Asymptotic analysis of solutions of the Chern-Simons $CP(1)$ model on a torus

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Outline

History

Main Theorem I: construction of stable bubbling solutions

Main Theorem II: asymptotic behavior of mountain pass solutions

Main Theorem III: construction of (unstable) solutions blowing up at one point

Chern-Simons-Higgs (CSH) model

$$\Delta u_\varepsilon + \frac{1}{\varepsilon^2} e^{u_\varepsilon}(1 - e^{u_\varepsilon}) = 4\pi \sum_{j=1}^{d} m_j \delta_{p_j} \text{ on } \mathbb{T} \quad (CSH).$$

• $\mathbb{T}$: flat 2-torus.

• p_j : the vortex point.

• $m_j \in \mathbb{N}$: the mass of vortex point.

• δ_p : the Dirac measure concentrated at p
($\delta_p(x) = 0$ if $x \neq p$, $\int_{\mathbb{T}} \psi(x)\delta_p(x)dx = \psi(p)$).

• $\varepsilon > 0$: the coupling constant.

• [J. Hong, Y. Kim, P.Y. Pac/ R. Jackiw, E.J. Weinberg (1990)]:
-(CSH) describes vortices in high temperature superconductivity.

Chern-Simons-Higgs (CSH) model

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- $u_\varepsilon(x) = 2m_j \ln|x - p_j| + O(1), \; x \to p_j.$
- $\lim_{x \to p_j} u_\varepsilon(x) = -\infty.$
- $u_\varepsilon < 0$ by the maximum principle.
- $\int_{\mathbb{T}} \frac{1}{\varepsilon^2} e^{u_\varepsilon}(1 - e^{u_\varepsilon}) dx = \int_{\mathbb{T}} \frac{1}{\varepsilon^2} |e^{u_\varepsilon}(1 - e^{u_\varepsilon})| dx = 4\pi \sum_{j=1}^{d} m_j.$

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Theorem (L.A. Caffarelli, Y.S. Yang(1995))

$\exists \varepsilon_* > 0$ *such that*

$$\begin{cases} \nexists \textit{ solution for } \varepsilon > \varepsilon_*, \\ \exists \textit{ solution for } \varepsilon \in (0, \varepsilon_*) - \textit{monotone scheme method.} \end{cases}$$

Chern-Simons-Higgs (CSH) model

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Theorem (K. Choe, N. Kim (2008): Brezis-Merle type alternatives)

One of the following holds true:

(i) $\lim_{\varepsilon\to 0}\left(\sup_K |u_\varepsilon|\right) = 0$, $\forall K \Subset \mathbb{T} \setminus \{p_1, ..., p_d\}$;

(ii) $\exists$ *a finite blow up points set* $S \neq \emptyset \subseteq \mathbb{T}$ *such that*

$$\frac{1}{\varepsilon^2} e^{u_\varepsilon}(1 - e^{u_\varepsilon}) \to \sum_{p \in S} \alpha_p \delta_p, \;\; \alpha_p \geq 8\pi;$$

(iii) $u_\varepsilon - 2\ln\varepsilon$ *is uniformly bounded in* $L^\infty_{loc}(\mathbb{T} \setminus \{p_1, ..., p_d\})$.

- [K. Choe/ M. Nolasco, G. Tarantello/ C.-S. Lin, S. Yan...]: study on blow up (bubbling) solutions.
- The other developments...

Uniqueness of topological solutions

Theorem (G. Tarantello (2007))

For small $\varepsilon > 0$, $\exists!$ topological solution of

$$\Delta u_\varepsilon + \frac{1}{\varepsilon^2} e^{u_\varepsilon}(1 - e^{u_\varepsilon}) = 4\pi \sum_{j=1}^{d} m_j \delta_{p_j} \text{ on } \mathbb{T} \quad (CSH).$$

- u_ε is called a topological solution if $u_\varepsilon \to 0$ a.e. in $\mathbb{T}$ as $\varepsilon \to 0$.

- u_ε is called a stable solution if the linearized equation of (CSH) at u_ε has nonnegative eigenvalues.

- A key point for proof: topological solution $\Rightarrow$ strictly stable solution.

- Question: Is the converse also true?

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- A key point for proof: topological solution $\Rightarrow$ strictly stable solution.

- Question: Is the converse also true?

Yes, [D. Bartolucci, L., C.-S. Lin, M. Onodera]:
stable solution $\Leftrightarrow$ topological solution.

