Noncompact variational problems involving complex unimodular maps

Petru Mironescu

Institut Camille Jordan, Université Lyon 1

Taipei, October 25th, 2013
France-Taiwan Joint Conference on Nonlinear PDEs

Overview

* Digression: a compact problem
* The main tool: a minimum of the modulus principle
* A non compact variational problem
* Uniqueness
* Existence for large ε
* Existence for small ε

"Compact motivation"

Theorem (Dong Ye –Feng Zhou 96)

Assume $\Omega \subset \mathbb{R}^2$ simply connected, $g : \partial\Omega \to \mathbb{S}^1$ smooth of degree 0
Consider the Ginzburg-Landau type (GL) energy

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (1 - |u|^2)^2$$

subject to $u = g$ on $\partial\Omega$

Then, for small ε, there is only one minimizer u_ε of E_ε

If, in addition, $\|g - 1\|_{C^2} \ll 1$, then uniqueness holds for every ε

Main lines of the proof

* For small ε, $|u_\varepsilon| \approx 1$
* Ω being simply connected, write $u_\varepsilon = \rho_\varepsilon e^{\imath \varphi_\varepsilon}$
* The equation of u_ε is “close” to the limiting equation $\Delta \varphi_\varepsilon = 0$
* Uniqueness of the solution of the limiting equation $\Longrightarrow$ uniqueness of the solution of the ε–equation

Technically...

- $*$ Proof relies on uniform (pointwise) bounds on $\nabla\varphi_\varepsilon$
- $*$ This is used to measure how “far” the ε–problem is from the limiting problem (essential for the perturbative argument)

However, natural assumption is $g \in H^{1/2}(\partial\Omega; \mathbb{S}^1)$

When we work with $g \in H^{1/2}(\partial\Omega; \mathbb{S}^1)$ maps...

- $*$ Maps $g \in H^{1/2}(\partial\Omega; \mathbb{S}^1)$ do have a degree (Boutet de Monvel, Gabber 91)
- $*$ For small ε and $\deg g = 0$, we still have $|u_\varepsilon| \approx 1$
- $*$ But the perturbative argument does not work anymore

Uniqueness in degree 0

Theorem (Farina, M 11)

Assume Ω simply connected and $g \in H^{1/2}(\partial\Omega; \mathbb{S}^1)$ of degree 0

Then, for small ε, E_ε has a unique minimizer

If, in addition, $|g - 1|_{H^{1/2}} \ll 1$, then E_ε has a unique minimizer for any ε

A basic ingredient

Minimum of the modulus principle in a nutshell

If an energy minimizer u has small energy, then $|u|$ is almost constant

Minimum of the modulus principle

Let u minimize E_ε wrt its own Dirichlet bc in a simply connected domain Ω

If $|u| = 1$ on $\partial\Omega$ and $|u| < s$ somewhere in Ω, then $\int_\Omega |\nabla u|^2 \geq f(s) > 0$ (with explicit f)

Example

When $s = 0$, the argument gives $f(0) = 2\pi$

Which is optimal: take $\varepsilon = \infty$, $\Omega = \mathbb{D}$, $u = \mathrm{Id}$

Remarks

* ε independent conclusion
* Also works for minimizers of $\int_\Omega |\nabla u|^2 + \int_\Omega F(|u|)$, with suitable F. The conclusion is F-independent
* *Does not* work in multiply connected domains (counterintuitive)

Proof of uniqueness for almost constant g

The proof of
"$|g-1|_{H^{1/2}} \ll 1 \implies$ uniqueness of the mininimizer of E_ε, $\forall\, \varepsilon$"
relies on Wente estimates (Wente 69)

Theorem (Bethuel, Ghidaglia 93)

Let $u \in H^1_0(\Omega)$ solve $-\Delta u = \nabla f \wedge \nabla g$. Then

- $\|u\|_{L^\infty} \leq 2\|\nabla f\|_{L^2}\|\nabla g\|_{L^2}$
- $\|\nabla u\|_{L^2} \leq \sqrt{2}\|\nabla f\|_{L^2}\|\nabla g\|_{L^2}$
- $\boxed{\left|\int_\Omega h\nabla f \wedge \nabla g\right| \leq \sqrt{2}\|\nabla f\|_{L^2}\|\nabla g\|_{L^2}\|\nabla h\|_{L^2}, \forall\, h \in H^1_0(\Omega)}$

Proof of uniqueness for almost constant g

and on...

