A Hamilton-Jacobi approach for a model of population structured by space and trait

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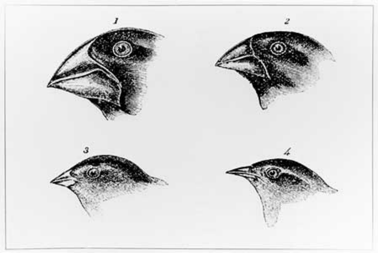
Taipei, Octobre 21–25, 2013

Darwinian evolution of a structured population density

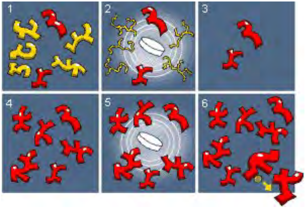
- We study the **Darwinian** evolution

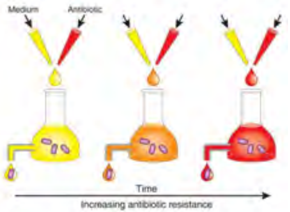
  of a population structured by phenotypical traits, and the position in space,

  under selection and mutation



Example of evolution: creation of antibiotic resistance of bacteria under drug selection

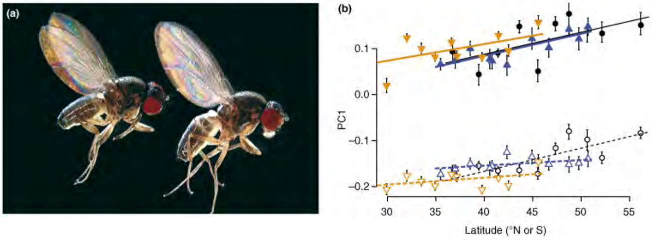




Morbidostat : a selective pressure is applied continuously to the bacterial population. It automatically tunes drug concentration such that a constant growth rate is maintained.

Following evolution of bacterial antibiotic resistance in real time, Rosenthal et Elowitz, Nature Genetics 2012

The latitudinal cline in wing size of Drosophila subobscura



(a) Drosophila subobscura from Barcelona, Spain (39 latitude, left) and Aarhus, Denmark (56 latitude, right) demonstrating the latitudinal cline in wing size that has evolved in the native European range, and also in the introduced North and South American ranges. (b) Latitudinal clines in wing size in different regions.

Figure from: Effects of exotic species on evolutionary diversification, Vellend et al. 2007.

Rapid Evolution of a Geographic Cline in Size in an Introduced Fly, Hulley et al. 2000.

Our objective: describing the spatial invasion and the phenotypical distribution of the population

An approach based on Hamilton-Jacobi equations has been used to study

- models structured only by a space variable (as KPP type equations) and helps to describe the invasion scenarios: Barles, Evans, Majda, Souganidis,...(89-94)

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Here, we try to combine these two methods to study models structured by a phenotypical trait and a space variable.

A model structured by a phenotypical trait and a space variable

$$\begin{cases} \partial_t n = \Delta_x n + \alpha \Delta_\theta n + r\, n\, [a(x,\theta) - \rho], & (x,\theta) \in \mathbb{R}^d \times \Theta, \\ \frac{\partial n}{\partial \mathbf{n}} = 0 \text{ on } \partial\Theta, \\ \rho(t,x) = \int_\Theta n(t,x,\theta) d\theta, & n(0,x,\theta) = n^0(x,\theta). \end{cases}$$

- $x \in \mathbb{R}^d$: position in space
- $\theta \in \Theta$: phenotypical trait
- $n(t,x,\theta)$: density of trait θ at position x
- $ra(x,\theta)$: intrinsic growth rate
- r: death rate due to competition (constant)
- α: mutation rate (constant)

What we expect ?

In general we expect that the population propagates in the x-direction and attains a certain distribution in θ in the invaded parts.