Entire solution of (CSH) in $\mathbb{R}^2$

$$\Delta u_\varepsilon + \frac{1}{\varepsilon^2} e^{u_\varepsilon}(1 - e^{u_\varepsilon}) = 4\pi \sum_{j=1}^{d} m_j \delta_{p_j} \text{ on } \mathbb{T} \quad (CSH).$$

- By considering the scaled function $u_\varepsilon(\varepsilon \cdot + q)$ for $q \in \mathbb{T}$,

$$\begin{cases} \Delta u + e^u(1 - e^u) = 4\pi\nu\delta_0 \text{ in } \mathbb{R}^2, \quad \nu \in \mathbb{N}; \\ e^u(1 - e^u) \in L^1(\mathbb{R}^2). \end{cases}$$

1) topological solution: $\lim_{|x| \to +\infty} u(x) = 0$,

2) nontopological solution: $\lim_{|x| \to +\infty} u(x) = -\infty$.

Radially symmetric entire solutions of (CSH) in $\mathbb{R}^2$

- Let $u = u(r; s)$ be the unique solution of

$$\begin{cases} u'' + \frac{1}{r}u' + e^u(1 - e^u) = 0, & 0 < r < \infty, \quad (CSH) \\ u(r) = 2m \ln r + s + o(1) \quad \text{as} \quad r \to 0. \end{cases}$$

- $\hat{\varphi}(r) \equiv \frac{\partial u(r;s)}{\partial s}$, $\hat{f}(u) \equiv e^u(1 - e^u)$.

- $\hat{\varphi} = \frac{\partial u(\cdot;s)}{\partial s}$ satisfies the linearized equation of (CSH) at $u(r; s)$:

$$\begin{cases} \hat{\varphi}'' + \frac{1}{r}\hat{\varphi}' + \hat{f}'(u)\hat{\varphi} = 0, \\ \hat{\varphi}(0) = 1. \end{cases}$$

Radially symmetric entire solutions of (CSH) in $\mathbb{R}^2$

- $\hat{\varphi}'' + \frac{1}{r}\hat{\varphi}' + \hat{f}'(u)\hat{\varphi} = \frac{(r\hat{\varphi}')'}{r} + \hat{f}'(u)\hat{\varphi} = 0, \quad \hat{\varphi}(0) = 1.$

- $\hat{\varphi}(r) \equiv \frac{\partial u(r;s)}{\partial s}, \quad \hat{f}(u) \equiv e^u(1 - e^u), \quad \hat{\beta}(s) \equiv \int_0^\infty \hat{f}(u(r;s))rdr.$

- $\hat{\beta}'(s) = \int_0^\infty \hat{f}'(u)\frac{\partial u(r;s)}{\partial s}rdr = \int_0^\infty \hat{f}'(u)\hat{\varphi}rdr = -\lim_{r\to+\infty} r\hat{\varphi}'(r).$

- $\hat{\beta}'(s) = 0 \Leftrightarrow \hat{\varphi}$ is bounded in $\mathbb{R}^2$.

- If the linearized equation at $u(r;s)$ is non-degenerate, then $\hat{\beta}'(s) \neq 0$.

Stable entire solutions of (CSH)

- $\hat{\varphi}'' + \frac{1}{r}\hat{\varphi}' + \hat{f}'(u)\hat{\varphi} = \frac{(r\hat{\varphi}')'}{r} + \hat{f}'(u)\hat{\varphi} = 0, \quad \hat{\varphi}(0) = 1.$

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- $\hat{\beta}'(s) = \int_0^\infty \hat{f}'(u)\frac{\partial u(r;s)}{\partial s}rdr = \int_0^\infty \hat{f}'(u)\hat{\varphi}rdr = -\lim_{r\to+\infty} r\hat{\varphi}'(r).$

- The sign of $\hat{\beta}'(s)$ is related to the stability of $u(r;s)$ ($\hat{\beta}'(s) > 0$: unstable).

- u is called a stable entire solution of (CSH),

if the linearized equation of (CSH) at u has only nonnegative eigenvalues in any compact subset of $\mathbb{R}^2$.

Stable entire solutions of (CSH)

- [H. Chan, C.-C. Fu, C.-S. Lin, (2002)]:

On $\{s \mid u(r;s)$ is a non-topological solution of (CHS)$\}$, $\hat{\beta}'(s) > 0$ (unstable).

- Any non-topological radially symmetric entire solution of (CSH) is unstable.

- [D. Bartolucci, L., C.-S. Lin, M. Onodera]:

1) Any non-topological entire solution (including non-radially symmetric) of (CSH) is unstable.

2) Stable entire solution of (CSH) in $\mathbb{R}^2$
$\Leftrightarrow$ topological solution, $\lim_{|x|\to+\infty} u(x) = 0$.

3) Stable solution of (CSH) on a torus
$\Leftrightarrow$ topological solution, $u_\varepsilon \to 0$ a.e. in $\mathbb{T}$ as $\varepsilon \to 0$.

