Refined Finite-dimensional Reduction Method and Applications to Nonlinear Elliptic Equations

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Introduction

We consider the following singularly perturbed elliptic problem

$$(LNT) \quad \varepsilon^2 \Delta u - u + u^p = 0 \,, \quad u > 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ with its unit outer normal ν, $n \geq 2$, $1 < p < \frac{n+2}{n-2}$ for $n \geq 3$, while $p > 1$ for $n = 2$, and $\varepsilon > 0$ is a small parameter.

Background

Problem (LNT) is known as the stationary equation of the Keller-Segel system in chemotaxes [KS, 1970].

$$\begin{cases} u_t = D_1 \Delta u - \chi \nabla \cdot (u \nabla \phi(v)) \quad \text{in} \quad \Omega \times (0, +\infty), \\ v_t = D_2 \Delta v + k(u, v) \text{ in } \quad \Omega \times (0, +\infty), \\ u(x, 0) = u_0 > 0, \quad v(x, 0) = v_0 > 0 \text{ in } \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu} \text{ on } \quad \partial \Omega, \end{cases} \tag{1.1}$$

where u denotes the population of amoebae , and v is the concentration of the chemical.

It can also be viewed as a limiting stationary equation for the Geirer-Meinhardt system in biological pattern formation [GM,1972].

$$\begin{cases} u_t = d_1\Delta u - u + \frac{u^p}{v^q} & \text{in} \quad \Omega\times[0,T), \\ v_t = d_2\Delta v - v + \frac{u^r}{v^s} & \text{in} \quad \Omega\times[0,T), \\ \frac{\partial u}{\partial\nu} = 0 = \frac{\partial v}{\partial\nu} & \text{on} \quad \partial\Omega\times[0,T), \end{cases} \tag{1.2}$$

where u, v represent respectively the concentrations of two substances, activator and inhibitor.

$$\begin{cases} d\Delta u - u + \frac{u^p}{v^q} = 0 \text{ in } \Omega, \\ D\Delta v - \xi v + \frac{u^r}{v^s} = 0 \text{ in } \Omega, \\ u > 0, \quad v > 0 \text{ in } \Omega, \\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0 \end{cases} \tag{1.3}$$

where p, q, r, s satisfy the condition $0 < \frac{p-1}{q} < \frac{r}{s+1}$.

Some Pioneering Works of Problem (LNT)

Point Concentration:

- Lin, Ni and Takagi 88,91, 93: Existence of least energy solution u_ε and asymptotic behavior (boundary spike).

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- *Boundary spike* are related to the mean curvature of the boundary $\partial\Omega$.

  Bates, Dancer and Shi, Dancer and Yan, del Pino, Felmer and Wei, Gui, Wei and Winter, Y. Y. Li, Wei, Wei and Winter$\cdots$

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- See the review paper: J. Wei, *Existence and Stability of Spikes for the Gierer-Meinhardt System*, HANDBOOK OF DIFFERENTIAL EQUATIONS, Stationary Partial Differential Equations, volume 5, 2008.

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- A. Malchiodi, W. Ni and J. Wei: For a unit ball in $\mathbb{R}^n$, radial solution concentrating at N spheres: $\bigcup_{j=1}^{N}\{|x| = r_j^\varepsilon\}$ (*Clustered layers*)

$$1 - r_1^\varepsilon \sim \varepsilon \log \frac{1}{\varepsilon}, \qquad r_{j-1}^\varepsilon - r_j^\varepsilon \sim \varepsilon \log \frac{1}{\varepsilon}, \quad j = 2, \cdots, N.$$

- Wei,Yang 07: Γ is a linesegment intersecting with the boundary orthorgonally under some non-degeneracy condition and a gap condition in $\mathbb{R}^2$. The gap condition was dropped for $\mathbb{R}^n$, Ao, Musso and Wei 10(Interior concentration).

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Remark: It seems that the upper bound of the number of interior spikes $\frac{C}{(\varepsilon|\log\varepsilon|)^n}$ *is almost "best" possible, where the constant C depend on* n, Ω, p*, because of the boundary mean curvature*.

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  Remark: It seems that the upper bound of the number of interior spikes $\frac{C}{(\varepsilon|\log\varepsilon|)^n}$ is almost "best" possible, where the constant C depend on n, Ω, p, because of the boundary mean curvature.
- We shall address the question of the maximal possible number of interior spikes??

- p is subcritical $\Longrightarrow$ solutions are uniformly bounded (Lin-Ni-Takagi [LNT]).

