Estimates of the mean field equations at critical parameters with integral sources

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France-Taiwan Joint Conference on Nonlinear Partial Differential Equations
Oct. 24,2013

Outline:

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I. Introduction

We consider the following mean field equation:

$$\Delta u(x) + \rho \left(\frac{h(x) e^{u(x)}}{\int_M h(x) e^{u(x)} dx} - \frac{1}{|M|} \right) = 4\pi \sum_{j=1}^{d} \alpha_j \left(\delta_{q_j} - \frac{1}{|M|} \right) \text{ in } M \tag{1.1}$$

where (M, g) is a compact Riemann surface and $|M|$ is the area.

- Δ stands the Beltrami-Laplacian operator on (M, g).
- $\alpha_j > -1$, δ_{q_j} is the Dirac measure at q_j and $\rho \in \mathbb{R}^+$.
- $h(x)$ is a positive smooth function on M.
- In geometry, equation (1.1) are related to the well-known Nirenberg problem when $\alpha_j = 0 \ \forall j$
- In general α_j, it related to the existence of the metric of the positive constant curvature with conic singularities
- Equation (1.1) can also be one of the limiting problem of Chern-Simons-Higgs model.

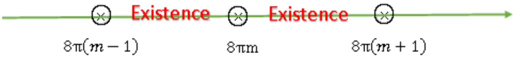
Question: Given $q_1, ..., q_d \in M$ and $\alpha_j \in \mathbb{N}$, $j = 1, ..., d$, whether the equation (1.1) has a solution for each $\rho \in \mathbb{R}^+$ or not.

The following result has been proved:

Theorem A. (Chen-Lin) *Suppose* $\alpha_j \in \mathbb{N}$, *and* $\chi\left(M\backslash\left\{q_1, ..., q_d\right\}\right) \leq 0$. *Then if* $\rho \neq 8m\pi$ *for all* $m \in \mathbb{N}^+$, *then the equation (1.1) always has a solution. Here* $\chi\left(M\backslash\left\{q_1, ..., q_d\right\}\right) = 2 - 2g - d$ is the Euler characteristic unmber of $M\backslash\left\{q_1, ..., q_d\right\}$.

Sketch:

- Let $K \subset\subset (8\pi(m-1), 8\pi m)$ and $\rho \in K$. Then if u_ρ is any solution of (*1.1*), u_ρ is uniformly bounded outside vortex points.
- The topology degree d_ρ is well-defined.
- When $\chi\left(M\backslash\left\{q_1, ..., q_d\right\}\right) \leq 0$, then $d_\rho > 0$. This implies the existence result for $\rho \neq 8m\pi$.
- For $\chi\left(M\backslash\left\{q_1, ..., q_d\right\}\right) \leq 0$, we have



Question: Can we extend **Theorem A** at $\rho = 8m\pi$ (**critical parameters)**?
Our main theorem gives a positive answer for large ρ:

Theorem 1.1

Let $\alpha_j \in \mathbb{N}$, $\chi(M\backslash\{q_1, ..., q_d\}) \leq 0$ and $h(x)$ is a positive C^2 function. Then there exists $\rho_0 \in \mathbb{R}$ such that for any $\rho > \rho_0$, equation (1.1) has a solution. Here $\rho_0 = \max_M (2K - \ln h + N^*)$, K is the Gaussian curvature and $N^* = 4\pi \sum_{j=1}^{d} \alpha_j$.

Strategy:

- Let $\rho_k \in (8\pi(m-\delta), 8\pi(m+\delta))$ and $\rho_k \to 8\pi m$. Let u_k be a solution of (1.1) with $\rho = \rho_k$. To prove there exists a positive constant C such that

$$|u_k| \leq C \text{ for } M \setminus \{q_1, ..., q_d\}. \tag{2.1}$$

  Then after passing limt, u_k will converge to u_∞ which is a solution at $\rho = 8\pi m$.
- Prove by contradiction: Suppose u_k blows up somewhere

Definition 2.1

u_k is called a sequence of blowup solution of (1.1) which blows up at $S = \{p_1, ..., p_m\}$ if there exists $\{x_{k,i}\}_{1\leq i\leq m}$ such that

$$x_{k,i} \to p_i \quad \text{and} \quad u_k(x_{k,i}) \to \infty \quad \text{as} \quad k \to \infty.$$

Remark 2.2

Let u_k be a sequence of blowup solutions. Then

$$\rho_k h e^{u_k} \to \sum_{i=1}^{m} 8\pi \left(1+\alpha\left(p_i\right)\right) \delta_{p_i}$$

in the sense of measure where $\alpha\left(p_i\right)=0$ if $p_i \neq q$ and $\alpha\left(p_i\right)=\alpha_i$ if $p_i=q_i$.