Our main equation

$$\Delta u_\varepsilon + \frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3} = 4\pi\sum_{j=1}^{d_1} m_{j,1}\delta_{p_{j,1}} - 4\pi\sum_{j=1}^{d_2} m_{j,2}\delta_{p_{j,2}} \text{ on } \mathbb{T} \quad (CP(1)).$$

• CP(1) model describes the planar ferromagnet and has rich vacuum and soliton structure.

• $\tau \in (0,\infty)$, $m_{j,i} \in \mathbb{N}$.

• $Z \equiv \{p_{j,i} \in \mathbb{T} \mid 1 \leq j \leq d_i \text{ and } i = 1,2\}$: the set of vortex points.

• $Z_i \equiv \{p_{j,i} \in \mathbb{T} \mid 1 \leq j \leq d_i\ \} \neq \emptyset$.

• $N_i \equiv \sum_{j=1}^{d_i} m_{j,i}$ for $i = 1,2$, $\quad N \equiv N_1 - N_2$.

Our main equation

$$\Delta u_\varepsilon + \frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3} = 4\pi\sum_{j=1}^{d_1} m_{j,1}\delta_{p_{j,1}} - 4\pi\sum_{j=1}^{d_2} m_{j,2}\delta_{p_{j,2}} \text{ on } \mathbb{T} \quad (CP(1)).$$

- $u_\varepsilon(x) = 2m_{j,1}\ln|x-p_{j,1}| + O(1),\ x \to p_j, 1.$
- $u_\varepsilon(x) = -2m_{j,2}\ln|x-p_{j,2}| + O(1),\ x \to p_j, 2.$
- $\lim_{x\to p_{j,1}} u_\varepsilon(x) = -\infty, \quad \lim_{x\to p_{j,2}} u_\varepsilon(x) = +\infty.$
- u_ε and $\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3}$ change sign.
- $\int_{\mathbb{T}} \frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3}dx = 4\pi N = 4\pi(N_1 - N_2).$

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- $\lim_{x\to p_{j,1}} u_\varepsilon(x) = -\infty, \quad \lim_{x\to p_{j,2}} u_\varepsilon(x) = +\infty.$
- u_ε and $\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3}$ change sign.
- $\int_{\mathbb{T}} \frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3}dx = 4\pi N = 4\pi(N_1 - N_2).$
- $\int_{\mathbb{T}} \left|\frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3}\right|dx < C$ for some constant $C > 0$.

Our main equation

$$\Delta u_\varepsilon + \frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3} = 4\pi\sum_{j=1}^{d_1} m_{j,1}\delta_{p_{j,1}} - 4\pi\sum_{j=1}^{d_2} m_{j,2}\delta_{p_{j,2}} \text{ on } \mathbb{T} \quad (CP(1)).$$

- The behavior of a topological solution of (CP(1)) is similar to (CSH): topological solution $\Rightarrow$ strictly stable solution.
- Question: stable solution $\Leftrightarrow$ topological solution ?

Entire solution of (CP(1)) in $\mathbb{R}^2$

$$\Delta u_\varepsilon + \frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3} = 4\pi\sum_{j=1}^{d_1} m_{j,1}\delta_{p_{j,1}} - 4\pi\sum_{j=1}^{d_2} m_{j,2}\delta_{p_{j,2}} \text{ on } \mathbb{T} \quad (CP(1)).$$

- By considering the scaled function $u_\varepsilon(\varepsilon\cdot+q)$ for $q\in\mathbb{T}$,

$$\begin{cases} \Delta u + \frac{e^u(1-e^u)}{(\tau+e^u)^3} = 4\pi\nu\delta_0 \text{ in } \mathbb{R}^2, \quad \nu\in\mathbb{Z}; \\ \frac{e^u(1-e^u)}{(\tau+e^u)^3} \in L^1(\mathbb{R}^2). \end{cases}$$

1) topological solution: $\lim_{|x|\to+\infty} u(x) = 0$,

2) nontopological solution of type I: $\lim_{|x|\to+\infty} u(x) = -\infty$,

3) nontopological solution of type II: $\lim_{|x|\to+\infty} u(x) = +\infty$.

