Bubbling Solutions for the Chern-Simons Model on a Torus

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(joint work with Chang-Shou Lin)

Consider the following Chern-Simons-Higgs equation:

(1)
$$\begin{cases} \Delta u + \frac{1}{\varepsilon^2} e^u (1 - e^u) = 4\pi \sum_{j=1}^{N} \delta_{p_j}, & \text{in } \Omega, \\ u \text{ is doubly periodic on } \partial\Omega, \end{cases}$$

where Ω is a parallelogram in $\mathbb{R}^2$, $p_j \in \Omega$, $j = 1, \cdots, N$.

Some simple facts:

- (1) has no solution if $\varepsilon > 0$ is large:

$$\int_\Omega e^u(1-e^u) = 4\pi N\varepsilon.$$

  But $e^u(1-e^u) \le \frac{1}{4}$. So $\varepsilon \le \frac{|\Omega|}{16\pi N}$.

- Maximum principle gives $u \le 0$. So

$$e^u(1-e^u) \ge 0.$$

We study (1) with $\varepsilon \to 0$.

Introduce

$$u_0(x) = -4\pi \sum_{j=1}^{N} G(x, p_j),$$

where $G(x, p_j)$ is the Green function:

$$\Delta G(x, p_j) = -\delta_{p_j} + \frac{1}{|\Omega|},$$

$$\int_{\Omega} G(x, p_j)\, dx = 0.$$

Near each vortex point p_j,

$$u_0(x) = 2m \ln |x - p_j| + O(1),$$

where m is the number of p_i satisfying $p_i = p_j$.

We can use the function u_0 to remove the singularities from (1).

Replace u by $u + u_0$ in (1), then u satisfies

(2)
$$\begin{cases} \Delta u + \frac{1}{\varepsilon^2} e^{u+u_0}(1 - e^{u+u_0}) = \frac{4\pi N}{|\Omega|}, & \text{in } \Omega, \\ u \text{ is doubly periodic on } \partial\Omega, \end{cases}$$

Note that u_0 has a singularity at p_j. But near p_j,

$$e^{u_0} \sim |x - p_j|^{2m}$$

and e^{u_0} is a smooth function.

Let u_ε be a solution of (2). Integrate (2):

$$\int_\Omega e^{u_\varepsilon+u_0}(1-e^{u_\varepsilon+u_0}) = 4\pi N\varepsilon \to 0.$$

Since $e^{u_\varepsilon+u_0}(1-e^{u_\varepsilon+u_0}) \geq 0$, we find either

$$u_\varepsilon + u_0 \to 0, \quad \text{a.e. in } \ \Omega,$$

or

$$u_\varepsilon + u_0 \to -\infty, \quad \text{a.e. in } \ \Omega.$$

Known Result for (2)**:**

For $\varepsilon_n \to 0$, then one of the following is true (K.Choe and N.Kim, 2008)

(a) $u_n + u_0 \to 0$ uniformly in any compact subset of $\Omega \setminus \{p_1, \cdots, p_N\}$;

(b) $u_n + \ln \frac{1}{\varepsilon_n^2}$ is bounded;

(c) there is a finite set $S = \{q_1, \cdots, q_L\} \subset \Omega$ and $x_{1,n}, \cdots, x_{L,n} \in \Omega$, such that as $n \to +\infty$, $x_{j,n} \to q_j$,

$$u_n(x_{j,n}) + \ln \frac{1}{\varepsilon_n^2} \to +\infty, \quad \forall\, j = 1, \cdots, L,$$

and

$$u_n(x) + \ln \frac{1}{\varepsilon_n^2} \to -\infty, \quad \text{uniformly on any compact subset of } \Omega \setminus S.$$

Moreover,

$$\frac{1}{\varepsilon_n^2} e^{u_n+u_0}(1 - e^{u_n+u_0}) \to \sum_{j=1}^{L} M_j \delta_{q_j}, \quad M_j \geq 8\pi.$$

Solution satisfying (a) is called a topological solution. Solution satisfying (b) or (c) is called a non-topological solution. Solution satisfying (c) is called a bubbling solution.