- Suppose there is a sequence of bubbling solutions u_k of (1.1) with ρ_k and

$$\lim_{k\to\infty} \rho_k = \rho_\infty = 8\pi m.$$

We want to find the sharp estimate of $(\rho_k - \rho_\infty)$ and

$$\rho_k - \rho_\infty > 0$$

under some suitable condition. This implies there are no blowing up solution provided $\rho_k < 8m\pi$.

Our goal here is to do bubbling analysis with **Integral Sources**.

Definition 3.1

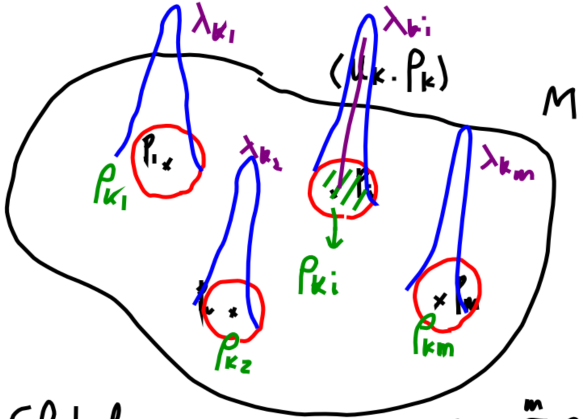
Let p be a blow-up point of u_k, and $r > 0$ such that in $B_{2r}(p) \setminus \{p\}$, u_k has no blow-up points. We define the local mass by

$$\rho_{k,p} = \frac{\rho_k \int_{B_r(p)} h(x) e^{u_k} dx}{\int_M h(x) e^{u_k} dx} \text{ and } \rho_{\infty,p} = \lim_{k \to \infty} \rho_{k,p} = 8\pi (1 + \alpha(p)). \tag{3.1}$$

Also, set

$$\lambda_k = u_k(p_k) = \max_{B_{r_0}(p)} u_k(x). \tag{3.2}$$

where p_k be the local maximum point of u_k near p.



ρ_k : Global mass

ρ_{ki} : local mass around p_i, $\rho_k \approx \sum_{i=1}^{m} \rho_{ki}$

First, we want to show the behavior of u_k is

$$u_k(x) = -\lambda_{k,p} + O(1) \text{ for } |x-p| = r_0 \tag{3.3}$$

for each blowup point p of u_k. This implies that we can compare any two bubbles. Namely,

$$\left|\lambda_{k,i} - \lambda_{k,j}\right| = O(1) \text{ for } i \neq j.$$

Question 1: *Is* $\left|\lambda_{k,i} - \lambda_{k,j}\right| = O(1)$ *for* $i \neq j$*?*

Without loss of generality, we may assume $p = 0$. For $\alpha = 0$ or $\alpha \notin \mathbb{N}$, (3.3) is a consequence of simple blowup property. **Simple blowup property** means that $u_k(x)$ can be locally well-controlled by the entire solutions of its limiting problem. More precisely, define

$$v_k(y) = u_k(\varepsilon_k y) - \lambda_k \text{ for } |y| \leq \frac{1}{\varepsilon_k} \text{ where } \varepsilon_k = e^{-\frac{\lambda_k}{2(1+\alpha)}}.$$

After scaling, a subsequence of $v_k(y)$ would converge to U in $C^2_{loc}(\mathbb{R}^2)$ where U is an entire solution to

$$\begin{cases} \Delta U + \rho_\infty h(0) |y|^{2\alpha} e^U = 0 \text{ in } \mathbb{R}^2 \\ \max U = 0 \end{cases}. \tag{3.4}$$

Parajapat and Tarantello have completely classified all solutions of (3.4), that is,

$$U(y; a_0) = -2 \ln \left(1 + \frac{\rho_\infty h(0)}{8(1+\alpha)^2} \left| y^{1+\alpha} - a_0 \right|^2 \right)$$

In fact,

$$a_0 = \lim_{k \to +\infty} \left(\frac{p_k}{\varepsilon_k} \right)^{1+\alpha}. \tag{3.5}$$

In particular, for $\alpha = 0$ or $\alpha \notin \mathbb{N}$, we have

$$\frac{p_k}{\varepsilon_k} \longrightarrow 0$$

That is

$$a_0 = 0.$$

Bartolucci, Chen, Lin and Tarantello (CPDE 2004) proved the simple blowup property.
Theorem B *Let* 0 *be a blowup point of* u_k *with* $\alpha(0) \notin \mathbb{N}$. Then

$$\left| u_k(x) - \lambda_k + 2\ln\left(1 + \frac{\rho_k h(0)}{8(1+\alpha)^2} e^{\lambda_k} |x|^{2(1+\alpha)}\right)\right| \leq C \text{ in } B_{r_0}(0).$$

For $\alpha \in \mathbb{N}$, there are two cases we need to consider :
Case 1: $\frac{|p_k|}{\varepsilon_k} = O(1)$ i.e. $\frac{p_k}{\varepsilon_k}$ converge as $k \to \infty$.
For $\alpha \in \mathbb{N}$, in general, $a_0 \neq 0$. $U(y; a_0)$ is **no longer radial symmetric** and this would cause lots of troubles in our analysis. Although, $U(y; a_0)$ is **not** radial symmetric, we still could prove the simple blowup peoperty in this case.