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Related works:

- L. Desvillettes, R. Ferriere and C. Prevost (2004)
- N. Champagnat and S. Méléard (2007)
- A. Arnold, L. Desvillettes, and C. Prevost (2012)

Other related works:

- O. Benichou, V. Calvez, N. Meunier and R. Voituriez (2012)
- E. Bouin, V. Calvez, N. Meunier, S. M., B. Perthame, G. Raoul and R. Voituriez (2012): A formal result for the invasion of cane toads using the **Hamilton-Jacobi** approach
- Alfaro, Coville, Raoul (2013) : Existence of propagating fronts with $a(x,\theta) = b(x \cdot e - \theta)$ in the form of

$$n(t,x,\theta) = u(x \cdot e - ct, \theta), \qquad c \geq c^*.$$

- H. Berestycki and G. Chapuisat, preprint: a local model.

Rescaling

$$\begin{cases} \varepsilon\,\partial_t n_\varepsilon = \varepsilon^2\,\Delta_x n_\varepsilon + \alpha\,\Delta_\theta n_\varepsilon + r\,n_\varepsilon\,\big(a(x,\theta)-\rho_\varepsilon\big), \quad (x,\theta)\in\mathbb{R}^d\times\Theta,\\ \dfrac{\partial n_\varepsilon}{\partial \mathbf{n}} = 0 \text{ on } \partial\Theta,\\ \rho_\varepsilon(t,x) = \int_\Theta n_\varepsilon(t,x,\theta)d\theta, \qquad n_\varepsilon(0,x,\theta) = n_\varepsilon^0(x,\theta). \end{cases}$$

ε is small:

- **small diffusion** in space and **long time**.

Lemma (Eigenvalue or cell problem)

For all $x \in \mathbb{R}^d$, there exists a unique eigenvalue $H(x)$ corresponding to a strictly positive eigenfunction $Q(x,\cdot)$ which satisfies

$$\begin{cases} \alpha \Delta_\theta Q + ra(x,\theta) Q = H(x) Q \\ \dfrac{\partial Q(x,\cdot)}{\partial n} = 0 \ on \ \partial\Theta. \end{cases}$$

The eigenvector is unique under the additional normalization assumption

$$\int_\Theta Q(x,\theta) d\theta = 1.$$

Moreover, H and Q are smooth functions.

Some assumptions

- $-Mx^2 + B \leq a(x,\theta) \leq a_\infty$
- $\rho^0(x) \leq a_\infty$
- $\exp\left(\frac{-C_1(x)}{\varepsilon}\right) \leq n_\varepsilon(0,x,\theta) \leq \exp\left(\frac{C_2}{\varepsilon}\right)$

and we use the **Hopf-Cole** transformation

$$n_\varepsilon = \exp\left(\frac{u_\varepsilon}{\varepsilon}\right).$$

Theorem (Asymptotic behavior)

(i) *The family* $(u_\varepsilon)_\varepsilon$ ***converges locally uniformly*** *to* $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ *the unique viscosity solution of*

$$\begin{cases} \max\left(\partial_t u - |\nabla_x u|^2 - H, u\right) = 0, & \textit{in } (0,\infty) \times \mathbb{R}^d, \\ u(0,x) = u_0(x). \end{cases}$$

(ii) *Uniformly on compact subsets of* $Int\{u < 0\} \times \Theta$, $\lim_{\varepsilon\to 0} n^\varepsilon = 0$,

(iii) *There exists* $C > 1$ *such that, uniformly on compact subsets of* $\mathrm{Int}\left(\{u(t,x) = 0\} \cap \{H(x) > 0\}\right)$,

$$\liminf_{\varepsilon\to 0} \rho_\varepsilon(t,x) \geq \frac{H(x)}{rC}.$$

Theorem (Convergence of n_ε in a particular case)

Let $a(x,\theta) = a(\theta) - b(x)$. *Then* $Q(x,\theta) = Q(\theta)$. *Let the initial data be*

$$n_\varepsilon(t=0,x,\theta) = m_\varepsilon(x)Q(\theta).$$

Then

(i) *For all* $t > 0$ *and* $(x,\theta) \in \mathbb{R} \times \Theta$,

$$n_\varepsilon(t,x,\theta) = m_\varepsilon(t,x)Q(\theta).$$

(ii) *For all* $(t,x,\theta) \in \{u(t,x) = 0\} \times \Theta$,

$$\lim_{\varepsilon \to 0} n_\varepsilon(t,x,\theta) = \frac{H}{r} Q(\theta).$$

Some qualitative properties

Let $a(x,\theta) = a(\theta)$.