  The energy functional associated with (LNT) is

$$E_\varepsilon(u) = \int_\Omega \varepsilon^2 |\nabla u|^2 + u^2 - \frac{1}{p+1} \int_\Omega u_+^{p+1} dz.$$

  By $z = \varepsilon x$,

$$E(u) = \varepsilon^n \int_{\frac{\Omega}{\varepsilon}} |\nabla u|^2 + u^2 - \frac{1}{p+1} u_+^{p+1} dx.$$

  Each spike contributes to at least $O(\varepsilon^n)$ energy$\Longrightarrow$ the number of interior spikes can not exceed $O(\varepsilon^{-n})$.

Main Theorem

Theorem 1: (Ao, Wei and Zeng)

There exists an $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, and any positive integer k satisfying

$$1 \leq k \leq \frac{\delta(\Omega, n, p)}{\varepsilon^n}, \tag{1.4}$$

where $\delta(\Omega, n, p)$ is a constant depending on n, Ω and p only, problem (LNT) has a solution u_ε that possesses exactly k local maximum points.

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3. Felmer-Martinez-Tanaka 06,del Pino-Felmer-Tanaka, 02 Extension to Gierer-Meinhardt system .

Sketch of the Proof

- Method: Localized energy method as in [GW],[LNW].

- There are two main difficulties. First, the distance between spikes is assumed only to be $O(\varepsilon)$. In the Lyapunov-Schmidt reduction process, we have to prove that all the estimates are uniform with respect to the integer k.

- Second, we have to detect the difference in the energy when spikes move to the boundary of the configuration space.

- By the following rescaling:

$$z = \varepsilon x,\ \ x \in \Omega_\varepsilon := \{\varepsilon x \in \Omega\},$$

equation (LNT) becomes

$$(LNT1)\quad \begin{cases} \Delta u - u + u^p = 0 \ \text{ in } \Omega_\varepsilon \\ u > 0 \text{ in } \Omega_\varepsilon,\ \ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \partial\Omega_\varepsilon. \end{cases}$$

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- Associated with problem (LNT1) is the energy functional:

$$J_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon}(|\nabla u|^2 + u^2) - \frac{1}{p+1}\int_{\Omega_\varepsilon} u_+^{p+1}, \quad u \in H^1(\Omega_\varepsilon),$$

where $u_+ = \max(u, 0)$.

Profile function

- We will use the solution w of

$$(GS)\quad \begin{cases} \triangle w - w + w^p = 0,\ w > 0 & \text{in } \mathbb{R}^n, \\ w(0) = \max_{y \in \mathbb{R}^n} w(y),\ w \to 0 & \text{at } \infty. \end{cases}$$

to build up the approximate solution for (LNT).

Approximate Solutions

Configuration space:

$$\Lambda_k := \left\{ (Q_1, \cdots, Q_k) \in \Omega^k \,\middle|\, \begin{array}{l} \min_{i \neq j} |Q_i - Q_j| \geq \rho\varepsilon, \\ \min_{i,j,d(Q_j,\partial\Omega)\leq\delta_c} |Q_i - Q_j^*| \geq \rho\varepsilon \end{array} \right\} \tag{1.6}$$

where $Q_j^* = Q_j + 2d(Q_j, \partial\Omega)\nu_{\bar{Q}_j}$.

For $Q \in \Omega$, define $w_{\varepsilon,Q}$ to be the unique solution of

$$\Delta v - v + w(\cdot - \frac{Q}{\varepsilon})^p = 0 \text{ in } \Omega_\varepsilon, \ \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega_\varepsilon.$$

Approximate solution: for $\mathbf{Q} = (Q_1, \cdots, Q_k) \in \Lambda_k$,

$$w_{\varepsilon,\mathbf{Q}} = \sum_{i=1}^{k} w_{\varepsilon,Q_i}, \ x \in \Omega_\varepsilon. \tag{1.7}$$

Lyapunov-Schmidt reduction

Using Lyapunov-Schmidt reduction, one can find a solution $u = w_{\varepsilon,\mathbf{Q}} + \phi_{\varepsilon,\mathbf{Q}}$:

$$\begin{cases} \Delta(w_{\varepsilon,\mathbf{Q}} + \phi) - (w_{\varepsilon,\mathbf{Q}} + \phi) + (w_{\varepsilon,\mathbf{Q}} + \phi)^p = \sum_{i=1,\cdots,k,j=1,\cdots,n} c_{ij} Z_{ij} \\ \frac{\partial \phi}{\partial \nu} = 0, \quad \text{on} \quad \partial\Omega_\varepsilon \\ \int_{\Omega_\varepsilon} \phi Z_{ij} = 0 \quad \text{for } i = 1, \cdots, k, \ j = 1, \cdots, n, \end{cases}$$

for ρ large.

$$\|h\|_* = \sup_{x \in \Omega_\varepsilon} |(\sum_{i=1}^{k} w^{\eta}_{\varepsilon,Q_i})^{-1} h(x)|, \text{ for } 0 < \eta < 1.$$