Theorem 3.2

Suppose $\lim_{k\to+\infty} \left(\frac{p_k}{\varepsilon_k}\right)^{1+\alpha} = a_0$. *Then we have*

$$|v_k(y) - U(y; a_0)| \leq C \text{ for all } x \in B_{\frac{1}{\varepsilon_k}}(0). \tag{3.6}$$

That is

$$\left| u_k(x) - \lambda_k + 2\ln\left(1 + \frac{\rho_k h_0(0)}{8(1+\alpha)^2} e^{\lambda_k} \left|x^{1+\alpha} - p_k^{1+\alpha}\right|^2\right)\right| \leq C \text{ for } |x| \leq 1.$$

Case 2: $\lim_{k\to+\infty}\frac{|p_k|}{\varepsilon_k}=+\infty$

- u_k is not simply blowing-up at $q=0$.This is a new phenomenon which might occur only at the case when $\alpha\in\mathbb{N}$. However, this phenomenon also appears in the study of $SU(3)$ Toda system. Studying this non-simple blowup case for the scalar equation should be useful for the **system** case. For this case, we could prove the behavior (3.3) also holds.

We use the following scaling technique: Let $u_k(x)$ be a sequence of blowup solutions and set

$$\delta_k=|p_k|. \tag{3.7}$$

Let

$$\hat{\mu}_k=\lambda_k+2(1+\alpha)\ln\delta_k=2(1+\alpha)\ln\frac{\delta_k}{\varepsilon_k}\to\infty \tag{3.8}$$

and

$$\hat{u}_k(y)=u_k(\delta_k y)+2(1+\alpha)\ln\delta_k \text{ for } |y|\le\frac{1}{\delta_k}. \tag{3.9}$$

Then

$$\Delta\hat{u}_k(y)+\rho_k|y|^{2\alpha}h_0(\delta_k y)e^{\hat{u}_k(y)}=0 \text{ in } B_{\frac{1}{\delta_k}}(0) \tag{3.10}$$

Remark 3.3

(i) After scaling, the equation (3.10) has the same vortex point at 0.

(ii) $\hat{u}_k\left(\frac{p_k}{|p_k|}\right) = \hat{\mu}_k \to \infty$. $e_1 = 1 \neq 0$ *is one of the blowup point of* $\hat{u}_k$. *Since* e_1 *is not the vortex point,* e_1 *would carry the local mass* 8π.

(iii) From Pohozaev identity, we know that the total mass is $8\pi(1+\alpha)$, *and this implies there exists a blowup set* $S = \{e_1, e_2, ..., e_{1+\alpha}\}$. *In fact,*

$$e_{\ell+1} = \exp\left(i\frac{2\pi}{1+\alpha}\ell\right), \ \ell = 0, ..., \alpha. \tag{3.11}$$

(iv) $e_i \neq 0 \ \forall i$, *that is,* $\hat{u}_k$ *is simple blowup at each* e_i, *meaning*

$$\left|\hat{u}_k(y) - \hat{U}(y)\right| = O(1) \ \text{ for } |y - e_i| \leq r_0$$

where

$$\hat{U}(y) = \ln \frac{e^{\hat{\mu}_k}}{\left(1 + e^{\hat{\mu}_k}|y - e_i|^2\right)^2}$$

In case 2 (i.e. $\lim_{k\to+\infty}\frac{|p_k|}{\varepsilon_k}=+\infty$), we have the following theorem:

Theorem 3.4

$\forall y\in B_{\frac{1}{\delta_k}}(0)\setminus\cup_{\ell=1}^{1+\alpha}B_{r_0}(e_\ell)$, we have

$$\hat{u}_k(y)=-\hat{\mu}_k-\sum_{\ell=1}^{1+\alpha}4\ln|y-e_\ell|+O(1). \tag{3.12}$$

By Theorem 3.4, on $\partial B_{\frac{1}{\delta_k}}(0)$, we have

$$\hat{u}_k(y)=-\hat{\mu}_k+4(1+\alpha)\ln\delta_k+O(1)\ \text{ for }\ |y|=\frac{1}{\delta_k} \tag{3.13}$$

and thus we have

$$\begin{aligned} u_k(x) &= 2(1+\alpha)\ln\frac{\varepsilon_k}{\delta_k}+2(1+\alpha)\ln\delta_k+O(1) \\ &= -\lambda_k+O(1)\ \text{ for }\ |x|=1. \end{aligned} \tag{3.14}$$

After (3.3) is established at each blowup point p, we are able to give a positive answer for Question 1.