$$\Longrightarrow H(x) = H, \quad Q(x,\theta) = Q(\theta).$$

Propagation in the x-direction:

$$\max\left(\partial_t u - |\nabla_x u|^2 - H, u\right) = 0.$$

Speed of propagation in space, starting from an initial data with compact support:

$$c = 2\sqrt{H}.$$

The **phenotypical distribution** in $\mathrm{Int}\{u(t,x) = 0\}$ (at the back of the front):

$$n_\varepsilon(t,x,\theta) \approx \frac{H}{r} Q(\theta).$$

A possible explanation for the case of Drosophila Subobscura

Let

$$r = 1, \qquad a(x,\theta) = A - D\,(\theta - bx)^2, \qquad x,\, \theta \in \mathbb{R}.$$

Then

$$Q(x,\theta) = C \exp\left(\frac{1}{2}\sqrt{\frac{D}{\alpha}}(\theta - bx)^2\right), \qquad H(x) = A - \sqrt{D\alpha}.$$

While the mutation rate α is small, the population concentrates on the line $\theta = bx$.

Some heuristics

We use the following ansatz

$$n_\varepsilon = \exp\left(\frac{u_\varepsilon(t,x,\theta)}{\varepsilon}\right).$$

$$\partial_t u_\varepsilon = \varepsilon\Delta_x u_\varepsilon + \frac{\alpha}{\varepsilon}\Delta_\theta u_\varepsilon + |\nabla_x u_\varepsilon|^2 + \frac{\alpha}{\varepsilon^2}|\nabla_\theta u_\varepsilon|^2 + r(a(x,\theta) - \rho_\varepsilon)$$

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A formal expansion of u_ε:

$$u_\varepsilon(t,x,\theta) = u_0(t,x,\theta) + \varepsilon u_1(t,x,\theta) + O(\varepsilon^2).$$

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A formal expansion of u_ε:

$$u_\varepsilon(t,x,\theta) = u_0(t,x,\theta) + \varepsilon u_1(t,x,\theta) + O(\varepsilon^2).$$

We replace this in the equation and keeping the terms of order ε^{-2} we obtain, for all (t,x,θ),

$$|\nabla_\theta u_0(t,x,\theta)|^2 = 0.$$

This suggests that

$$u_0(t,x,\theta) = u_0(t,x).$$

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Next, we keep the zero order terms (terms with coefficient ε^0):

$$-\alpha\Delta_\theta u_1 - \alpha|\nabla_\theta u_1|^2 - ra(x,\theta) = \left[-\partial_t u_0 + |\nabla_x u_0|^2 - r\rho_0\right](t,x).$$

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$$\Downarrow$$

$$\left[\partial_t u_0 - |\nabla_x u_0|^2 + r\rho_0\right](t,x) = H(x),$$

$$u_1(t,x,\theta) = \ln Q(x,\theta) + \mu(t,x).$$

We can now write

$$n_\varepsilon(t,x,\theta) \approx e^{\frac{u_0(t,x)}{\varepsilon}+u_1(t,x,\theta)}, \qquad \rho_\varepsilon(t,x) \approx e^{\mu(t,x)+\frac{u_0(t,x)}{\varepsilon}}.$$

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Therefore

$$\rho_0 \leq C \quad \Longrightarrow \quad u_0 \leq 0, \quad \text{and} \quad \rho_0 > 0 \quad \Longrightarrow \quad u_0 = 0.$$

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We deduce that

$$\begin{cases} \rho_0(t,x) = 0 \quad \Longrightarrow \quad \partial_t u_0(t,x) - |\nabla_x u_0|^2(t,x) - H(x) = 0, \\ \\ \rho_0(t,x) > 0 \quad \Longrightarrow \quad \begin{array}{c} u_0(t,x) = 0 \quad \text{and} \\ r\exp(\mu(t,x)) = r\rho_0(t,x) = H(x) \end{array}, \end{cases}$$