The limit problem is given by either

$$\Delta u + |x|^{2m} e^u = 0, \quad \text{in } \mathbb{R}^2, \quad \lim_{|x| \to \infty} u(x) = -\infty; \tag{3}$$

or

$$\Delta u + |x|^{2m} e^u (1 - |x|^{2m} e^u) = 0, \quad \text{in } \mathbb{R}^2, \quad \lim_{|x| \to \infty} u(x) = -\infty. \tag{4}$$

Here, $m = 0$ if the blow-up point is different from any of the vortex points p_j, and m is the number of vortex points which coincide with q_i if the blow-up point is q_i.

- All the entire solutions of (11) have been classified.
- For (4) with $m > 0$, only the entire radial solutions have been classified.
- All the solutions of either (11) or (4) carry an energy of at least 8π. This is the reason that in (c) of Theorem A, the weight $M_j \geq 8\pi$.

Question:

- Are there solutions satisfying each case (a), or (b), or (c)?
- In (c), can M_j be any constant bigger than 8π?
- Simple blow up: is there just one bubble near each blow up point?
- Where is the blow up points?
- Can we count the exact number of the solutions for (2)?

Simple result: If $N = 1$, then $\frac{1}{\varepsilon^2}\int_\Omega e^u(1 - e^u) = 4\pi$. So (c) can not occur in this case.

Topological Solution

- Caffarelli and Y.Yang (1995): If $\varepsilon > 0$ is small, (2) always has a topological solution.
- Tarantello (2007): (2) has a unique topological solution provided that $\varepsilon > 0$ is small enough.

Non-topological Solutions

Type (b) solutions:

- Tarantello (1996): If $N = 1$, (2) has a solution satisfying (b).
- Nolasco and Tarantello (1999): For $N = 2$, (2) has a solution satisfying (b) if

(5)
$$\inf\Big\{\frac{1}{2}\int_\Omega |Du|^2 - 8\pi \ln \int_\Omega e^{u_0+u} : \int_\Omega u = 0 \text{ and } u \text{ is periodic in } \partial\Omega\Big\}$$

is achieved.

If (b) occurs, then $u_n + \ln \frac{1}{\varepsilon_n^2}$ converges to u in C^1 and u satisfies the following mean field equation:

(6)
$$\begin{cases} -\Delta u = e^{u+u_0} - \frac{4N\pi}{|\Omega|}, \quad \text{in } \Omega, \\ u \;\text{ is doubly periodic on } \partial\Omega, \end{cases}$$

Note that if (6) has no solution, then (2) has no type (b) solution.

Single Bubbling Solutions

Problem (raised by Tarantello): Can one find solutions u_ε, such that u_ε concentrates as $\varepsilon \to 0$ at any given critical point of e^{u_0}?

Critical point x_0 of e^{u_0}:

- x_0 is a critical point of u_0;
- x_0 is one of the singular point p_j.

In this talk, we will give the following answer for the above question:

- Yes, if x_0 is a critical point of u_0 and $N \geq 3$;
- Yes, if x_0 is one of the singular point p_j and $N \geq 5$;
- No, if $N = 2$;

Existence of Single Bubbling Solutions

- Nolasco and Tarantello (1999): For $N = 2$, (2) has a solution satisfying (c) if

  (7)
  $$\inf\{\frac{1}{2}\int_\Omega |Du|^2 - 8\pi \ln \int_\Omega e^{u_0+u} : \int_\Omega u = 0 \text{ and } u \text{ is periodic in } \partial\Omega\}$$

  is not achieved. Moreover, $L = 1$ and $p = p_1$ satisfies $u_0(p) = \max_{x\in\Omega} u_0(x)$.

- Choe (2007): For $N = 3$, (2) has a solution satisfying (c) with $L = 1$ and $p = p_1$ satisfies $u_0(p) = \max_{x\in\Omega} u_0(x)$.