Next, we want to discuss the estimate of $\rho_k - \rho_\infty$. Let $\lambda_k = \max_{1\le j\le m} \lambda_{k,j}$ and $\rho_{k,i}$ be the local mass defined by (3.1) at p_i. By (3.3), we have good boundary information near each blowup point, then we could localize the problem, namely,

$$\rho_k - \rho_\infty = \sum_{i=1}^{m} \left[\rho_{k,i} - 8\pi\left(1+\alpha\left(p_i\right)\right)\right] + O\left(e^{-\lambda_k}\right), \tag{4.1}$$

$$\rho_\infty = 8\pi \sum_{i=1}^{m} \left(1+\alpha\left(p_i\right)\right), \tag{4.2}$$

where $\alpha\left(p_i\right) = 0$ if $p_i \notin \{q_1, ..., q_d\}$ and $\alpha\left(p_i\right) = \alpha_i$ if $p_i = q_i$.

Question 2: *Estimate the difference* $\rho_{k,p} - \rho_{\infty,p}$

(i) If $p \neq q$ (i.e.$\alpha = 0$), then there exists a function $Q(x)$ such that

$$\nabla Q(p) = 0$$

and the sharp estimate of $\rho_{k,p} - \rho_{\infty,p}$ has already derived by **Chen and Lin (CPAM 2002)**:

Theorem D *Let* (u_k, ρ_k) *be a sequence of solutions of (1.1) which blows up at* $\{p_1, ..., p_m\}$*. Suppose* $\alpha(p) = 0$*. Then we have*

$$\rho_{k,p} - 8\pi = \frac{16\pi}{\rho_\infty h_0(p)} \left(\Delta \ln h(p) - N^* + \rho_\infty - 2K(p)\right) \varepsilon_k^2 \left|\ln \varepsilon_k\right| + O\left(\varepsilon_k^2\right)$$

where $K(x)$ *denotes the Gaussian curvature and* $N^* = 4\pi \sum_{j=1}^{d} \alpha_j$*.*

(ii) If p (the blowup point) is one of the vortex point q. Suppose that $\alpha(q) \notin \mathbb{N}$. Then $\nabla Q(p)$ may not 0. With the help of $a_0 = 0$, Chen and Lin proved

Theorem E (Chen-Lin DCDS-A 2010) *Let (u_k, ρ_k) be a sequence of solutions of (1.1) which blows up at $\{p_1, ..., p_m\}$. Suppose $\alpha(p) \notin \mathbb{N}$. Then we have*

$$\rho_{k,q} - 8\pi(1 + \alpha(q)) = d(q,\alpha)(\Delta \ln h(p) - N^* + \rho_\infty - 2K(q))\varepsilon_k^2 + o(1)\varepsilon_k^2$$

where $d(q,\alpha)$ is a positive constant depending on q and α.

Again, for $p = q = 0$ and $\alpha \in \mathbb{N}$, we have two cases need to consider, **Simple blowup** and **Non-simple blowup**.

Case 1(Simple blowup):

Without loss of generality, we may assume $\nabla Q(0) = (Q_1(0), 0)$. To obtain the formula, we need to approximate the bubbling solution $u_k(x)$ (or $v_k(z)$) : More precisely, we define the error term $\tilde{\eta}_k(z)$ by

$$\tilde{\eta}_k(z) = v_k(z) - U(z; a_0) - (G_k^*(\varepsilon_k z) - G_k^*(0))$$

We want to prove

$$\tilde{\eta}_k(z) = \varepsilon_k Q_1(0)\, \psi_{11}(z; a_0) + o(\varepsilon_k). \tag{4.3}$$

where $\psi_{11}(z; a_0)$ be the solution of

$$\begin{cases} \Delta\psi_{11}(z; a_0) + \rho_k h_0(0) |z|^{2\alpha} e^{U(z;a_0)} (\psi_{11}(z; a_0) + z_1) = 0 \text{ in } \mathbb{R}^2 \\ |\psi_{11}(z; a_0)| = O\left(\frac{1}{|z|}\right) \text{ at } \infty \text{ and } \partial_z^{1+\alpha}\psi_{11}\left(a_0^{\frac{1}{1+\alpha}}; a_0\right) = 0 \end{cases} \tag{4.4}$$

- We need to solve (4.4).