Identity (Lassoued, M 99)

Let $u, v \neq 0$, with u critical point of E_ε. Write $u = \rho e^{\imath\varphi}$, $v = u\eta e^{\imath\psi}$. Then

$$E_\varepsilon(v) = E_\varepsilon(u) + \approx \underbrace{\int_\Omega |\nabla\psi|^2}_{\text{good term}} + \underbrace{\approx \int |\nabla\eta|^2}_{\text{good term}} +$$

$$+ \underbrace{\int_\Omega (\eta^2 - 1)\rho^2 \nabla\varphi \cdot \nabla\psi}_{\text{bad term}} + \approx \underbrace{\frac{1}{\varepsilon^2} \int_\Omega (1 - \eta^2)^2}_{\text{good potential term}}$$

Strategy for uniqueness

* Let u, v be minimizers
* Prove that $u \neq 0$ and $v \neq 0$
* Control the bad term in order to arrive at $E_\varepsilon(v) - E_\varepsilon(u) > 0$ except when $u = v$

Proof of uniqueness for almost constant g

Sketch

* g almost constant $\implies$ $\exists$ an almost constant competitor u_0 of modulus 1
* Thus minimal energy satisfies (1) $E_\varepsilon(u_\varepsilon) \leq E_\varepsilon(u_0) \ll 1$
* By the minimum of the modulus principle, (2) $|u_\varepsilon| - 1 \ll 1$
* GL equation $\implies$ the bad term can be rewritten as

$$(3) \int_\Omega (\eta^2 - 1)\rho^2 \nabla\varphi \cdot \nabla\psi = \int_\Omega (\eta^2 - 1)\nabla H \wedge \nabla\psi$$

  for some H
* (1) + (2) + (3) + Wente estimates $\implies$ bad term is controlled by the good terms

Proof of asymptotic uniqueness for arbitrary g

The proof of
"$\deg g = 0 \implies$ uniqueness of the mininiminizer of E_ε for small ε"
relies on "asymptotic Wente estimates"

Baby estimate

Let $U_\varepsilon \in H^1_0(\Omega)$ satisfy

$$-\Delta U_\varepsilon + \frac{1}{\varepsilon^2} U_\varepsilon = \nabla f_\varepsilon \wedge \nabla g_\varepsilon$$

If (f_ε) converges in $H^1(\Omega)$, then

$$\|\nabla U_\varepsilon\|_{L^2} = o(1)\|\nabla g_\varepsilon\|_{L^2}$$

Proof of asymptotic uniqueness for arbitrary g

Sketch

* Prove that minimizers satisfy $|u_\varepsilon| \to 1$ as $\varepsilon \to 0$ (blow up argument)
* Determine the equation satisfied by $\eta^2 - 1$
* Use this equation + asymptotic Wente estimates to control the bad term via the good terms (surprise: in the final computation, no need of the good potential term)

A non compact problem: GL with semi-stiff BC

Semi-stiff GL problem

Find minimizers/critical points of E_ε in a (generally multiply connected) domain $\Omega \subset \mathbb{R}^2$ with bc:
$|u| = 1$ on $\partial\Omega$ and $\deg(u, \Gamma_j) = d_j$ given (with Γ_j component of $\partial\Omega$)

Main features (partly conjectural)

* Allows boundary vortices (another model: Kurzke 06 for thin magnetic films)
* Critical points always exist (for small ε)
* Minimizers sometimes do exist, sometimes do not exist
* Non compact problem

Golovaty, Berlyand 02, Berlyand, M 06, Golovaty, Berlyand, Rybalko 09, Dos Santos 09, Berlyand, Rybalko 10, Farina, M 11, Berlyand, M, Sandier, Rybalko 12, Lamy, M 13

A existence + uniqueness case

Theorem (Golovaty, Berlyand 02)

Let $\Omega = \mathbb{D}_R \setminus \mathbb{D}$, with $R - 1 \ll 1$ (thin circular annulus)

Then E_ε attains its minimum in the class

$$\{u : \Omega \to \mathbb{C}; |u| = 1 \text{ on } \partial\Omega, \deg u = 1 \text{ on } C_R \text{ and } C_1\}$$

In addition, “the” minimizer is unique and radial

Uniqueness in thin domains

Theorem (Farina, M 11)

There is some $\delta > 0$ s.t., if $\inf E_\varepsilon(u) < \delta_0$, then E_ε has a "unique" minimizer (with prescribed degrees)

Remark

δ_0 does not depend on the prescribed degrees

Sketch of proof

* Prove compactness of minimizing sequences. Otherwise, formation of bubbles. But not enough energy
* This leads to existence of minimizers
* Extend the minimum of the modulus principle to prescribed degrees minimizers
* Then proceed as for the uniqueness of minimizers in case of almost constant g