The following theorems are about the existence of single bubbling solutions.

Theorem 0.1. *(C.S. Lin and S. Yan, CMP (2010)) Assume that* $N \geq 5$*. Then there is an* $\varepsilon_0 > 0$*, such that for any* $\varepsilon \in (0, \varepsilon_0)$ *and* p_i *satisfying* $p_i \neq p_j$ *for all* $j \neq i$*,* (2) *has a solution* u_ε*, satisfying*

$$\frac{1}{\varepsilon^2} e^{u_\varepsilon} \left(1 - e^{u_\varepsilon}\right) \to 4\pi N \delta_{p_i},$$

as $\varepsilon \to 0$*.*

Remark.

- We need to assume that $p_i \neq p_j$ for all $j \neq i$ because we can only prove that the radial solution of

$$\Delta u + |x|^{2m} e^u (1 - |x|^{2m} e^u) = 0, \quad \text{in } \mathbb{R}^2, \quad \lim_{|x| \to \infty} u(x) = -\infty$$

  is non-degenerate for $m = 1$;

- We believe that $p_i \neq p_j$ for all $j \neq i$ can be relaxed, but it can not be removed totally. Due to the energy constraint, the number m of the p_j with $p_j = p_i$ must satisfy $2(m+1) \leq N$.

Theorem 0.2. *(C.S. Lin and S. Yan, CPAM, 2013) Suppose that $N \geq 3$. Assume that p_0 is an isolated critical point of the function u_0 satisfying $\deg(Du_0, p_0) \neq 0$. Then there is an $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$,* (1) *has a solution u_ε, satisfying*

$$\frac{1}{\varepsilon^2} e^{u_\varepsilon} \left(1 - e^{u_\varepsilon}\right) \to 4\pi N \delta_{p_0},$$

as $\varepsilon \to 0$.

Theorem 0.3. *(C.-S.Lin and S. Yan: ARMA, 2013) Suppose that* $N = 2$*. Assume that* p_0 *is an isolated critical point of the function* u_0 *satisfying* $\deg(Du_0, p_0) \neq 0$ *and*

(8)
$$\begin{aligned} D(p_0) := \int_\Omega & \frac{e^{8\pi(\gamma(y,p_0)-\gamma(p_0.p_0))+(u_0(y)-u_0(p_0))} - 1}{|y-p_0|^4}\, dy \\ & - \int_{\mathbb{R}^2\setminus\Omega} \frac{1}{|y-p_0|^4}\, dy < 0, \end{aligned}$$

where $\gamma(y, x)$ *is the regular part of* $G(y, x)$*. Then there is an* $\varepsilon_0 > 0$*, such that for any* $\varepsilon \in (0, \varepsilon_0)$*,* (2) *has a solution* u_ε*, blowing up at* p_0 *as* $\varepsilon \to 0$*.*

- Del Pino, Musso and Esposito: Construct bubbling solutions blowing at a vortex point in the case $N = 4$;
- Condition (8) is nearly necessary.

The exact number of solutions in the case $N = 2$

We now discuss the number of solutions that (2) can have.

From the above discussion, we know that if $N \geq 3$, then the number of solutions for (2) is at least the number of the critical points of $u_0 + 1$. Actually, it has more solutions than this.

The case $N = 2$ is different.

Now we introduce a recent result obtained by C.S.Lin and myself:

Theorem 0.4. *Suppose that Ω is a rectangle, $N = 2$ and $p_1 = p_2 = p$. If $\varepsilon > 0$ is small, then* (2) *has exactly two solutions. One is the topological solution; the other is the bubbling solution blowing up at the maximum point q of u_0.*

Note that under the condition of Theorem 0.4,

$$u_0 = 8\pi G(x, p)$$

has three critical points (two saddle points and one maximum point), all of which are non-degenerate. Theorem 0.4 shows two of these critical points (in fact, the two saddle points) can not generate a bubbling solution.