Remark 4.1

(i) For $\alpha \notin \mathbb{N}$, then $\frac{\rho_k}{\varepsilon_k} \to 0$ and thus $a_0 = 0$. The solution $\psi_{11}(z;0)$ can be solved explicitly

$$\psi_{11}(z;0) = -\frac{2(1+\alpha)}{\alpha} \frac{z_1}{1 + \frac{\rho_k h_0(0)}{8(1+\alpha)^2} |z|^{2(1+\alpha)}}. \tag{4.5}$$

We see that $\psi_{11}(z;0) = O\left(\frac{1}{|z|^{2\alpha+1}}\right)$.
(ii) However, for $\alpha \in \mathbb{N}$, in general, $\nabla Q(0) \neq 0$ and $a_0 \neq 0$, thus it is hard to write down the solution explicitly. That means we need to solve (4.4) under $U(z;a_0)$ is not radial symmetric. Moreover, if $a_0 \neq 0$, $\psi_{11}(z;a_0)$ does not have this decay $O\left(\frac{1}{|z|^{2\alpha+1}}\right)$. In fact, the decay rate $O\left(\frac{1}{|z|}\right)$ is **optimal** *for $a_0 \neq 0$.*

After we solved the equation (4.4), we could estimate the difference $\rho_{k,q} - 8\pi(1+\alpha(q))$ as follows:

Theorem 4.2

Suppose $\lim_{k\to+\infty}\left(\frac{p_k}{\varepsilon_k}\right)^{1+\alpha} = a_0$ *holds. Then we have*

$$\rho_{k,q} - 8\pi(1+\alpha(q)) = \frac{1}{4}F_1(a_0;\alpha(q))(\Delta \ln h(q) - N^* + \rho_\infty - 2K(q))\varepsilon_k^2 + o(1)\varepsilon_k^2$$

where $F_1(a_0;\alpha(q)) > 0$.

Case 2(Non-simple blowup):
Since this is a new phenomenon, the method for this case would be different from Case 1. We use not only a good approximation of $\hat{u}_k(y)$ inside each simply bubbling regions $B_{r_0}(e_i)$ but also the behavior outside the bubbling region $B_{\frac{1}{\delta_k}}(0) \setminus \cup_{i=1}^{1+\alpha} B_{r_0}(e_i)$. More precisely,

- In $B_{r_0}(e_i)$, we need a good approximation, i.e. the error term

$$\hat{u}_k(y) - \hat{U}(y) - \left(\hat{H}(y) - \hat{H}(e_i)\right) = O\left(\hat{\mu}_k e^{-\hat{\mu}_k}\right)$$

- For $y \in B_{\frac{1}{\delta_k}}(0) \setminus \cup_{\ell=1}^{1+\alpha} B_{r_0}(e_\ell)$, we need

$$\hat{u}_k(y) = -\hat{\mu}_k - \sum_{\ell=1}^{1+\alpha} 4 \ln |y - e_\ell| + \hat{C}_k + O\left(\hat{\mu}_k e^{-\hat{\mu}_k}\right).$$

where $\hat{C}_k$ is some constant and $\hat{C}_k$ is bounded as $k \to \infty$.

Theorem 4.3

Suppose that $\lim_{k\to+\infty}\frac{|p_k|}{\varepsilon_k}=+\infty$ *for* $p=q$. *Then*

$$\rho_{k,q}-8\pi\left(1+\alpha\left(q\right)\right)=C\left(q,\alpha\right)\left(\Delta\ln h\left(q\right)-N^{*}+\rho_{\infty}-2K\left(q\right)\right)\delta_k^2\sigma_k^2\left|\ln\sigma_k\right|+O\left(\delta_k^2\sigma_k^2\right)$$

where $C\left(q,\alpha\right)>0$, $\delta_k=|p_k|$ *and* $\sigma_k=e^{-\frac{\hat{\mu}_k}{2}}$.

The Linearized equation

- For $\alpha \in \mathbb{N}$, solving the linearized equation (4.4) is the main difference from $\alpha = 0$ or $\alpha \notin \mathbb{N}$, and many difficulties come from the linearized equation. Thus, we want to discuss the existence and asymptotic behavior of the linearized equation (4.4).

$$\begin{cases} \Delta\psi_{11}(z;a_0) + \rho_k h_0(0)|z|^{2\alpha} e^{U(z;a_0)}(\psi_{11}(z;a_0)+z_1) = 0 \text{ in } \mathbb{R}^2 \\ |\psi_{11}(z;a_0)| = O\left(\frac{1}{|z|}\right) \text{ at } \infty \text{ and } \partial_z^{1+\alpha}\psi_{11}\left(a_0^{\frac{1}{1+\alpha}};a_0\right) = 0 \end{cases}$$

Lemma 5.1

There exists a unique solution $\psi_{11}(z;a_0)$ which depends on a_0 of the equation (4.4) where $a_0^{\frac{1}{1+\alpha}}$ is one of the complex $(1+\alpha)$-roots of a_0.