Radially symmetric entire solution of (CP(1)) in $\mathbb{R}^2$

Let $u = u(r; s)$ be the unique solution of

$$\begin{cases} u'' + \frac{1}{r}u' + \frac{e^u(1-e^u)}{(\tau+e^u)^3} = 0, \quad 0 < r < \infty, \quad (CP(1)) \\ u(r) = -2m \ln r + s + o(1) \quad \text{as} \quad r \to 0. \end{cases}$$

- $\varphi(r) \equiv \frac{\partial u(r;s)}{\partial s}, \quad f(u) \equiv \frac{e^u(1-e^u)}{(\tau+e^u)^3}, \quad \beta(s) \equiv \int_0^\infty f(u(r;s))rdr.$
- $\varphi'' + \frac{1}{r}\varphi' + f'(u)\varphi = \frac{(r\varphi')'}{r} + f'(u)\hat{\varphi} = 0, \quad \varphi(0) = 1.$
- $\beta'(s) = \int_0^\infty f'(u)\frac{\partial u(r;s)}{\partial s} rdr = \int_0^\infty f'(u)\varphi rdr = -\lim_{r\to+\infty} r\varphi'(r).$
- The sign of $\beta'(s)$ is related to the stability of $u(r;s)$ ($\beta'(s) > 0$: unstable).

Radially symmetric entire solution of (CP(1)) in $\mathbb{R}^2$

- For nontopological solution of (CSH), $\beta'(s) > 0$ (unstable).

Theorem (K. Choe, J. Han, C.-S. Lin, T.-C. Lin)

(a) *If* $m = 0$, *then* $\beta'(s) > 0$ *for* $\forall s \in \mathbb{R} \setminus \{0\}$.

(b) *If* $m \geq 1$, *then* $\exists s_*$ *such that* $\beta' > 0$ *on* $(s_*, +\infty)$.

- (b-1) *If* $m = 1$ *or* $\tau \leq 1$, *then* $\beta'(s) > 0$, $\forall s \in \mathbb{R} \setminus \{s_*\}$.
- (b-2) *If* $m > 1$, *then* $\exists\ \tau_* > 1$ *such that* $\{s < s_* \mid \beta'(s) < 0\} \neq \emptyset$.

- (b-2) makes a difference between (CSH) and (CP(1)).

Stable entire solutions of CP(1)

- $\beta'(s) < 0$: stable.

Theorem (K. Choe, J. Han, C.-S. Lin, T.-C. Lin)

(b-2) *If $m > 1$, then $\exists\ \tau_* > 1$ such that $\{s < s_* \mid \beta'(s) < 0\} \neq \emptyset$.*

$$\begin{cases} \Delta u + \frac{e^u(1-e^u)}{(\tau+e^u)^3} = -4\pi m\delta_0 \text{ on } \mathbb{R}^2 \qquad (CP(1)), \\ u(x) = (-2m-2N)\ln|x| + O(1) \text{ as } |x| \to +\infty, \\ -2m-2N < -2. \end{cases}$$

- If $m > 1$ and $\tau \gg 1$, a non-topological stable entire solution of (CP(1)) might exist.

- Question: Is there a stable bubbling solutions of (CP(1)) on a torus whenever $m_{j,i} > 1$ for some i, j?

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- If $m > 1$ and $\tau \gg 1$, a non-topological stable entire solution of (CP(1)) might exist.

- Question: Is there a stable bubbling solutions of (CP(1)) on a torus whenever $m_{j,i} > 1$ for some i, j?

No, there exists a global condition which allows $m_{j,i} > 1$ for some i, j for the non-existence result of stable bubbling solutions of (CP(1)) on a torus.

Stable solutions and topological solutions on a torus

Theorem (D. Bartolucci, L., C.-S. Lin, M. Onodera)

$$\Delta u_\varepsilon + \frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3} = 4\pi\sum_{j=1}^{d_1} m_{j,1}\delta_{p_{j,1}} - 4\pi\sum_{j=1}^{d_2} m_{j,2}\delta_{p_{j,2}} \text{ on } \mathbb{T} \quad (CP(1))$$

(i) *topological solution* $\Longrightarrow$ *strictly stable solution.*

(ii) *Assume*

(H1) $N_1 \neq N_2$,

(H2) *either* $\tau = 1$, *or*

if $N_1 > N_2$, *then* $m_{j,1} \in [0,1]$ *for* $\forall j$, *or*

if $N_2 > N_1$, *then* $m_{j,2} \in [0,1]$ *for* $\forall j$.

Then, stable solution $\Longrightarrow$ *topological solution.*

- Non-existence result of stable bubbling solutions of (CP(1)) under a global condition (H2) which allows $m_{j,i} > 1$ for some i, j.
- Question: Is the condition (H2) is necessary?

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$$\Delta u_\varepsilon + \frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3} = 4\pi\sum_{j=1}^{d_1} m_{j,1}\delta_{p_{j,1}} - 4\pi\sum_{j=1}^{d_2} m_{j,2}\delta_{p_{j,2}} \ \textit{on}\ \mathbb{T} \quad (CP(1))$$

(i) *topological solution* $\Longrightarrow$ *strictly stable solution.*

(ii) *Assume*

(H1) $N_1 \neq N_2$,

(H2) *either* $\tau = 1$, *or*

if $N_1 > N_2$, *then* $m_{j,1} \in [0,1]$ *for* $\forall j$, *or*

if $N_2 > N_1$, *then* $m_{j,2} \in [0,1]$ *for* $\forall j$.