Remark

Recall that, in multiply connected domains, the minimum of the modulus principle requires *some* extra assumption in addition to u being a minimizer of E_ε wrt its own Dirichlet bc

Existence issues

What is known

* In multiply connected domains, critical points do exist for small ε (Berlyad, Rybalko 10 for doubly connected domain, Dos Santos 09 for general multiply connected domains)
* Such solutions are built as local minimizers of E_ε
* Construction does not work in simply connected domains; similar with help from the topology of the domain in case of the critical Sobolev exponent (Coron 84, Bahri – Coron 88)

Existence issues

In simply connected domains

* Fact: no minimizer (except for $\varepsilon = \infty$)
* Critical points do exist (partial results)

Proposition

If $\varepsilon < \infty$ and prescribed degree $\neq 0$, then no minimizer

Proof.

Asume e.g. $\Omega = \mathbb{D}$ and $d = 1$. Then $|\nabla u|^2 \geq 2\text{Jac}\, u \implies$

$$\frac{1}{2}\int_{\mathbb{D}} |\nabla u|^2 \geq \int_{\mathbb{D}} \text{Jac}\, u = \pi \deg(u, \partial\mathbb{D}) = \pi$$

so that $E_\varepsilon(u) > \pi$
Now test $E_\varepsilon(M_a)$ (Moebius transform centered at a) and let $|a| \nearrow 1$ to obtain $\inf E_\varepsilon(u) \leq \pi$ □

Remark

In general, $\inf E_\varepsilon = \pi d$, with d the prescribed degree, and inf is not attained

Analysis for large ε

Theorem (Berlyand, M, Rybalko, Sandier 12)

Assume Ω simply connected, and prescribe degree 1 on the boundary
Then E_ε has critical points for *large* ε

Remarks

* Probably holds also for degree ≥ 2, but no proof
* We may work on $\Omega = \mathbb{D}$ (price to pay: a weight in the potential)
* Rough plan: start from $\varepsilon = \infty$, and perturb the problem
* Help from the minimum of the modulus principle

A tool: almost Blaschke products

Let

$$M_a = \frac{z-a}{1-\overline{a}z},\ a \in \mathbb{D}, z \in \overline{\mathbb{D}} \text{ (Moebius transform)}$$

$$B_{\alpha, a_1, \ldots, a_d} = \alpha \prod_{j=1}^{d} M_{a_j},\ \alpha \in \mathbb{S}^1,\ a_1, \ldots, a_d \in \mathbb{D} \text{ (Blaschke product)}$$

Blaschke products

Proposition

Blaschke products are precisely critical points of E_∞ in $\mathbb{D}$ with prescribed degree $d > 0$

Hint

Compute the Hopf differential of critical points

Almost Blaschke products

Remarks

- Thus when $\varepsilon = \infty$, critical points = energy minimizers (all Blaschke products B have energy $E_\infty(B) = \pi\, d$)
- Search of critical points when $\varepsilon \gg 1$ leads to maps of energy $E_\varepsilon(u) - \pi d \ll 1$ (and thus $E_\infty(u) - \pi d \ll 1$)
- Energetically, these are “almost” Blaschke products
- Their structure? Crucial matter for the existence of critical points
- Objects better fitted for analysis: traces on $\mathbb{S}^1$, thus $g : \mathbb{S}^1 \to \mathbb{S}^1$ s.t. $\deg g = d$ and $|g|^2_{H^{1/2}} - \pi\, d \ll 1$, where

$$|g|^2_{H^{1/2}} := \frac{1}{2}\int_{\mathbb{D}} |\nabla u|^2, \text{ with } u \text{ the harmonic extension of } g$$

Structure of almost Moebius transforms

Theorem (Berlyand, M, Rybalko, Sandier 12)

Assume $|g|^2_{H^{1/2}} \leq \pi + \delta < 2\pi$ (i.e., no room for 2 Moebius transforms M_a) and $\deg g = 1$
Then we may write $u = M_a e^{\imath\varphi}$ with (1) $|\varphi|_{H^{1/2}} \leq F(\delta)$

In addition, for small δ we may pick a s.t. $g \mapsto a$ is continuous

Remark

The phase control part of the statement (estimate (1)) is not intuitively clear: if we take $g : \mathbb{S}^1 \to \mathbb{S}^1$ smooth and of zero degree, then its smooth phase φ does not satisfy (1)

Structure of almost Moebius transforms

Sketch of proof for small δ

* Minimum of the modulus principle $\Longrightarrow$ there is no room for two zeros of the harmonic extension u of g
* From this, control the region where $|u| \not\approx 1$, then the phase of u, then (by taking traces) the one of g
* We may take $a =$the zero of u. Continuity comes essentially from uniqueness