Let $\varphi(z)$ satisfy

$$\Delta\varphi(z) + \rho_k h_0(0) |z|^{2\alpha} e^{U(z)} \varphi(z) = 0 \text{ in } \mathbb{R}^2. \tag{5.1}$$

Define $\varphi_j(z)$, $j = 0, 1, 2$ as follows:

$$\varphi_0(z) = \frac{1 - C_k |z^{1+\alpha} - a_0|^2}{1 + C_k |z^{1+\alpha} - a_0|^2}, \varphi_1(z) = \frac{C_k \mathrm{Re}(z^{1+\alpha} - a_0)}{1 + C_k |z^{1+\alpha} - a_0|^2}, \varphi_2(z) = \frac{C_k \mathrm{Im}(z^{1+\alpha} - a_0)}{1 + C_k |z^{1+\alpha} - a_0|^2}$$

where $C_k = \frac{\rho_k h_0(0)}{8(1+\alpha)^2}$. Then $\varphi_j(z)$ are three kernels of $\left(\Delta + \rho_k h_0(0) |z|^{2\alpha} e^{U(z)}\right)$ and **Del Pino** etc. proved the following nondegeneracy result:

Lemma A *Let $\varphi(z)$ be any solution of (5.1). Suppose that there exists a constant $C > 0$ such that $|\varphi(z)| \leq C(1 + |z|)^\tau$ for some $\tau \in [0, 1)$. Then $\varphi(z) = \sum_{j=0}^{2} c_j \varphi_j(z)$ for some constants c_j, $j = 0, 1, 2$.*

Existence:

- The idea is to use **Fredholm property** in some suitable weighted function space.
- We could prove the non-homogeneous term $\rho_k h_0(0)|z|^{2\alpha} e^{U(z)} z_1$ is perpendicular to those three kernels $\varphi_j(z)$. Then the existence follows.

The crucial part is to derive the asympotic behavior of $\psi_{11}(z)$.
Asymptotic behavior:

- Let $\psi_{11}(z)$ be a solution of (4.4) and
$f(z) = \rho_k h_0(0)|z|^{2\alpha} e^{U(z)} (\psi_{11}(z) + z_1) \in L^1(\mathbb{R}^2)$, $\gamma = \int_{\mathbb{R}^2} f(z)\, dz$,
$g(z) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \left(\ln \left(\frac{|y|}{|y-z|} \right) \right) f(y)\, dy$.
- First, we have $\psi_{11}(z) = g(z) + C$ for some constant and $g(z) = \frac{1}{2\pi}\gamma \ln|z| + O(1)$ as $|z| \to \infty$, meaning, $\psi_{11}(z) = \frac{1}{2\pi}\gamma \ln|z| + O(1)$ as $|z| \to \infty$.
- We can show that $\gamma = \int_{R^2} \rho_k h_0(0)|z|^{2\alpha} e^{U(z)} \varphi_0(z) z_1 dy = 0$. That is, $g(z) = o(1)\ln|z| + O(1)$ as $|z| \to \infty$ so does $\psi_{11}(z)$.

Asymptotic behavior:

- Consider the Kelvin transformation of $g(z)$. That is $\hat{g}(z) = g\left(\frac{z}{|z|^2}\right)$. Then we can show that $\hat{g}(z)$ is C^1 at $z = 0$ which implies $g(z) = C + O\left(\frac{1}{|z|}\right)$ as $|z| \to \infty$. Thus $\psi_{11}(z) = C + O\left(\frac{1}{|z|}\right)$ as $|z| \to \infty$.
- By considering $\psi_{11}(z) + C\varphi_0$, we have $\psi_{11}(z) = O\left(\frac{1}{|z|}\right)$ for $|z| \to +\infty$.
- Choose C_1 and C_2 such that $\psi_{11}(z) + C_1\varphi_1(z) + C_2\varphi_2(z)$ has the property $\partial_z^{1+\alpha}\psi_{11}\left(a_0^{\frac{1}{1+\alpha}}; a_0\right) = 0$.

Uniqueness:

- Uniqueness is easy from the condition $|\psi_{11}(z; a_0)| = O\left(\frac{1}{|z|}\right)$ at ∞ and $\partial_z^{1+\alpha}\psi_{11}\left(a_0^{\frac{1}{1+\alpha}}; a_0\right) = 0$.

- The difficulties coming from $a_0 \neq 0$ and $\nabla Q(0) \neq 0$ are not only the existence of the solution to the linearized equation but also in the sharp estimates. We explain it as follows: Recall

$$\tilde{\eta}_k(z) = \varepsilon_k \partial_1 Q(0) \psi_{11}(z; a_0) + o(\varepsilon_k) \tag{5.2}$$

Lemma 5.2

Let $\Omega_k = B_{\frac{r_0}{\varepsilon_k}}(0)$

$$\rho_{k,0} - 8\pi(1+\alpha) = \int_{\Omega_k} \varphi_0(z) \Delta\tilde{\eta}_k(z) - \tilde{\eta}_k(z) \Delta\varphi_0(z)\, dz + o\left(\varepsilon_k^2\right).$$