Idea of the proof of Theorem 0.4:

By a result of C.-S. Lin and C.-L.Wang (Ann. Math., 2010), if Ω is a rectangle, $N = 2$ and $p_1 = p_2 = p$, then

(9)
$$\begin{cases} -\Delta u = e^{u+u_0} - \frac{8\pi}{|\Omega|}, & \text{in } \Omega, \\ u \ \text{ is doubly periodic on } \partial\Omega, \end{cases}$$

has no solution. So, (2) just has a unique topological solution and some bubbling solutions. To prove Theorem 0.4, we just need to prove the uniqueness of bubbling solution.

Uniqueness of Bubbling Solution

(1) The limit equation.

Integrating (2), we find

$$\frac{1}{\varepsilon^2}\int_\Omega e^{u_\varepsilon+u_0}(1-e^{u_\varepsilon+u_0}) = 8\pi. \tag{10}$$

From this energy identity, we can deduce that any bubbling solution must have just one bubble, and the limit equation for this solution is given by

$$\Delta u + e^u = 0, \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^u < +\infty. \tag{11}$$

(2) A necessary condition for the blow-up point.

Let x_ε be the blow-up point of the bubbling solution u_ε. We have

Theorem 0.5. *Suppose that $x_\varepsilon \to q$. Then q is a critical point of u_0, and*

$$D(q) := \int_\Omega \frac{e^{8\pi(\gamma(y,q)-\gamma(q,q))+(u_0(y)-u_0(q))} - 1}{|y-q|^4}\, dy - \int_{\mathbb{R}^2\setminus\Omega} \frac{1}{|y-q|^4}\, dy \leq 0, \tag{12}$$

where $\gamma(y,x)$ is the regular part of $G(y,x)$.

Under the condition of Theorem 0.4, u_0 has three critical points, all of which are non-degenerate. Two of them are saddle points at which the quantity D is positive. At the maximum point q of u_0, the quantity D is negative. So only the maximum point of u_0 can generate a bubbling solution. We need to show that this maximum point can only generate one bubbling solution.

(3) Uniqueness of bubbling solution.

Theorem 0.6. *Suppose that* q *is a non-degenerate critical point of* u_0 *satisfying* $D(q) < 0$*. Then there exists a unique bubbling solution for* (1)*, blowing up at* q*.*

Remark: The assumption that Ω is a rectangle and $p_1 = p_2 = q$ is not essential. What we need is the following:

- The following problem

(13) $$\begin{cases} -\Delta u = e^{u+u_0} - \frac{8\pi}{|\Omega|}, & \text{in } \Omega, \\ u \ \text{ is doubly periodic on } \partial\Omega, \end{cases}$$

  has no solution. This will exclude type (b) solution

- At all the critical point q, $D(q) \neq 0$. There is just one critical point q, which is non-degenerate and $D(q) < 0$. This will guarantee the uniqueness of bubbling solution.

Multi-bubbling Solutions

For simplicity, we consider

$$\begin{cases} \Delta u + \frac{1}{\varepsilon^2} e^{u+u_0}(1 - e^{u+u_0}) = \frac{4\pi N}{|\Omega|}, \quad \text{in } \Omega, \\ u \text{ is doubly periodic on } \partial\Omega, \end{cases} \tag{14}$$

with $u_0 = -4\pi N G(x, p)$.

Theorem 0.7. *(C.-S. Lin and Yan)*

Let u_ε *be a solution of* (14) *blowing up at* k *points* $q_1, \cdots, q_k$*. Then,* $q_j \neq p$ *and* $M_j = \frac{4\pi N}{k} \geq 8\pi$ *for* $j = 1, 2, \cdots, k$*. In addition, all the blow up points are simple. Moreover, it holds*

$$kDG(q_j, p) = \sum_{h \neq j} DG(q_j, q_h), \quad j = 1, \cdots, k. \tag{15}$$

Conversely, if $M_j = \frac{4\pi N}{k} > 8\pi$*, then for any* $(q_1, \cdots, q_k)$ *satisfying* (15)*, which is also non-degenerate, then* (14) *has a unique solution blowing up at* $(q_1, \cdots, q_k)$*.*

Extra necessary condition in the case $N = 2k$.