Maximization Problem

Fix $\mathbf{Q} \in \Lambda_k$, we define a new functional $\mathcal{M}_\varepsilon : \Lambda_k \to \mathbb{R}$ by

$$\mathcal{M}_\varepsilon(\mathbf{Q}) = J_\varepsilon(u_{\varepsilon,\mathbf{Q}}) = J_\varepsilon[w_{\varepsilon,\mathbf{Q}} + \phi_{\varepsilon,\mathbf{Q}}]. \quad (1.8)$$

Define

$$C_k^\varepsilon = \max_{\mathbf{Q}\in\Lambda_k}\{\mathcal{M}_\varepsilon(\mathbf{Q})\}. \quad (1.9)$$

Lemma : If $\mathbf{Q}_k$ is a critical point of $\mathcal{M}_\varepsilon(\mathbf{Q}_k)$ in Λ_k, then $u_{\varepsilon,\mathbf{Q}_k} = w_{\varepsilon,\mathbf{Q}_k} + \phi_{\varepsilon,\mathbf{Q}_k}$ is a critical point of the energy functional $J_\varepsilon(u)$, i.e. a solution of equation (LNT1).

Maximization Problem

Proposition 1 :

$$(R)\quad C_{k+1}^{\varepsilon} \geq C_k^{\varepsilon} + I(w) + O(e^{-(1+\xi)\rho}),$$

for some $\xi > 0$ independent of k, ρ, ε, and $I(w)$ is the energy of w:

$$I(w) = \frac{1}{2}\int_{\mathbb{R}^n}(|\nabla w|^2 + w^2) - \frac{1}{p+1}\int_{\mathbb{R}^n} w^{p+1}. \qquad (1.10)$$

Maximization Problem

Remark: We need to compare $J_\varepsilon(u_{\varepsilon,Q_1,\cdots,Q_{k+1}})$ and $J_\varepsilon(u_{\varepsilon,Q_1,\cdots,Q_k})$.

$$u_{\varepsilon,Q_1,\cdots,Q_{k+1}} = u_{\varepsilon,Q_1,\cdots,Q_k} + u_{\varepsilon,Q_{k+1}} + \varphi_{k+1}. \quad (1.11)$$

One can get $\|\varphi_{k+1}\|_* \le Ce^{-\frac{1+\xi}{2}\rho}$, if we use this estimate to estimate the L^2 norm of φ_{k+1}, we get

$$\int_{\Omega_\epsilon} |\varphi_{k+1}|^2 \le Ce^{-(1+\xi)\rho} \int_{\Omega_\epsilon} W^2 \le Cke^{-(1+\xi)\rho}.$$

This estimate depends linearly on k.

The Secondary Lyapunov-Schmidt Reduction

Lemma : It holds

$$(K)\int_{\Omega_\varepsilon}(|\nabla\varphi_{k+1}|^2+\varphi_{k+1}^2)\leq Ce^{-(1+\xi)\rho},$$

for some constant $C>0,\xi>0$ independent of ε,ρ,k and $\mathbf{Q}\in\Lambda_{k+1}$.

Idea for the Key Lemma

To prove (K), we need to perform a further decomposition of the function φ_{k+1}.In the proof, we perform a secondary Lyapunov-Schmidt reduction.

$$\bar{L}\varphi_{k+1} = \bar{S} + \sum_{i=1,\cdots,k+1, j=1,\cdots,n} c_{ij} Z_{ij} \tag{1.12}$$

for some constants $\{c_{ij}\}$, where

$$\bar{L} = \Delta - 1 + p\tilde{W}^{p-1},$$

$$\tilde{W}^{p-1} = \begin{cases} \frac{(\bar{W}+\varphi_{k+1})^p - \bar{W}^p}{p\varphi_{k+1}}, & \text{if } \varphi_{k+1} \neq 0 \\ \bar{W}^{p-1}, & \text{if } \varphi_{k+1} = 0, \end{cases}$$

$$\bar{S} = (u_{\varepsilon,Q_1,\cdots,Q_k} + u_{\varepsilon,Q_{k+1}})^p - u^p_{\varepsilon,Q_1,\cdots,Q_k} - u^p_{\varepsilon,Q_{k+1}}.$$

Around each spike, we project φ_{k+1} into the orthogonal space of the unstable eigenfunctions and kernels.

$$\varphi_{k+1} = \psi + \sum_{i=1}^{k+1} c_i \phi_i + \sum_{i=1,\cdots,k+1, j=1,\cdots,n} d_{ij} Z_{ij} \qquad (1.13)$$

for some c_i, d_{ij} such that

$$\int_{\Omega_\varepsilon} \psi \phi_i dx = \int_{\Omega_\varepsilon} \psi Z_{ij} dx = 0, \ i = 1, ..., k+1, \ j = 1, ..., n. \quad (1.14)$$