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We deduce that

$$\begin{cases} \rho_0(t,x) = 0 \implies \partial_t u_0(t,x) - |\nabla_x u_0|^2(t,x) - H(x) = 0, \\ \rho_0(t,x) > 0 \implies \begin{matrix} u_0(t,x) = 0 \quad \text{and} \\ r\exp(\mu(t,x)) = r\rho_0(t,x) = H(x) \end{matrix}, \end{cases}$$

$$\Downarrow$$

$$\max\left(u_0\,,\, \partial_t u_0 - |\nabla_x u_0|^2 - H(x) \right) = 0,$$

Therefore, we expect that

$$n_\varepsilon(t,x,\theta) \longrightarrow \begin{cases} \frac{H(x)}{r} Q(x,\theta) & \text{if } u_0(t,x) = 0 \\ 0 & \text{if } u_0(t,x) < 0 \end{cases}$$

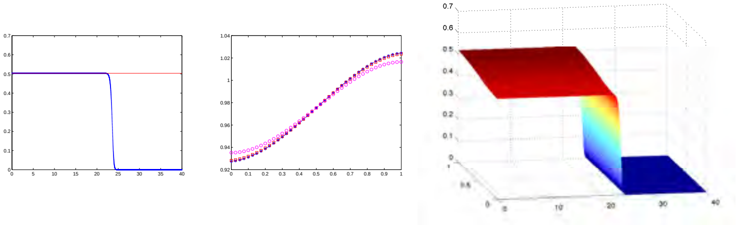
with

$$\begin{cases} \alpha \Delta_\theta Q + ra(x,\theta) Q = H(x) Q. \\ \\ \dfrac{\partial Q(x,\cdot)}{\partial n} = 0 \text{ on } \partial\Theta, \qquad \displaystyle\int_\Theta Q(x,\theta) d\theta = 1. \end{cases}$$

and

$$\max\left(u_0 \, , \, \partial_t u_0 - |\nabla_x u_0|^2 - H(x) \right) = 0.$$

The numerical resolution of the main problem



- The values of $\rho_\varepsilon(x)$ and $\frac{H(x)}{r}$
- The **trait distribution** at the edge and at the back of the front that we compare to Q
- The density $n^\varepsilon(t,x,\theta)$

$$\alpha = 1, \quad r = 2, \quad a_1(\theta) = \frac{1}{4} + \frac{\theta}{2}$$

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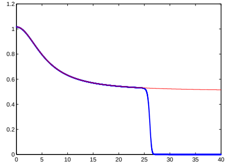

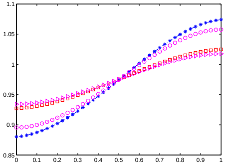

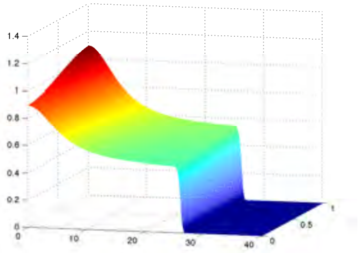

$$a_2(x,\theta) = \left(\frac{1}{4} + \frac{\theta}{2}\right)\left(1 + \frac{1}{1 + 0.05x^2}\right)$$

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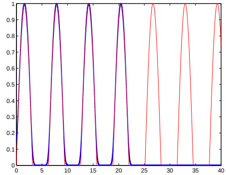

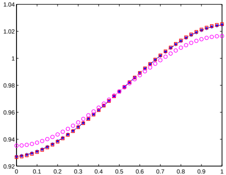

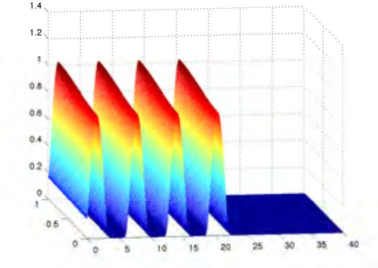

$$a_3(\theta) = \frac{1}{4} + \frac{\theta}{2} + \left(\sin(x) - \frac{1}{2}\right)$$

Difficulties

- We don't have a priori Lipschitz estimate in x
- No maximum principle due to the **nonlocal term**

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Regularizing effect in θ:

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Convergence to the effective Hamilton-Jacobi equation:

- half-relaxed limits method for viscosity solutions
- perturbed test functions

Outlines of the proof

Theorem (Convergence to the Hamilton-Jacobi equation)