After long computation, we have

$$\int_{\Omega_k} \varphi_0(z)\,\Delta\tilde{\eta}_k(z) - \tilde{\eta}_k(z)\,\Delta\varphi_0(z)\,dz$$
$$= \int_{\Omega_k} -\rho_k h_0(0)\,|z|^{2\alpha}\,e^{U(z;a_0)}\varphi_0(z)\left[\frac{\varepsilon_k^2}{2}\nabla^2 Q(0)\,z^2\right.$$
$$\left.+\,\frac{1}{2}\left(\tilde{\eta}_k(z)+\varepsilon_k Q_1(0)\,z_1\right)^2\right]dz + o\left(\varepsilon_k^2\right).$$

Then by (5.2)

$$\left(\tilde{\eta}_k(z)+\varepsilon_k Q_1(0)\,z_1\right)^2 = \left(\left(\psi_{11}(z;a_0)+z_1\right)\right)^2\left(Q_1(0)\right)^2\varepsilon_k^2 + o(1)\,\varepsilon_k^2,$$

we have

$$\int_{\Omega_k} \varphi_0(z)\,\Delta\tilde{\eta}_k(z) - \tilde{\eta}_k(z)\,\Delta\varphi_0(z)\,dz$$
$$= \int_{\Omega_k} -\rho_k h_0(0)\,|z|^{2\alpha}\,e^{U(z;a_0)}\varphi_0(z)\left[\frac{\varepsilon_k^2}{2}\nabla^2 Q(0)\,z^2\right.$$
$$\left.+\,\frac{1}{2}\left(\left(\psi_{11}(z;a_0)+z_1\right)\right)^2\left(Q_1(0)\right)^2\varepsilon_k^2\right]dz + o\left(\varepsilon_k^2\right).$$

(i)

$$\int_{\Omega_k} -\rho_k h_0(0)\,|z|^{2\alpha} e^{U(z;a_0)} \varphi_0(z) \frac{1}{2} \nabla^2 Q(0) z^2 dz$$
$$= \frac{1}{4} F_1(a_0;\alpha)\, \Delta Q(0) + o(1)$$

where

$$F_1(a_0;\alpha) = \int_{R^2} -\rho_\infty h_0(0)\,|z|^{2\alpha+2} e^{U(z;a_0)} \varphi_0(z)\, dz > 0$$

and

$$\Delta Q(0) = \Delta \ln h(q) - N^* + \rho_\infty - 2K(q)$$

(ii) It is easy to see that

$$\int_{\Omega_k} -\rho_k h_0(0)|z|^{2\alpha} e^{U(z;a_0)} \varphi_0(z) \frac{1}{2} \left(\left(\psi_{11}(z;a_0) + z_1 \right) Q_1(0) \right)^2 dz$$
$$= E(a_0;\alpha) \left(Q_1(0) \right)^2 + o(1).$$

where

$$E(a_0;\alpha) = \frac{1}{2} \int_{\mathbb{R}^2} -\rho_\infty h_0(0)|z|^{2\alpha} e^{U(z;a_0)} \varphi_0(z) \left(\psi_{11}(z;a_0;\alpha) + z_1 \right)^2 dz$$

Thus, we have

$$\begin{aligned} & \rho_{k,0} - 8\pi(1+\alpha) \\ = & \left\{ \frac{1}{4} F_1(a_0;\alpha) \left(\Delta \ln h(q) - N^* + \rho_\infty - 2K(q) \right) + E(a_0;\alpha) \left(Q_1(0) \right)^2 \right\} \varepsilon_k^2 + o(1) \varepsilon_k^2. \end{aligned}$$

Remark 5.3

(i) If $\alpha = 0$*, then we know that* $\nabla Q(0) = 0$*. Thus* $E(a_0;\alpha)(Q_1(0))^2 = 0$*.*

(ii) If $\alpha \notin \mathbb{N}$*, then* $a_0 = 0$*, and it is easy to compute* $E(0;\alpha) = 0$*.*

(iii) In our case, that is, $\alpha \in \mathbb{N}$*. Then* $\nabla Q(0)$ *and* a_0 *both may not be* 0*. Thus we meet the difficulty that how to compute the integral*

$$E(a_0;\alpha) = \frac{1}{2}\int_{\mathbb{R}^2} -\rho_\infty h_0(0)|z|^{2\alpha} e^{U(z;a_0)}\varphi_0(z)(\psi_{11}(z;a_0;\alpha)+z_1)^2\,dz$$

which involving the solution of linearized equation.