In this way, we obtain a linear operator which is positively definite, i.e.

$$\int_{\Omega_\varepsilon} [-\bar{L}(\psi)\psi] \geq c_0 \|\psi\|^2_{H^1(\Omega_\varepsilon)}$$

Thus we need to estimate three components of φ_{k+1}: the coefficients of projections to the unstable eigenfunction c_i and kernels d_{ij}, and the orthogonal part ψ.

$$\bar{L}\psi+\sum_{i=1}^{k+1} c_i\bar{L}\phi_i+\sum_{i=1,\cdots,k+1,j=1,\cdots,N} d_{ij}\bar{L}Z_{ij}=\bar{S}+\sum_{i=1,\cdots,k+1,j=1,\cdots,N} c_{ij}Z_{ij}. \tag{1.15}$$

Estimate of d_{ij} :

$$\varphi_{k+1} = \phi_{\varepsilon,Q_1,\cdots,Q_{k+1}} - \phi_{\varepsilon,Q_1,\cdots,Q_k} - \phi_{\varepsilon,Q_{k+1}}, \tag{1.16}$$

$$\begin{aligned} d_{ij} &= \int_{\Omega_\varepsilon} \varphi_{k+1} Z_{ij} \\ &= \int_{\Omega_\varepsilon} (\phi_{\varepsilon,Q_1,\cdots,Q_{k+1}} - \phi_{\varepsilon,Q_1,\cdots,Q_k} - \phi_{\varepsilon,Q_{k+1}}) Z_{ij} \\ &= -\int_{\Omega_\varepsilon} \phi_{\varepsilon,Q_{k+1}} Z_{ij} \end{aligned}$$

$$\begin{cases} |d_{ij}| \leq c e^{-(1+\xi)\frac{\rho}{2}} e^{-\eta\frac{|Q_i - Q_{k+1}|}{\varepsilon}} \text{ for } i = 1, \cdots, k \\ |d_{k+1,j}| \leq c e^{-(1+\xi)\frac{\rho}{2}} \sum_{i=1}^{k} e^{-\eta\frac{|Q_i - Q_{k+1}|}{\varepsilon}} \end{cases} \tag{1.17}$$

for some $\eta > 0$.

Estimate of c_i : Multiply the above equation (1.15) by ϕ_i.

$$c_i \int_{\Omega_\varepsilon} (\bar{L}(\phi_i))\phi_i = -\sum_{j=1}^{n} d_{ij} \int_{\Omega_\varepsilon} \bar{L}(Z_{ij})\phi_i + \int_{\Omega_\varepsilon} \bar{S}\phi_i - \int_{\Omega_\varepsilon} \bar{L}\psi\phi_i,$$

$$\int_{\Omega_\varepsilon} (\bar{L}(\phi_i))\phi_i = -\lambda_1 \int_{\mathbb{R}^n} \phi_0^2 + O(e^{-(1+\xi)\frac{\rho}{2}}). \tag{1.18}$$

$$\begin{cases} |c_i| \le c e^{-(1+\xi)\frac{\rho}{2}} e^{-\eta \frac{|Q_i - Q_{k+1}|}{\varepsilon}} + e^{-\xi\rho} \|\psi\|_{H^1(B_{\frac{\rho}{2}}(\frac{Q_i}{\varepsilon}))}, \quad i = 1, ..., k \\ |c_{k+1}| \le c e^{-(1+\xi)\frac{\rho}{2}} \sum_{i=1}^{k} e^{-\eta \frac{|Q_i - Q_{k+1}|}{\varepsilon}} + e^{-\xi\rho} \|\psi\|_{H^1(B_{\frac{\rho}{2}}(\frac{Q_{k+1}}{\varepsilon}))}, \end{cases} \tag{1.19}$$

for some $\eta > 0, \xi > 0$.

Estimate of ψ:

$$\int_{\Omega_\varepsilon} [-\bar{L}(\psi)\psi] \geq c_0 \|\psi\|^2_{H^1(\Omega_\varepsilon)}$$

$$\|\psi\|_{H^1(\Omega_\epsilon)} \leq c\Big(\sum_{ij} |d_{ij}| + e^{-\frac{\rho}{2}(1+\xi)} + \|\bar{S}\|_{L^2(\Omega_\epsilon)}\Big). \quad (1.20)$$

$$\begin{aligned} \|\varphi_{k+1}\|_{H^1(\Omega_\epsilon)} &\leq c(e^{-\frac{\rho}{2}(1+\xi)} + \|\bar{S}\|_{L^2}) \quad (1.21) \\ &\leq ce^{-\frac{\rho}{2}(1+\xi)}. \end{aligned}$$

Using the relation (R) between C_k^ε and C_{k+1}^ε, we have the following:

Proposition: The maximization problem

$$\max_{\mathbf{Q}\in\bar{\Lambda}_k} \mathcal{M}_\varepsilon(\mathbf{Q}) \tag{1.22}$$

has a solution $\mathbf{Q}^\varepsilon \in \Lambda_k^\circ$, i.e., the interior of Λ_k.