Under the previous assumptions and

$$\lim_{\varepsilon \to 0} u_\varepsilon(0, x, \theta) = u^0(x),$$

as ε *vanishes,* $(u_\varepsilon)_\varepsilon$ *converges locally uniformly to* u *the unique viscosity solution of*

$$\begin{cases} \max\left(\partial_t u - |\nabla_x u|^2 - H, u\right) = 0, & \text{in } (0,\infty) \times \mathbb{R}^d \\ u(0,x) = u^0(x). \end{cases}$$

By replacing

$$n_\varepsilon = \exp\left(\frac{u_\varepsilon}{\varepsilon}\right).$$

in the equation on n_ε we obtain

$$\begin{cases} \partial_t u_\varepsilon = \varepsilon \Delta_x u_\varepsilon + \frac{\alpha}{\varepsilon}\Delta_\theta u_\varepsilon + |\nabla_x u_\varepsilon|^2 + \frac{\alpha}{\varepsilon^2}|\nabla_\theta u_\varepsilon|^2 + r(a(x,\theta) - \rho_\varepsilon) \\ \frac{\partial u_\varepsilon(t,x,\cdot)}{\partial \mathbf{n}} = 0 \text{ on } \partial\Theta, \\ u_\varepsilon(0,x,\theta) = u_\varepsilon^0(x,\theta). \end{cases}$$

Theorem (Regularity results for u_ε)

*Under the previous assumptions, the family $(u_\varepsilon)_{\varepsilon>0}$ is **locally uniformly bounded** in $\mathbb{R}^+ \times \mathbb{R}^d \times \Theta$.*
Let $\gamma > 0$ and

$$v_\varepsilon := \sqrt{C(t) + \gamma^2 - u_\varepsilon}.$$

Then,

$$|\nabla_\theta v_\varepsilon| \leq \frac{\varepsilon}{2\sqrt{\alpha t}} + \left(\frac{rC_0\varepsilon^2}{\alpha\gamma}\right)^{\frac{1}{3}}.$$

*In particular, this gives a **regularizing effect in** θ for all $t > 0$, and the fact that $|\nabla_\theta v_\varepsilon| \to 0$ locally uniformly as ε goes to 0.*

Bounds on ρ_ε and u_ε

$$\varepsilon\partial_t\rho_\varepsilon = \varepsilon^2\Delta_x\rho_\varepsilon + r\left(\int_\Theta n_\varepsilon(t,x,\theta)a(x,\theta)d\theta - \rho_\varepsilon^2\right)$$

$$\leq \varepsilon^2\Delta_x\rho_\varepsilon + r\rho_\varepsilon\left(a_\infty - \rho_\varepsilon\right).$$

$$\Downarrow$$

$$\rho_\varepsilon \leq a_\infty,$$

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$$\Downarrow$$

$$\rho_\varepsilon \leq a_\infty,$$

and by maximum/comparison principle

$$-r(M|x|^2+B)t - C_1(x) - r\varepsilon M t^2 \leq u_\varepsilon(t,x,\theta) \leq C_2 + r a_\infty t.$$

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$$-r(M|x|^2 + B)t - C_1(x) - r\varepsilon M t^2 \leq u_\varepsilon(t,x,\theta) \leq C_2 + r a_\infty t.$$

Recall (assumptions):

$$-Mx^2 + B \leq a(x,\theta) - a_\infty \leq 0,$$

$$\rho_\varepsilon^0(x) \leq a_\infty, \qquad -C_1(x) \leq u_\varepsilon(0,x,\theta) \leq C_2.$$

Regularity in θ

Let

$$v_\varepsilon := \sqrt{C_2 + ra_\infty t + \gamma^2 - u_\varepsilon} \geq \gamma.$$

We replace this in the equation on u_ε and obtain

$$\partial_t v_\varepsilon = \varepsilon \Delta_x v_\varepsilon + \frac{\alpha}{\varepsilon}\Delta_\theta v_\varepsilon + \left(\frac{\varepsilon}{v_\varepsilon} - 2v_\varepsilon\right)|\nabla_x v_\varepsilon|^2$$

$$+ \left(\frac{\alpha}{\varepsilon v_\varepsilon} - \frac{2\alpha v_\varepsilon}{\varepsilon^2}\right)|\nabla_\theta v_\varepsilon|^2 - \frac{1}{2v_\varepsilon} r(a(x,\theta) - a_\infty - \rho_\varepsilon).$$