Then, stable solution $\Longrightarrow$ *topological solution.*

• Non-existence result of stable bubbling solutions of (CP(1)) under a global condition (H2) which allows $m_{j,i} > 1$ for some i, j.

• Question: Is the condition (H2) is necessary?
Yes, we will construct a stable bubbling solution when (H2) does not hold.

Main Goal

$$\Delta u_\varepsilon + \frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3} = 4\pi\sum_{j=1}^{d_1} m_{j,1}\delta_{p_{j,1}} - 4\pi\sum_{j=1}^{d_2} m_{j,2}\delta_{p_{j,2}} \text{ on } \mathbb{T} \quad (CP(1))$$

Our main goal is to investigate asymptotic behavior of various bubbling solutions of (CP(1)) on a torus.

- Morse index $= 0$: stable solutions,
- Morse index $= 1$: mountain pass solutions,
- Solutions which blow up at one point.

Outline

History

Main Theorem I: construction of stable bubbling solutions

Main Theorem II: asymptotic behavior of mountain pass solutions

Main Theorem III: construction of (unstable) solutions blowing up at one point

Main Theorem I: construction of stable bubbling solutions

- (H3) $m_{1,2} > 1$, $1 - m_{1,2} < N = N_1 - N_2 < 0$.

Theorem (K. Choe, J. Han, L., C.-S. Lin)

Let $N_1 < N_2$, $m_{j,2} = 1$ for $j \neq 1$, (H3) hold.

Assume $\exists$ a non-degenerate stable entire solution u of (CP(1)) for some $\tau \gg 1$.

Then, $\exists$ stable solutions u_ε of (CP(1)) on a torus which blows up at $p_{1,2} \in Z_2$.

- (H2) says that if $N_1 < N_2$, then $m_{j,2} \in [0, 1]$ for $\forall j$.
Thus, (H3) is an example which does not satisfy (H2).

- (H2) is a (necessary) condition for non-existence result of stable bubbling solutions of (CP(1)) on a torus.

Main Theorem I: construction of stable bubbling solutions

- (H3) $m_{1,2} > 1$, $1 - m_{1,2} < N = N_1 - N_2 < 0$.

Theorem (K. Choe, J. Han, L., C.-S. Lin)

Let $N_1 < N_2$, $m_{j,2} = 1$ for $j \neq 1$, (H3) hold.

Assume $\exists$ a non-degenerate stable entire solution u of (CP(1)) for some $\tau \gg 1$.

Then, $\exists$ stable solutions u_ε of (CP(1)) on a torus which blows up at $p_{1,2} \in Z_2$.

- [K. Choe, J. Han, C.-S. Lin, T.-C. Lin]:
Under (H3), there might exist a non-degenerate stable entire solution u of (CP(1)) for some $\tau > 1$.

- Idea of proof: the argument of [H. Chan, C.-C. Fu, C.-S. Lin, (2002)] and [C.-S. Lin, S. Yan (2010)].

Outline

History

Main Theorem I: construction of stable bubbling solutions

Main Theorem II: asymptotic behavior of mountain pass solutions

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Mountain pass solutions of (CSH)

Theorem (K. Choe (2007))

Let u_ε be mountain pass solutions of

$$\Delta u_\varepsilon + \frac{1}{\varepsilon^2} e^{u_\varepsilon}(1 - e^{u_\varepsilon}) = 4\pi \sum_{j=1}^{d} m_j \delta_{p_j} \text{ on } \mathbb{T} \quad (CSH).$$

If $\hat{N} \equiv \sum_{j=1}^{d} m_j > 2$, u_ε blows up at a maximum point of $\hat{u}_0$.

- $\hat{u}_0$ is the unique solution of

$$\Delta \hat{u}_0 = -\frac{4\pi \hat{N}}{|\mathbb{T}|} + 4\pi \sum_{j=1}^{d} m_j \delta_{p_j} \text{ on } \mathbb{T}, \int_{\mathbb{T}} \hat{u}_0 dx = 0. \text{ on } \mathbb{T} \quad (CSH).$$

- Question: What is an asymptotic behavior of mountain pass solutions of CP(1) model?

Existence of mountain pass solutions of (CP(1))

- $\Delta u + \frac{1}{\varepsilon^2}\frac{e^u(1-e^u)}{(\tau+e^u)^3} = 4\pi\sum_{j=1}^{d_1} m_{j,1}\delta_{p_{j,1}} - 4\pi\sum_{j=1}^{d_2} m_{j,2}\delta_{p_{j,2}}$ on $\mathbb{T}$ $\ (CP(1))$.