Remark 2:

- As to the solution to (LNT) in the theorem , an interesting problem is to study the *homogenization* of the measure $\varepsilon^{-n}|\nabla u|^2 dx$. We expect that it will approach some kind of Lebesgue measure. As $\varepsilon \to 0$, the locations of the maximum points should approach to some sphere-packing positions.

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- We consider the interior spike solutions. A natural and interesting question is whether we can get similar result for the boundary spike solutions $O(\frac{1}{\varepsilon^{n-1}})$.

Other Applications

Nonlinear Schrödinger Equation:

$$\Delta u - V(x)u + u^p = 0, \quad u > 0 \quad \text{in } \mathbb{R}^n, \quad u \in H^1(\mathbb{R}^n), \qquad (1.23)$$

Equation (1.23) arises in the study of solitary waves in nonlinear equations of the Klein-Gordon or Schrödinger type,

$$ih\frac{\partial \tilde{\psi}}{\partial t} = -h^2 \Delta \tilde{\psi} + \tilde{V}\tilde{\psi} - |\tilde{\psi}|^{p-1}\tilde{\psi}$$

Let $\tilde{\psi} = e^{-i\lambda t/h}\psi(x)$, then $u(x) = \psi(hx)$, with $V(x) = \tilde{V}(hx) - \lambda$ will satisfy (1.23).

Known Results

:

- Rabinowitz 92: Existence of least energy solution

$$0 < \inf_{x \in \mathbb{R}^n} V(x) \leq V(x) \leq \lim_{|x| \to \infty} \inf V(x),$$

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- Bahri and Li 90: Existence of positive solution using minimax methods under a suitable decay condition on V at infinity.

- For multiplicity of positive solutions of (1.23):

  Coti-Zelati and Rabinowitz 92: Existence of infinitely many positive solutions, when V is a periodic.

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- Wei and Yan 2010: Existence of infinitely many non-radial positive bump solutions for (1.23) under the following assumption at infinity:

$$V(x) = V(|x|) = V_\infty + \frac{a}{|x|^m} + O(\frac{1}{|x|^{m+\sigma}}), \ \ m > 1, V_\infty, a, \sigma > 0.$$

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- Del Pinol, Wei and Yao 2013:

$$V(x) = V_\infty + \frac{a}{|x|^m} + O(\frac{1}{|x|^{m+\sigma}}), \quad m > 1, V_\infty, a, \sigma > 0.$$

Cerami-Passasseo-Solimini (2012) developed a localized Nehari's manifold argument and localized variational method to prove the existence of infinitely many positive solutions of the following equation

$$\begin{cases} \Delta u - (1+\delta V)u + u^p = 0 \quad \text{in } \mathbb{R}^n \\ u > 0 \text{ in } \mathbb{R}^n, \; u \in H^1(\mathbb{R}^n) \end{cases} \tag{1.24}$$

where the potential V satisfies suitable decay assumption and p is subcritical for δ small.

$$\begin{cases} V(x) \to 0 \quad \text{as } |x| \to \infty, \\ \exists \; 0 < \bar{\eta} < 1, \lim_{|x|\to\infty} V(x)\, e^{\bar{\eta}|x|} = +\infty, \\ \sup_{x\in\mathbb{R}^n} \|V\|_{L^{\frac{n}{2}}(B_1(x))} < \infty \end{cases} \tag{1.25}$$

$$\begin{cases} \Delta u - (1+\delta V(x))u + f(u) = 0 \quad \text{in } \mathbb{R}^n \\ u > 0 \text{ in } \mathbb{R}^n, \quad u \in H^1(\mathbb{R}^n) \end{cases} \tag{1.26}$$

Assumption on f:

(f_1) $f : \mathbb{R} \to \mathbb{R}$ is of class $C^{1+\sigma}$ for some $0 < \sigma \leq 1$ and $f(u) = 0$ for $u \leq 0$, $f'(0) = 0$.

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(f_2) The equation

$$\begin{cases} \Delta w - w + f(w) = 0, \; w > 0 \text{ in } \; \mathbb{R}^n \\ w(0) = \max_{y \in \mathbb{R}^n} w(y), \; w \to 0 \text{ as } |y| \to \infty \end{cases} \tag{1.27}$$

has a non-degenerate solution w.