We differentiate this equation with respect to θ and multiply it by by $\frac{\nabla_\theta v_\varepsilon}{|\nabla_\theta v_\varepsilon|}$ to obtain

$$\begin{aligned}
\partial_t|\nabla_\theta v_\varepsilon| \quad &\leq \varepsilon\Delta_x|\nabla_\theta v_\varepsilon| + \frac{\alpha}{\varepsilon}\Delta_\theta|\nabla_\theta v_\varepsilon| + 2\left(\frac{\varepsilon}{v_\varepsilon} - 2v_\varepsilon\right)\nabla_x v_\varepsilon \cdot \nabla_x|\nabla_\theta v_\varepsilon| \\
&+2\left(\frac{\alpha}{\varepsilon v_\varepsilon} - \frac{2\alpha v_\varepsilon}{\varepsilon^2}\right)\nabla_\theta v_\varepsilon \cdot \nabla_\theta|\nabla_\theta v_\varepsilon| + \left(-\frac{\varepsilon}{v_\varepsilon^2} - 2\right)|\nabla_x v_\varepsilon|^2|\nabla_\theta v_\varepsilon| \\
&+\left(-\frac{\alpha}{\varepsilon v_\varepsilon^2} - \frac{2\alpha}{\varepsilon^2}\right)|\nabla_\theta v_\varepsilon|^3 + \frac{r|\nabla_\theta a(x,\theta)|}{2v_\varepsilon},
\end{aligned}$$

$$\partial_t|\nabla_\theta v_\varepsilon| \leq \varepsilon\Delta_x|\nabla_\theta v_\varepsilon| + \frac{\alpha}{\varepsilon}\Delta_\theta|\nabla_\theta v_\varepsilon| + 2\left(\frac{\varepsilon}{v_\varepsilon} - 2v_\varepsilon\right)\nabla_x v_\varepsilon \cdot \nabla_x|\nabla_\theta v_\varepsilon|$$

$$+2\left(\frac{\alpha}{\varepsilon v_\varepsilon} - \frac{2\alpha v_\varepsilon}{\varepsilon^2}\right)\nabla_\theta v_\varepsilon \cdot \nabla_\theta|\nabla_\theta v_\varepsilon| + \left(-\frac{\varepsilon}{v_\varepsilon^2} - 2\right)|\nabla_x v_\varepsilon|^2|\nabla_\theta v_\varepsilon|$$

$$+\left(-\frac{\alpha}{\varepsilon v_\varepsilon^2} - \frac{2\alpha}{\varepsilon^2}\right)|\nabla_\theta v_\varepsilon|^3 + \frac{r|\nabla_\theta a(x,\theta)|}{2v_\varepsilon},$$

Therefore, $w_\varepsilon := |\nabla_\theta v_\varepsilon|$ is a subsolution of

$$\partial_t w_\varepsilon \leq \varepsilon\Delta_x w_\varepsilon + \frac{\alpha}{\varepsilon}\Delta_\theta w_\varepsilon + 2\left(\frac{\varepsilon}{v_\varepsilon} - 2v_\varepsilon\right)\nabla_x v_\varepsilon \cdot \nabla_x w_\varepsilon$$

$$+2\left(\frac{\alpha}{\varepsilon v_\varepsilon} - \frac{2\alpha v_\varepsilon}{\varepsilon^2}\right)\nabla_\theta v_\varepsilon \cdot \nabla_\theta w_\varepsilon - \frac{2\alpha}{\varepsilon^2}|w_\varepsilon|^3 + \frac{rC_0}{2\gamma}.$$

We can find a supersolution for the previous equation

$$z(t) := \frac{\varepsilon}{2\sqrt{\alpha t}} + \left(\frac{rC_0\varepsilon^2}{\alpha\gamma}\right)^{\frac{1}{3}}.$$

