.

Remark

When $N = 1$ and u_1, u_2 are both positive, this result $u_2 = Cu_1$ has been proved by Wei and Yao (CPAA2012).

An nonexistence result of semi-nodal solutions

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An open question

Since

$$\int_{\Omega} u_1 u_2 \left[(\lambda_2 - \lambda_1) + (\mu_1 - \beta) u_1^2 + (\beta - \mu_2) u_2^2 \right] \, dx = 0,$$

when $\lambda_1 \leq \lambda_2$ and $\mu_1 \geq \beta \geq \mu_2$ and either $\mu_1 > \mu_2$ or $\lambda_1 < \lambda_2$, system (2) has no nontrivial positive solutions.

An large open question

when $\lambda_1 \leq \lambda_2$ and $\mu_1 \geq \beta \geq \mu_2$ and either $\mu_1 > \mu_2$ or $\lambda_1 < \lambda_2$, does system (2) have nontrivial solutions? So far, this question is completely unknown. Clearly, if exists, they must be sign-changing or semi-nodal.

Main difficulty

1. Problem (2) has infinitely many semi-trivial solutions $(u_{1,n}, 0)$ and $(0, u_{2,n})$, where $u_{i,n}$ are sign-changing solutions of the scalar equation $-\Delta u + \lambda_i u = \mu_i u^3$. We have to eliminate all these solutions when seeking sign-changing solutions.
2. In the case $\beta < -\sqrt{\mu_1 \mu_2}$, we already know that problem (2) may have infinitely many positive solutions $(u_{1,n}, u_{2,n})$ (see Dancer-Wei-Weth(ANIHPC2009) for example). We also have to eliminate all these solutions when seeking sign-changing solutions.
3. We need to give a proper variational framework to overcome these difficulties.

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Main ideas

1. Our proof is mainly inspired by Tavares and Terracini (ANIHPC2012), where a new notion of *vector genus* introduced by them will be used to define appropriate minimax values.
2. To obtain nontrivial solutions of (2), the first step is turning to study a new problem J with two constraints. *This idea, which seems new to problem (2), is crucial in our proof*. Then we define a sequence of minimax values of J by using vector genus. Here, in order to obtain sign-changing solutions, we also need to use cones of positive/negative functions as in some previous papers (such as Conti, Merizzi and Terracini (NoDEA1999)), by which, these minimax values are actually sign-changing critical values.

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Previous ideas

The energy functional $E : H := H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$

$$E(u_1, u_2) = \sum_{i=1}^{2} \int_\Omega \left[\frac{1}{2}(|\nabla u_i|^2 + \lambda_i u_i^2) - \frac{\mu_i}{4} u_i^4 \right] - \frac{\beta}{2} \int_\Omega u_1^2 u_2^2.$$

To obtain nontrivial solutions of (2), in many papers (see Lin-Wei (CMP2005) and Sirakov (CMP2007) for example), people usually turn to study nontrivial critical points of E under the following Nehari manifold type constraint

$$\mathcal{N} := \left\{ (u_1, u_2) \in H \ : \ u_i \neq 0, E'(u_1, u_2)(u_1, 0) = E'(u_1, u_2)(0, u_2) = 0 \right\}.$$

Clearly, all nontrivial solutions belong to $\mathcal{N}$. Besides, all critical points of $E|_{\mathcal{N}}$ are nontrivial solutions of (2), provided $\beta < \sqrt{\mu_1 \mu_2}$.