Without loss of generality, we may assume that $a \in \mathbb{R}$. We need to transform the integral $E(a;\alpha)$ on **entire space** into an integral **at infinity**. Write

$$\psi_{11}(z;a;\alpha) = A_1(a;\alpha)\frac{z_1}{|z|^2} + B_1(a;\alpha)\frac{z_2}{|z|^2} + O\left(\frac{1}{|z|^2}\right) \text{ as } |z| \to \infty.$$

Let $\lambda \in \mathbb{R}$ and

$$U(z;a;\alpha;\lambda) = \lambda - 2\ln\left(1 + \frac{\rho_\infty h(0)}{8(1+\alpha)^2} e^{\lambda} \left|z^{1+\alpha} - a\right|^2\right)$$

Then $U(z;a;\alpha;\lambda)$ also satisfy

$$\Delta U(z;a;\alpha;\lambda) + \rho_\infty h(0)|z|^{2\alpha} e^{U(z;a;\alpha;\lambda)} = 0.$$

Let $\psi_{11}(z;a;\alpha;\lambda)$ be the corresponding solution to

$$\begin{cases} \Delta\psi_{11}(z;a;\alpha;\lambda) + \rho_k h_0(0)|z|^{2\alpha} e^{U(z;a;\alpha;\lambda)}(\psi_{1\ell}(z;a;\alpha;\lambda) + z_1) = 0 \text{ in } \mathbb{R}^2 \\ |\psi_{11}(z;a;\alpha;\lambda)| = O\left(\frac{1}{|z|}\right) \text{ at } \infty \text{ and } \partial_z^{1+\alpha}\psi_{1\ell}\left(a^{\frac{1}{1+\alpha}};a;\alpha;\lambda\right) = 0 \end{cases}$$

Then

$$\psi_{11}(z;a;\alpha;0) = \psi_{11}(z;a;\alpha)$$

and

$$\psi_{11}(z;a;\alpha;\lambda) = e^{-\frac{\lambda}{2(1+\alpha)}}\psi_{11}\left(e^{\frac{\lambda}{2(1+\alpha)}}z;e^{\frac{\lambda}{2}}a;\alpha;0\right).$$

Thus

$$\psi_{11}(z;a;\alpha;\lambda) = e^{-\frac{\lambda}{1+\alpha}}A_1\left(e^{\frac{\lambda}{2}}a;\alpha\right)\frac{z_1}{|z|^2} + e^{-\frac{\lambda}{1+\alpha}}B_1\left(e^{\frac{\lambda}{2}}a;\alpha\right)\frac{z_2}{|z|^2} + O\left(\frac{1}{|z|^2}\right)$$

as $|z| \to \infty$.

Then we have the following lemma:

Lemma 5.4

$$
\begin{aligned}
E(a;\alpha) &= \frac{1}{2}\int_{\mathbb{R}^2} -\rho_\infty h_0(0)\,|z|^{2\alpha} e^{U(z;a)} \varphi_0(z)\left(\psi_{11}(z;a;\alpha)+z_1\right)^2 dz \\
&= \frac{1}{2}\lim_{R\to\infty}\int_{\partial B_R} z_1\left(\frac{\partial}{\partial \nu}\partial_\lambda \psi_{11}(z;a;\alpha;\lambda)\right) - \partial_\lambda \psi_{11}(z;a;\alpha;\lambda)\left(\frac{\partial}{\partial \nu} z_1\right) d\sigma\Big|_{\lambda=0} \\
&= \frac{\pi}{4}\left[A_1(a;\alpha) - \frac{2}{1+\alpha}\left(\partial_a A_1(a;\alpha)\right) a\right]
\end{aligned}
$$

So we have to study the asymptotic behavior in a deeper way, namely, we need to find out the leading coefficient $A_1(a;\alpha)$. More precisely, we have

Lemma 5.5

(i) For $\alpha = 1$, $A_1(a;1) = 4a$ *and* $B_1(a;1) = 0$.
(ii) For $\alpha > 1$, $A_1(a;\alpha) = B_1(a;\alpha) = 0$.

From Lemma 5.4 and Lemma 5.5, we conclude that

$$E\left(a;\alpha\right)=0.$$

Thus

$$\begin{aligned} & \rho_{k,0}-8\pi\left(1+\alpha\right) \\ = \quad & \frac{1}{4}F_1\left(a;\alpha\right)\left(\Delta\ln h\left(q\right)-N^{*}+\rho_{\infty}-2K\left(q\right)\right)\varepsilon_k^2+o\left(1\right)\varepsilon_k^2. \end{aligned}$$

Another delicate issue: *To locate the position of* a_0

One of the most interesting examples is the following: We consider u_k to be a blowup solution satisfying

$$\Delta u_k + e^{u_k} = 4\pi \sum_{i=1}^{N} \alpha_{k,i}\delta_{q_i} \text{ in } T \tag{5.3}$$

where $T = \mathbb{C}/\Lambda$ and $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$.

$$\rho_k = 4\pi \sum_{i=1}^{N} \alpha_{k,i}$$

If $\alpha_{k,i} = \alpha_i$ and $\sum_{i=1}^{N} \alpha_i$ is even integer, then u_k is simple blowup and $a_0 = 0$.

Question: *In general, for* $4\pi\sum_{i=1}^{N} \alpha_{k,i} \to 8\pi m$*, whether* u_k *is simple blowup or not. If* u_k *is simple blowup, can we determine the position of* a_0*?*

Thank you very much !