Assumption on V:

$$\begin{cases} (H1) \quad V(x) \to 0 \quad \text{as } |x| \to \infty, \\ (H2) \quad \exists \ 0 < \bar{\eta} < 1, \lim_{|x|\to\infty} V(x)\, e^{\bar{\eta}|x|} = +\infty, \\ (H3) \quad V \ \text{is continuous in } \mathbb{R}^n. \end{cases} \tag{1.28}$$

Theorem 2 (Ao and Wei):

Let f satisfy assumptions $(f_1) - (f_2)$ and the potential V satisfy assumptions $(H1) - (H3)$. Then there exists a positive constant δ_0, such that for $0 < \delta < \delta_0$, problem (1.26) has infinitely many positive solutions.

Remark:

We note that the function

$$f(w) = w^p - aw^q, \quad \text{for} \quad w \geq 0 \tag{1.29}$$

with a constant $a \geq 0$ satisfies the above assumptions $(f_1) - (f_2)$ if $1 < q < p < (\frac{N+2}{N-2})_+$. We should point out that f need not to be superlinear, only existence and non-degeneracy are needed.

$$f(w) = \frac{w^2}{1 + w^2}, \quad \text{for} \quad w \geq 0 \tag{1.30}$$

$$\Lambda_1 = \mathbb{R}^n, \ \ \Lambda_k := \{(Q_1, \cdots, Q_k) \in \mathbb{R}^{nk} | \min_{i \neq j} |Q_i - Q_j| \geq \rho\}, \forall k > 1. \tag{1.31}$$

Fixing $\mathbf{Q}_k = (Q_1, \cdots, Q_k) \in \Lambda_k$, we define the sum of k spikes as

$$w_{Q_1, \cdots, Q_k} = \sum_{i=1}^{k} w_{Q_i} \ \text{ where } w_{Q_i} = w(x - Q_i). \tag{1.32}$$

One can show that there exists a constant ρ_0, such that for $\rho \geq \rho_0$, and $\delta < c_\rho$, there is a solution $u = w_{\mathbf{Q}_k} + \phi_{\mathbf{Q}_k}$ of

$$\begin{cases} \Delta u - (1 + \delta V)u + f(u) = \sum_{i=1,\cdots,k}^{j=1,\cdots,n} c_{ij} Z_{ij}, \\ \int_{\mathbb{R}^n} \phi_{\mathbf{Q}_k} Z_{ij} = 0, \text{ for } i = 1, \cdots, k, j = 1, \cdots, n. \end{cases}$$

In the step 5, one need to maximize $\mathcal{M}(\mathbf{Q}_k)$ over $\bar{\Lambda}_k$, where

$$\begin{aligned} \mathcal{M}(\mathbf{Q}_k) &= J(u_{\mathbf{Q}_k}) = J[w_{\mathbf{Q}_k} + \phi_{\mathbf{Q}_k}] \\ &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + (1 + \delta V)u^2 dx - \int_{\mathbb{R}^N} F(u) dx, \end{aligned}$$

where $F(u) = \int_0^u f(s) ds$.

In order to get the relation between C_k and C_{k+1}, one need to get the globel estimate of φ_{k+1}, where $u_{Q_1,\cdots,Q_{k+1}} = u_{Q_1,\cdots,Q_k} + w_{Q_{k+1}} + \varphi_{k+1}$.

Lemma 2: There exists $\rho_0 > 0$, such that for $\rho > \rho_0$ and $\delta < c_\rho$, it holds

$$\begin{aligned} \int_{\mathbb{R}^n} (|\nabla \varphi_{k+1}|^2 + \varphi_{k+1}^2) &\leq Ce^{-\xi\rho} \sum_{i=1}^{k} w(|Q_{k+1} - Q_i|) \qquad (1.33) \\ &+ C\delta^2 (\int_{\mathbb{R}^n} V^2 w_{Q_{k+1}}^2 dx + (\int_{\mathbb{R}^n} |V| w_{Q_{k+1}} dx)^2), \end{aligned}$$

for some constants $C > 0, \xi > 0$ independent of ρ, k and $\mathbf{Q}_{k+1} \in \Lambda_{k+1}$.

Proposition 2: There exists $\mathbf{Q}_k = (Q_1, Q_2, \cdots, Q_k) \in \Lambda_k$ such that

$$\mathcal{C}_k = \mathcal{M}(\mathbf{Q}_k); \tag{1.34}$$

There holds

$$\mathcal{C}_{k+1} > \mathcal{C}_k + I_1(w), \tag{1.35}$$

where $I_1(w)$ is the energy of the solution w of (1.27):

$$I_1(w) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla w|^2 + w^2) - \int_{\mathbb{R}^N} F(w)dx. \tag{1.36}$$

Proof:

- Mathematical induction.