Note that

$$z(0) = \infty.$$

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$$z(0) = \infty.$$

By the comparison principle (with Neumann boundary condition):

$$|\nabla_\theta v_\varepsilon| = w_\varepsilon \leq \frac{\varepsilon}{2\sqrt{\alpha t}} + \left(\frac{rC_0\varepsilon^2}{\alpha\gamma}\right)^{\frac{1}{3}}.$$

In particular, for all $t > 0$, as $\varepsilon \to 0$,

$$|\nabla_\theta v_\varepsilon(t,x,\theta)| \to 0, \quad \Rightarrow \quad |\nabla_\theta u_\varepsilon(t,x,\theta)| \to 0.$$

Convergence to the Hamilton-Jacobi equation

Let's first suppose that we also have regularity estimates in x, such that we know a priori that

$$u_\varepsilon \longrightarrow u, \qquad \text{locally uniformly as } \varepsilon \to 0.$$

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Let's also suppose that

$$\rho_\varepsilon(t,x) \longrightarrow \rho(t,x), \qquad \text{locally uniformly as } \varepsilon \to 0.$$

Then, we prove that u is a viscosity solution of

$$\partial_t u - |\nabla_x u|^2 - H + r\rho = 0.$$

Perturbed test function method

Let $\varphi : \mathbb{R}^+ \times \mathbb{R}^d$ be a smooth function such that $u - \varphi$ has a maximum at (t_0, x_0).
Then define

$$\varphi_\varepsilon(t, x, \theta) = \varphi(t, x) + \varepsilon\chi(x, \theta).$$

with $\chi(x, \theta) = \ln Q(x, \theta)$ and

$$-\alpha\Delta_\theta Q = a(x, \theta)Q - H(x)Q.$$

Then, for all $\varepsilon > 0$, there exists $(t_\varepsilon, x_\varepsilon, \theta_\varepsilon)$ such that $u_\varepsilon - \varphi_\varepsilon$ has a maximum at this point and

$$(t_\varepsilon, x_\varepsilon) \to (t_0, x_0), \qquad \text{as } \varepsilon \to 0.$$

By the **viscosity sub-solution criterion**, at the point $(t_\varepsilon, x_\varepsilon, \theta_\varepsilon)$,

$$\partial_t \varphi_\varepsilon \leq \varepsilon \Delta_x \varphi_\varepsilon + \frac{\alpha}{\varepsilon} \Delta_\theta \varphi_\varepsilon + |\nabla_x \varphi_\varepsilon|^2 + \frac{\alpha}{\varepsilon^2} |\nabla_\theta \varphi_\varepsilon|^2 + r(a(x_\varepsilon, \theta_\varepsilon) - \rho_\varepsilon)$$

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$$\Downarrow$$

$$\begin{aligned}\partial_t \varphi(t_\varepsilon, x_\varepsilon) \quad & -\varepsilon \Delta_x (\varphi + \varepsilon\chi)(t_\varepsilon, x_\varepsilon, \theta_\varepsilon) - |\nabla_x (\varphi + \varepsilon\chi)|^2 (t_\varepsilon, x_\varepsilon, \theta_\varepsilon) \\ & \leq \alpha \Delta_\theta \chi(x_\varepsilon, \theta_\varepsilon) + \alpha |\nabla_\theta \chi|^2 (x_\varepsilon, \theta_\varepsilon) + r(a(x_\varepsilon, \theta_\varepsilon) - \rho_\varepsilon(t_\varepsilon, x_\varepsilon)) \\ & = H(x_\varepsilon) - r\rho_\varepsilon(t_\varepsilon, x_\varepsilon).\end{aligned}$$

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$\Rightarrow$ as $\varepsilon \to 0$,

$$\partial_t \varphi(t_0, x_0) - |\nabla_x \varphi|^2 \leq H(x_0) - r\rho(t_0, x_0).$$

$\Rightarrow$ u is a **subsolution**. (Similar argument for supersolution).

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- We don't have a priori convergence of ρ_ε:

  We only prove

$$\max(\partial_t u - |\nabla_x u|^2 - H, u) = 0, \qquad \text{in } (0,\infty) \times \mathbb{R}^d.$$

Thank you for your attention !