New idea

Let $\beta < 0$. Define a new functional $J : H \to \mathbb{R} \cup \{+\infty\}$ by

$$J(u_1, u_2) := \max_{t_1, t_2 > 0} E(t_1 u_1, t_2 u_2)$$
$$= \frac{1}{4} \frac{\mu_2 |u_2|_4^4 \|u_1\|_{\lambda_1}^4 + 2|\beta| \|u_1\|_{\lambda_1}^2 \|u_2\|_{\lambda_2}^2 \int_\Omega u_1^2 u_2^2 + \mu_1 |u_1|_4^4 \|u_2\|_{\lambda_2}^4}{[\mu_1 \mu_2 |u_1|_4^4 |u_2|_4^4 - |\beta|^2 (\int_\Omega u_1^2 u_2^2)^2]_+}.$$

Define a set with two constraints

$$\mathcal{M} := \left\{ (u_1, u_2) \in H \ : \ \int_\Omega |u_1|^4 = \int_\Omega |u_2|^4 = 1, \ \mu_1 \mu_2 - |\beta|^2 \left(\int_\Omega u_1^2 u_2^2 \right)^2 > 0 \right\}.$$

The crucial observation: Clealy any critical point of $J|_{\mathcal{M}}$ is not a solution of (2). However, it can turn to be a *nontrivial* solution under a proper transformation.

New idea

Precisely, let $\vec{u} = (u_1, u_2) \in \mathcal{M}$ be a critical point of $J|_{\mathcal{M}}$, then there exists unique $t_1, t_2 > 0$ such that

$$E(t_1 u_1, t_2 u_2) = \max_{s_1, s_2 > 0} E(s_1 u_1, s_2 u_2) = J(u_1, u_2).$$

Then $(t_1 u_1, t_2 u_2)$ is a nontrivial critical point of E and so a nontrivial solution of (2).

Hence, to obtain nontrivial solutions of (2), it suffices to study $J|_{\mathcal{M}}$, a problem with two constraints. Somewhat surprisingly, up to our knowledge, *this natural idea has never been used for (2) in the literature*.

Cones of positive/negative functions

Define cones of positive/negative functions by

$$\mathcal{P}_i := \{\vec{u} = (u_1, u_2) \in H : u_i \geq 0\}, \quad \mathcal{P} := \bigcup_{i=1}^{2} (\mathcal{P}_i \cup -\mathcal{P}_i). \tag{7}$$

Define $\mathcal{P}_\delta := \{\vec{u} \in H \,:\, \mathrm{dist}_4(\vec{u}, \mathcal{P}) < \delta\}$ as neighborhoods of $\mathcal{P}$, where

$$\mathrm{dist}_4(\vec{u}, \mathcal{P}) := \min\left\{\mathrm{dist}_4(u_i, \mathcal{P}_i),\ \mathrm{dist}_4(u_i, -\mathcal{P}_i), \quad i = 1, 2\right\}, \tag{8}$$

$$\mathrm{dist}_4(u_i, \pm\mathcal{P}_i) := \inf\left\{ |u_i - v|_4 := \left(\int_\Omega |u_i - v|^4\right)^{1/4} \,:\, v \in \pm\mathcal{P}_i \right\}.$$

Denote $u^\pm := \max\{0, \pm u\}$, then $\mathrm{dist}_4(u_i, \mathcal{P}_i) = |u_i^-|_4$. That is, $\vec{u} = (u_1, u_2)$ satisfies $\mathrm{dist}_4(\vec{u}, \mathcal{P}) > 0$ if and only if both u_1 and u_2 change sign. Therefore, we will seek nontrivial solutions outside of the cone $\mathcal{P}$.

Vector genus

Define the transformations

$$\sigma_i : H \to H \quad \text{by} \quad \sigma_1(u_1, u_2) := (-u_1, u_2),\;\; \sigma_2(u_1, u_2) := (u_1, -u_2).$$

We consider the class of invariant sets

$$\mathcal{F} = \{A \subset \mathcal{M} : A \text{ is closed and } \sigma_i(\vec{u}) \in A \;\forall\, \vec{u} \in A,\;\; i = 1, 2\},$$

and for any integers $k_1, k_2 \geq 2$, we denote

$$\Gamma^{(k_1,k_2)} := \{A \in \mathcal{F} : \vec{\gamma}(A) \geq (k_1, k_2)\}.$$

Here, the definition of vector genus $\vec{\gamma}$ is seen in Tavares and Terracini(ANIHPC2012).