- $\Delta u_0 = -\frac{4\pi N}{|\mathbb{T}|} + 4\pi\sum_{j=1}^{d_1} m_{j,1}\delta_{p_{j,1}} - 4\pi\sum_{j=1}^{d_2} m_{j,2}\delta_{p_{j,2}}, \quad \int_{\mathbb{T}} u_0 dx = 0.$

- Letting $v = u - u_0$, we can rewrite (CP(1)) as

$$\Delta v + \frac{1}{\varepsilon^2}\frac{e^{u_0+v}(1-e^{u_0+v})}{(\tau+e^{u_0+v})^3} = \frac{4\pi N}{|\mathbb{T}|} \quad \text{on} \quad \mathbb{T} \qquad (CP(1)_R).$$

- $v \in W^{1,2}(\mathbb{T})$ satisfies $(CP(1)_R)$ if and only if
it is a critical point of the associated functional

$$I_\varepsilon(v) \equiv \int_{\mathbb{T}}\Big(\frac{1}{2}|\nabla v|^2 - \frac{e^{u_0+v}((1-\tau)e^{u_0+v}+2\tau)}{2\varepsilon^2\tau^2(\tau+e^{u_0+v})^2} + \frac{4\pi N v}{|\mathbb{T}|}\Big)dx.$$

Existence of mountain pass solutions of (CP(1))

- [D. Chae, H.S. Nam, (1999)]:

$v_{1,\varepsilon} \equiv u_{1,\varepsilon} - u_0$, where $u_{1,\varepsilon}$ is the topological (stable) solution of (CP(1)).

- $\mathcal{P}_\varepsilon \equiv \{\zeta \in C^0([0,1], W^{1,2}(\mathbb{T})) \mid \zeta(0) = v_{1,\varepsilon},\ \zeta(1) = \overline{C}_\varepsilon\}$,

where $\overline{C}_\varepsilon \in \mathbb{R}$ and $I_\varepsilon(\overline{C}_\varepsilon) < I_\varepsilon(v_{1,\varepsilon})$.

- If $N \neq 0$, then I_ε satisfies the Palais-Smale condition.

$\therefore$ $\exists$ a mountain pass solution $u_\varepsilon = v_\varepsilon^* + u_0$ of $(CP(1))$ such that

$$I_\varepsilon(v_\varepsilon^*) = \inf_{\zeta \in \mathcal{P}_\varepsilon} \sup_{t \in [0,1]} I_\varepsilon(\zeta(t)).$$

Main Theorem II: mountain pass solutions of (CP(1))

Theorem (K. Choe, J. Han, L., C.-S. Lin)

Assume $\tau = 1$ and $N \neq 0$.

Let u_ε be mountain pass solutions of $(CP(1))$.

Then,

(i) u_ε blows up at one point $p \in Z$.

(ii) $p \in Z_1$ if and only if $N_2 > N_1$.

• To prove this theorem, we need to understand the asymptotic behavior of solutions of (CP(1)) on $\mathbb{T}$ in detail.

The asymptotic behavior of solutions of (CP(1))

Theorem (D. Bartolucci, L., C.-S. Lin, M. Onodera)

Let u_ε be solutions of (CP(1)) on a torus.

One of the following holds true:

(a) $\lim_{\varepsilon\to 0}\left(\sup_K |u_\varepsilon|\right) = 0,\ \forall K \Subset \mathbb{T}\setminus Z;$

(b) $\lim_{\varepsilon\to 0}\left(\sup_K u_\varepsilon\right) < 0,\ \forall K \Subset \mathbb{T}\setminus Z_2;$

(c) $\lim_{\varepsilon\to 0}\left(\inf_K u_\varepsilon\right) > 0,\ \forall K \Subset \mathbb{T}\setminus Z_1.$

• We need to improve the above theorem to show the asymptotic behavior of mountain pass solutions u_ε of $(CP(1))$.

• $\Delta u + \frac{e^u(1-e^u)}{(\tau+e^u)^3} = -\left[\Delta(-u) + \frac{e^{-u}(1-e^{-u})}{\tau^3(\tau^{-1}+e^{-u})^3}\right].$

• If u_ε is a solution of (CP(1)), then $-u_\varepsilon$ is a solution of a similar equation. Thus it is enough to assume that u_ε satisfies (b).