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- Mathematical induction.
- First, we need to show that the maximum points in $\bar{\Lambda}_k$ will not go to infinity. This is guaranteed by the slow decay assumption on the potential V.
- Second, we have to detect the difference in the energy when the spikes move to the boundary of the configuration space. We have to detect the difference in energy between the k-spikes energy and the $k+1$-spikes energy.

Nonlinear Schrodinger system:

$$\begin{cases} -\Delta u + (1 + \delta a(x))u = \mu_1 u^3 + \beta v^2 u \\ -\Delta v + (1 + \delta b(x))v = \mu_2 v^3 + \beta u^2 v \end{cases} \text{ in } \mathbb{R}^n \qquad (1.37)$$

for $n \leq 3$, where μ_1, μ_2 and δ are positive constants, $\beta \in \mathbb{R}$ and the potentials $a(x), b(x)$ are continuous functions satisfying suitable decay assumption, but without any symmetry property.

Peng and Wang 2013: They proved the existence of infinitely many solutions of synchronized type to (1.37) for radially symmetric potentials $a(|x|), b(|x|)$ satisfying some algebraic decay assumption.

Assumption on a, b:

$$\begin{cases} (H_1') & a(x),\ b(x) \text{ are continuous functions in } \mathbb{R}^N, \\ (H_2') & a(x),\ b(x) \to 0 \text{ as } |x| \to \infty,\quad a(x), b(x) \geq 0 \text{ as } |x| \to \infty, \\ (H_3') & \exists\, 0 < \bar{\eta} < 1,\ \lim_{|x|\to\infty} (\alpha^2 a(x) + \gamma^2 b(x)) e^{\bar{\eta}|x|} = +\infty, \end{cases} \tag{1.38}$$

where $\alpha = \sqrt{\frac{\mu_2 - \beta}{\mu_1\mu_2 - \beta^2}}$ and $\gamma = \sqrt{\frac{\mu_1 - \beta}{\mu_1\mu_2 - \beta^2}}$.

Theorem 3 (Ao and Wei)

Let potentials a, b satisfy assumptions $(H_1') - (H_3')$. Then there exists $\beta^* > 0$, and $\delta_0 > 0$, such that for $\beta \in (-\beta^*, 0) \cup (0, \min\{\mu_1, \mu_2\}) \cup (\max\{\mu_1, \mu_2\}, \infty)$, and $0 < \delta < \delta_0$, system (1.37) has infinitely many positive solutions.

$$\begin{cases} -\Delta u + u = \mu_1 u^3 + \beta v^2 u, \\ -\Delta v + v = \mu_2 v^3 + \beta u^2 v. \end{cases} \tag{1.39}$$

It is easy to see that the following pair

$$(U, V) = (\alpha w, \gamma w) \tag{1.40}$$

solves (1.39) provided that $\beta > \max\{\mu_1, \mu_2\}$ or $-\sqrt{\mu_1 \mu_2} < \beta < \min\{\mu_1, \mu_2\}$, where

$$\alpha = \sqrt{\frac{\mu_2 - \beta}{\mu_1 \mu_2 - \beta^2}} > 0, \ \gamma = \sqrt{\frac{\mu_1 - \beta}{\mu_1 \mu_2 - \beta^2}} > 0. \tag{1.41}$$

and w is the unique solution of

$$\begin{cases} \Delta w - w + w^3 = 0, \\ w(0) = \max_{y \in \mathbb{R}^N} w(y), \ w(y) \to 0 \text{ as } |y| \to \infty. \end{cases} \tag{1.42}$$

Lemma

There exists $\beta^* > 0$, such that for $\beta \in (-\beta^*, 0) \cup (0, \min\{\mu_1, \mu_2\}) \cup (\max\{\mu_1, \mu_2\}, \infty)$, (U, V) is non-degenerate for the system (1.39) in $H^1(\mathbb{R}^n)$ in the sense that the kernel is given by

$$\text{Span}\{(\frac{\partial U}{\partial x_j}, \frac{\partial V}{\partial x_j}) | j = 1, \cdots, n\}. \tag{1.43}$$

Thank You!