Sign-changing minimax values

Lemma 1

For any $\delta < 2^{-1/4}$ and any $A \in \Gamma^{(k_1,k_2)}$ there holds $A \setminus \mathcal{P}_\delta \neq \emptyset$. Furthermore, there exist $A \in \Gamma^{(k_1,k_2)}$ such that $c^{k_1,k_2} := \sup_A J < +\infty$.

For every $k_1, k_2 \geq 2$ and $0 < \delta < 2^{-1/4}$, we define

$$c_\delta^{k_1,k_2} := \inf_{A \in \Gamma_0^{(k_1,k_2)}} \sup_{\vec{u} \in A \setminus \mathcal{P}_\delta} J(\vec{u}), \quad \text{where} \tag{9}$$

$$\Gamma_0^{(k_1,k_2)} := \left\{ A \in \Gamma^{(k_1,k_2)} \ : \ \sup_A J < c^{k_1,k_2} + 1 \right\}. \tag{10}$$

It suffices to prove that $c_\delta^{k_1,k_2}$ is a sign-changing critical value of $J|_{\mathcal{M}}$ provided that $\delta > 0$ is sufficiently small.

Remark

In order to prove that $c_\delta^{k_1,k_2}$ is a sign-changing critical value, we need to seek a decreasing deformation flow $\eta : [0,+\infty) \times \mathcal{M} \to \mathcal{M}$ such that for $\delta > 0$ small enough,

$$\eta(t,\vec{u}) \in \mathcal{P}_\delta \text{ whenever } u \in \mathcal{M} \cap \mathcal{P}_\delta,\ J(u) \le c^{k_1,k_2} + 1,\ t > 0. \quad (11)$$

This property is crucial to guarantee that $c_\delta^{k_1,k_2}$ is a *sign-changing* critical value. Remark that (11) may not hold without restriction $J(u) \le c^{k_1,k_2} + 1$ (a uniform bound). Hence, *in the definition of $c_\delta^{k_1,k_2}$, it does not seem that we could replace $\Gamma_0^{(k_1,k_2)}$ either by $\Gamma^{(k_1,k_2)}$ or by $\widetilde{\Gamma}^{(k_1,k_2)}$*, where

$$\widetilde{\Gamma}^{(k_1,k_2)} := \left\{ A \in \Gamma^{(k_1,k_2)} \ : \ \sup_A J < +\infty \right\}.$$

Least energy sign-changing solutions

Remark the first minimax value $c_\delta^{2,2}$ is precisely the least energy level that corresponds to the least energy sign-changing solutions. To see this, let us define

$$\tilde{c} := \inf_{\vec{u}\in\mathcal{S}} E(\vec{u}), \quad \text{where}$$

$$\mathcal{S} := \Big\{\vec{u} = (u_1, u_2) \in H \ : \ \text{both } u_1 \text{ and } u_2 \text{ change sign,}$$
$$E'(\vec{u})(u_1^{\pm}, 0) = 0,\ E'(\vec{u})(0, u_2^{\pm}) = 0\Big\}.$$

Then any sign-changing solutions belong to $\mathcal{S}$. We can prove $\tilde{c} = c_\delta^{2,2}$, and so a sign-changing critical point of $c_\delta^{2,2}$ must be a least energy sign-changing solution.

Remark

By using minimizing skills, one can prove the existence of a minimizer $(u_1, u_2) \in \mathcal{S}$ such that $E(u_1, u_2) = \tilde{c}$ directly. However, *it seems very difficult to prove that such a minimizer (u_1, u_2) is critical point of E*. For example, since the operators $u \in H_0^1(\Omega) \mapsto \int_\Omega |\nabla(u^{\pm})|^2\, dx$ are not C^1, the method of Lagrange multipliers, which is very powerful in obtaining least energy solutions, does not apply here; besides, since problem (2) is a system and $\mathcal{S}$ has four constraints, previous ideas, which are used for scalar equations to obtain least energy sign-changing solutions, do not seem to work here either. Here we do not need to use minimizing skills.

Thank you for your attention!