The detail of asymptotic behavior of solutions of (CP(1))

Theorem (K. Choe, J. Han, L., C.-S. Lin)

Let u_ε be solutions of (CP(1)) satisfying

$$\lim_{\varepsilon \to 0} \left(\sup_K u_\varepsilon \right) < 0, \quad \forall K \Subset \mathbb{T} \setminus Z_2.$$

Then, $\exists$ a finite blow up points set $S \subseteq \mathbb{T} \setminus Z_2$ such that

$$\frac{1}{\varepsilon^2} \frac{e^{u_\varepsilon}(1 - e^{u_\varepsilon})}{(\tau + e^{u_\varepsilon})^3} \to \sum_{p \in S \cup Z_2} \alpha_p \delta_p.$$

- S can also be empty set.
- For (CSH) equation, the corresponding result was presented in [K. Choe, N. Kim (2008)].

Comparison between (CP(1)) and (CSH)

Theorem (K. Choe, J. Han, L., C.-S. Lin)

Assume $m_{j,i} \in \mathbb{N}$.
Let u_ε *be solutions of (CP(1)) on a torus.*
One of the following holds true:
(i) $\lim_{\varepsilon \to 0} \left(\sup_K |u_\varepsilon| \right) = 0, \ \forall K \Subset \mathbb{T} \setminus Z;$

(ii) $\exists$ *a finite blow up points set* $S \subseteq \mathbb{T} \setminus Z$.

Moreover, either $\lim_{\varepsilon \to 0} \left(\sup_K u_\varepsilon \right) < 0$, *or* $\lim_{\varepsilon \to 0} \left(\inf_K u_\varepsilon \right) > 0$ *for* $\forall K \Subset \mathbb{T} \setminus Z$.

Theorem (K. Choe, N. Kim (2008))

Let u_ε *be solutions of (CSH) on a torus.*
One of the following holds true:
(i) $\lim_{\varepsilon \to 0} \left(\sup_K |u_\varepsilon| \right) = 0, \ \forall K \Subset \mathbb{T} \setminus Z;$

(ii) $\exists$ *a finite blow up points set* $S \neq \emptyset \subseteq \mathbb{T};$

(iii) $u_\varepsilon - 2 \ln \varepsilon$ *is uniformly bounded in* $L^\infty_{loc}(\mathbb{T} \setminus Z)$.

Remark of asymptotic behavior of solutions of (CP(1))

Remark

Since we assume $m_{j,i} \in \mathbb{N}$,

we could exclude the possibility $u_\varepsilon - 2\ln\varepsilon$ *is uniformly bounded in* $L^\infty_{loc}(\mathbb{T} \setminus Z)$ *for (CP(1)).*

If $0 \leq m_{j,2} < 1$ *for* $\forall j$, *in [D. Bartolucci, L., C.-S. Lin, M. Onodera],*

it was shown that it might happen that

$u_\varepsilon - 2\ln\varepsilon$ *is uniformly bounded in* $L^\infty_{loc}(\mathbb{T} \setminus Z)$, *converges to* w *in* $C^2_{loc}(\mathbb{T} \setminus Z_2)$,

$$\Delta w + \frac{e^w}{\tau^3} = 4\pi \sum_{j=1}^{d_1} m_{j,1}\delta_{p_{j,1}} - 4\pi \sum_{j=1}^{d_2} m_{j,2}\delta_{p_{j,2}} \quad \text{on } \mathbb{T}.$$

The detail of asymptotic behavior of solutions of (CP(1))

Theorem (K. Choe, J. Han, L., C.-S. Lin)

Let u_ε be solutions of (CP(1)) satisfying

$$\lim_{\varepsilon\to 0}\left(\sup_K u_\varepsilon\right) < 0, \quad \forall K \Subset \mathbb{T}\setminus Z_2, \quad \frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3} \to \sum_{p\in S\cup Z_2}\alpha_p\delta_p.$$

Then,
(i) $\alpha_p \geq 8\pi$ *if* $p \in S$;

(ii) $\alpha_p \geq \min\{0,(4-4m_{j,2})\pi\}$ *if* $p = p_{j,2} \in Z_2$;

(iii) if $\tau \in (0,1]$, *then* $\alpha_p \geq 0$ *if* $p = p_{j,2} \in Z_2$.

- If $\tau \in (0,1]$ or $m_{j,2} = 1$ for $\forall j$, then $\alpha_p \geq 0$ for $\forall p \in S \cup Z_2$,

$$4\pi(N_1 - N_2) = \int_{\mathbb{T}} \frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3}dx \to \sum_{p\in S\cup Z_2}\alpha_p \geq 0, \; N_1 \geq N_2.$$

Main Theorem II: mountain pass solutions of (CP(1))

Theorem (K. Choe, J. Han, L., C.-S. Lin)

Let u_ε be solutions of (CP(1)) satisfying

$$\lim_{\varepsilon \to 0} \left(\sup_K u_\varepsilon \right) < 0, \quad \forall K \Subset \mathbb{T} \setminus Z_2.$$

If $\tau \in (0, 1]$ or $m_{j,2} = 1$ for $\forall j$, then $N_1 \geq N_2$.