We first analyze $w_{\varepsilon,Q}$. To this end, set

$$\varphi_{\varepsilon,Q} = w(\frac{z-Q}{\varepsilon}) - w_{\varepsilon,Q}(\frac{z}{\varepsilon}). \qquad (1.44)$$

We state the following lemma on the properties of $\varphi_{\varepsilon,Q}$:

Lemma

There exists ε_0 small enough, and ρ_0 large enough, such that for $0 < \varepsilon < \varepsilon_0, \rho > \rho_0$, and $Q \in \Omega$ such that $c\varepsilon \leq d(Q, \partial\Omega) \leq C\varepsilon|\ln\varepsilon|$, where $c \geq \frac{\rho}{2}$, $C > 0$ are constants, we have

$$\varphi_{\varepsilon,Q} = -(A_0 + O(\frac{1}{\rho^{\frac{1}{2}}}))K(\frac{z-Q^*}{\varepsilon}) + O(e^{-2\rho}). \qquad (1.45)$$

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$$\begin{cases} \text{(KS1)} & u_t = D_1 \Delta u - \chi \nabla \cdot (u \nabla \phi(v)) \quad \text{in} \quad \Omega \times (0, +\infty), \\ \text{(KS2)} & v_t = D_2 \Delta v + k(u, v) \text{ in} \quad \Omega \times (0, +\infty), \\ \text{(IC)} & u(x, 0) = u_0 > 0, \quad v(x, 0) = v_0 > 0 \text{ in} \quad \Omega, \\ \text{(BC)} & \frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu} \text{ on} \quad \partial \Omega, \end{cases}$$

where D_1, D_2 and χ are positive constants; ϕ is the so-called sensitivity function which is a smooth function such that $\phi'(r) > 0$ for $r > 0$; k is a smooth function with $k_u \geq 0$ and $k_v \leq 0$.

$\phi(v) = \ln v$ and $k(u, v) = -av + bu$,

$$\begin{cases} D_1 \Delta u - \chi \nabla \cdot (u \nabla \ln v) = 0 \quad \text{in} \quad \Omega, \\ D_2 \Delta v - av + bu = 0 \text{ in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu} \text{ on} \quad \partial\Omega, \\ |\Omega|^{-1} \int_\Omega u(x) dx = \bar{u}, \end{cases} \tag{1.46}$$

$(u, v) = (\bar{u}, \bar{v})$ with $\bar{v} = a^{-1} b\bar{u}$ is a solution.

The first equation: $\nabla \cdot \{D_1 u \nabla[\ln u - \chi D_1^{-1} \ln v]\} = 0$ and using boundary condition we see that $u = \lambda v^{\frac{\chi}{D_1}}$ for some positive constant λ. Thus (1.46) is equivalent to the following system for (v, λ) :

$$\begin{cases} D_2 \Delta v - av + b\lambda v^{\frac{\chi}{D_1}} = 0 \text{ in } \quad \Omega, \\ \frac{\partial v}{\partial \nu} = 0 \text{ on } \quad \partial\Omega, \\ |\Omega|^{-1} \int_\Omega v(x)dx = \bar{v}. \end{cases} \tag{1.47}$$

$p = \frac{\chi}{D_1}, \varepsilon^2 = \frac{D_2}{a}, \gamma = (a^{-1} b\lambda)^{\frac{1}{q-1}}$, and $w(x) = \gamma \cdot v(x)$,

$$\varepsilon^2 \Delta w - w + w^p = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \tag{1.48}$$

Conversely, if w is a positive solution to (1.48) then

$$u(x) = |\Omega|\bar{u}\left(\int_{\Omega} w^p dx\right)^{-1} w^p(x)$$

and

$$v(x) = |\Omega|\bar{v}\left(\int_{\Omega} w dx\right)^{-1} w(x)$$

satisfy (1.46) with $p = \frac{\chi}{D_1}, \varepsilon^2 = \frac{D_2}{a}$.

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$$\begin{cases} u_t = d_1 \Delta u - u + \frac{u^p}{v^q} & \text{in} \quad \Omega \times [0, T), \\ v_t = d_2 \Delta v - v + \frac{u^r}{v^s} & \text{in} \quad \Omega \times [0, T), \\ \frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu} & \text{on} \quad \partial\Omega \times [0, T), \end{cases} \tag{1.49}$$

where u, v represent respectively the concentrations of two substances, activator and inhibitor.

$$\begin{cases} d_1 \Delta u - u + \frac{u^p}{v^q} = 0 & \text{in} \quad \Omega, \\ d_2 \Delta v - v + \frac{u^r}{v^s} = 0 \text{ in} & \Omega, \\ \frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu} \text{ on} & \partial\Omega. \end{cases} \tag{1.50}$$

Heuristically, v approaches a constant $\xi > 0$ as $d_2 \to +\infty$, and we are led to the shadow system of (1.50):

$$\begin{cases} d_1 \Delta u - u + \frac{u^p}{\xi^q} = 0 \text{ in } \Omega, \\ -\xi + \xi^{-s} |\Omega|^{-1} \int_\Omega u^r \, dx = 0, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.51}$$

Writing $w(x) = u(x)\xi^{-\frac{q}{p-1}}$, we see that w satisfies

$$d_1 \Delta w - w + w^p = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \tag{1.52}$$

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