Suppose that $M_j = \frac{4\pi N}{k} = 8\pi$. For $\mathbf{q} = (q_1, \cdots, q_k)$, we define the quantity $D(\mathbf{q})$ (similar to (12)) as follows.

$$D(\mathbf{q}) = \sum_{i=1}^{k} \rho_i \Big(\int_{\Omega_i} \frac{e^{f_{\mathbf{q},i}} - 1}{|y - q_i|^4} - \int_{\mathbb{R}^2 \setminus \Omega_i} \frac{1}{|y - q_i|^4} \Big), \tag{16}$$

where Ω_i is any open set satisfying with $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$, $\cup_{i=1}^{k} \bar{\Omega}_i = \bar{\Omega}$, $B_\delta(q_i) \subset\subset \Omega_i, i = 1, \cdots, k$,

$$
\begin{aligned}
f_{\mathbf{q},i}(y) =& 8\pi\Big(\gamma(y,q_i) - \gamma(q_i,q_i) + \sum_{j\neq i}\big(G(y,q_j) - G(q_i,q_j)\big)\Big) \\
&+ u_0(y) - u_0(q_i),
\end{aligned}
$$

and

$$
\rho_i = e^{8\pi\left(\gamma(q_i,q_i)+\sum_{j\neq i} G(q_i,q_j)\right)+u_0(q_i)}.
$$

Remark. The quantity $D(\mathbf{q})$ is independent of the decomposition. If $k = 1$, then $D(\mathbf{q})$ becomes $D(q)$ defined in (12).

Theorem 0.8. *(C.-S. Lin and Yan)*

Let u_ε *be a solution of* (14) *blowing up at* k *points* $q_1, \cdots, q_k$ *and* $N = 2k$*. Then, both* (15) *holds and* $D(\mathbf{q}) \leq 0$*.*

Conversely, if $M_j = \frac{4\pi N}{k} = 8\pi$*, then for any* $(q_1, \cdots, q_k)$ *satisfying* (15)*, which is also non-degenerate, and* $D(\mathbf{q}) < 0$*, then* (14) *has a unique solution blowing up at* $(q_1, \cdots, q_k)$*.*

Theorems 0.7 and (0.8) establish a relation between the number of the bubbling solutions and the number of some points satisfying (15). To count the number of bubbling solutions, we need to know

- the number of points satisfying (15) for each $k \leq \frac{N}{2}$; whether they are all non-degenerate;
- the number of points satisfying (15) and $D < 0$; whether it always holds $D \neq 0$.

If (6) has no solution, then (14) has no solution satisfying case (b). So, in this situation, the exact number of the solutions for (14) is the number of bubbling solution plus one.

Assume that Ω is a rectangle.

Results already known (C.-C.Chen, C.-S. Lin and G. Wang, 2003): If $k = 1$, then the number of points satisfying (15) is 3; all of them are all non-degenerate; If $N = 2$, the number of points satisfying (15) and $D < 0$ is one, and in the other two saddle points, $D > 0$.

What can be proved (C.-S.Lin and Yan): If $k = 2$, then the number of points satisfying (15) is 5; all of them are all non-degenerate; If $N = 4$, the number of points satisfying (15) and $D < 0$ is two, and in the other three points, $D > 0$. So, if $N = 4$, then (14) has exactly 6 solutions: one is topological solution; three are single bubbling solutions; and two are solutions with two bubbles.

Our conjecture: For any k, the number of points satisfying (15) is $2k+1$; all of them are all non-degenerate; If $N=2k$, the number of points satisfying (15) and $D<0$ is k, and in the other $k+1$ points, $D>0$. If this can be proved, then (14) has exactly $k(k+1)$ solutions for the case $N=2k$.

The proof of the above result is related to the non-existence of solution for the mean field equation (6).