Theorem (K. Choe, J. Han, L., C.-S. Lin)

Assume $\tau = 1$ and $N \neq 0$.
Let u_ε be mountain pass solutions of $(CP(1))$.
Then,
(i) u_ε blows up at one point $p \in Z$.
(ii) $p \in Z_1$ if and only if $N_2 > N_1$.

Remark

If $N \neq 0$, $m_{j,i} = 1$, $\forall i, j$, then the result of the above theorem holds for any $\tau \in (0, +\infty)$.

Proof of asymptotic behavior of mountain pass solutions of (CP(1))

• Idea of proof ($N > 0$):

1) An upper bound for $I_\varepsilon(v_\varepsilon^*)$:

By using a radially symmetric profile near $p_{k,2}$, we construct a curve $\zeta = \zeta_\varepsilon \in \mathcal{P}_\varepsilon$ such that

$$\sup_{t\in[0,1]} I_\varepsilon(\zeta(t,\cdot)) \leq 4\pi N(N + 2m_{k,2}) \ln \varepsilon + C \quad \text{as} \ \ \varepsilon \to 0,$$

where $m_{k,2} = \max\{m_{j,2} \mid 1 \leq j \leq d_2\}$.

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where $m_{k,2} = \max\{m_{j,2} \mid 1 \le j \le d_2\}$.
2) A lower bound for $I_\varepsilon(v_\varepsilon^*)$:
Estimate the value of I_ε for all possible asymptotic behavior by using Green's representation formula.
Then

$$I_\varepsilon(v_\varepsilon^*) \ge 4\pi N\big(N + 2m_{k,2} + o(1)\big) \ln \varepsilon \quad \text{as} \ \ \varepsilon \to 0,$$

and the equality holds only if,

$$\frac{1}{\varepsilon^2} \frac{e^{u_0+v_\varepsilon^*}(1-e^{u_0+v_\varepsilon^*})}{(1+e^{u_0+v_\varepsilon^*})^3} \to 4\pi N \delta_{p_{k,2}}. \quad \square$$

Outline

Main Theorem III: construction of (unstable) solutions blowing up at one point

- If $\exists$ solutions u_ε of (CP(1)) satisfying

$$\frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3} \to 4\pi N\delta_{z_0} \quad \text{in the sense of measure,}$$

then the well-known Pohozaev-type identity implies that

$$z_0 \in Z \cup \{x \in \mathbb{T} \mid \nabla u_0(x) = 0\}.$$

- Conversely, for $z_0 \in Z \cup \{x \in \mathbb{T} \mid \nabla u_0(x) = 0\}$,

we want to construct bubbling solutions which blows up at z_0.

Construction of (unstable) solutions blowing up at one point

Theorem (K. Choe, J. Han, L., C.-S. Lin)

Let $\tau > 0$, $N \geq 1$, $m_{j,i} = 1$, $\forall i, j$. *Assume*

$$\begin{cases} z_0 \in Z \cup \{x \in \mathbb{T} \mid \nabla u_0(x) = 0,\ det[D^2 u_0](x) \neq 0\} & when \quad N \geq 5, \\ z_0 \in Z_2 \cup \{x \in \mathbb{T} \mid \nabla u_0(x) = 0,\ det[D^2 u_0](x) \neq 0\} & when \quad N = 3, 4, \\ z_0 \in Z_2 \quad when \quad N = 1, 2. \end{cases}$$

Then, $\exists$ *solutions* u_ε *of (CP(1)) which blow up at* z_0.

• Idea of proof: the argument of [H. Chan, C.-C. Fu, C.-S. Lin, (2002)] and [C.-S. Lin, S. Yan (2010), (2013)]

• We only construct bubbling solutions from entire solutions of $CP(1)$ equation, but not the mean field equation.

Conclusions

On a torus,

• Stable solutions:

(CSH) stable solution $\Leftrightarrow$ topological solution.

(CP(1)) $\exists$ stable bubbling solutions which blow up at $p \in Z$ under some conditions: $\tau > 1$, $N_2 > N_1$, $m_{1,2} > 1$, $p = p_{1,2} \in Z_2$.
-$m_{1,2} = \max\{m_{j,2}\}$.

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• Mountain pass solutions:

(CSH) blows up at a maximum point of u_0.

(CP(1)) blows up at $p \in Z$: $\tau = 1$, $N_2 > N_1 \Leftrightarrow p = p_{1,1} \in Z_1$.
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• Solutions which blow up at one point $p \in Z \cup \{x \in \mathbb{T} \mid \nabla u_0(x) = 0\}$.

-Entire radially symmetric solutions.

-Green's representation formula.

-Pohozaev-type identity ...

Thank you for your attention!