An application: killing the Moebius group

The set

$$X := \{g \in H^{1/2}(\mathbb{S}^1;\mathbb{S}^1);\ \deg g = 1\}$$

is not weakly closed (troubles come from the action of the Moebius group)
However:

$$\{g \in X; |g|^2_{H^{1/2}} \leq \pi + \delta < 2\pi\}/\text{Moebius group}$$

is weakly closed

Structure of almost Blaschke products

Theorem (Berlyand, M, Rybalko, Sandier 12)

Let $d \geq 2$

Then there exist $\varepsilon, C > 0$ such that

$$g : \mathbb{S}^1 \to \mathbb{S}^1, |g|^2_{H^{1/2}} \leq \pi d + \varepsilon, \text{ deg } u = d \Longrightarrow$$

$u = B_{1, a_1, \ldots, a_d} e^{\imath\varphi}$, with $|\varphi|_{H^{1/2}} \leq C$

Remark

Probably $\forall\, \varepsilon < 2\pi$ works...

Idea of proof

Induction, relying on the case $d = 1$ + Wente estimates in order to obtain almost orthogonal decomposition of the energy

Back to the existence of critical points of E_ε in simply connected domains. Recall

Theorem (Berlyand, M, Rybalko, Sandier 12)

Assume Ω simply connected, and prescribe degree 1 on the boundary
Then E_ε has critical points for *large* ε

Sketch of proof

* Min-max method: consider

$$\min_{F} \max_{a \in \mathbb{D}} \{ E_\varepsilon(F(a));\ F \in C(\mathbb{D}; H^1), F(a) = M_a \text{ for } |a| \approx 1$$
$$|F(a)| = 1 \text{ on } \mathbb{S}^1,\ \forall\, a \in \mathbb{D}\}$$

* Establish mountain pass geometry (relies on the structure of almost Moebius maps)
* Prove that the energy functional is C^1 (small miracle)
* Next establish behavior of Palais-Smale (PS) sequences. Requires killing the Moebius group (rescaling)
* Establish decomposition of the energy (bubbling). Relies on Wente estimates
* Identify all possible limits. Relies on the minimum of the modulus principle
* Establish compactness of PS sequences (and conclude)

All steps but the identification of the limit and can be performed for arbitrary ε, degrees and multiply connected domains (via additional Wente type estimates)
This leads to bubbling analysis à la Brezis-Coron or Struwe, but not to compactness

A non local critical problem

How much it takes to wind once (or more)

Find

$$m_p = \min\{|g|^p_{W^{1/p,p}};\ g : \mathbb{S}^1 \to \mathbb{S}^1, \deg g = 1\}$$

with

$$|g|^p_{W^{1/p,p}} = \iint_{\mathbb{S}^1\times\mathbb{S}^1} \frac{|g(x)-g(y)|^p}{|x-y|^2}\, dxdy$$

Main difficulty: non compact (energy invariant by the Moebius group)

Easy cases

$p = 1$: $m_1 = 2\pi$, minimizers are $W^{1,1}$-maps with non decreasing phases
$p = 2$: $m_2 = 4\pi^2$, minimizers are Moebius maps

Proof when $p = 2$

Write $g = \sum a_n e^{\imath n\theta}$. Then

$$|g|^2_{H^{1/2}} = 4\pi^2 \sum |n||a_n|^2$$

and

$$\deg g = \sum n|a_n|^2$$

□

Theorem (M 13)

There exists some $\varepsilon > 0$ such that m_p is attained when $p \in (2-\varepsilon, 2)$

Sketch of proof

For such p,

$$\{g : \mathbb{S}^1 \to \mathbb{S}^1;\ \deg g = 1,\ |g|^p_{W^{1/p,p}} \approx m_p\}/\text{Moebius group}$$

is weakly closed $\square$

I do not know what happens when $\deg g \geq 2$ (even for p close to 2)

Analysis for small ε

Theorem (Lamy, M 13)

Let Ω be simply connected, and $d \geq 1$. Then, for small ε

* Under some (explicit) non degeneracy assumptions on Ω, E_ε has critical points u_ε with prescribed degree d
* In particular, critical points do exist when $d = 1$ and Ω is close to a disc
* When $d = 1$, the non degeneracy assumptions are "generically" satisfied

Remark

The non degeneracy assumptions look like generic ones. But we do not know whether they are indeeed generic when $d \geq 2$ or in multiply connected domains

Strategy of the proof

* Assume existence of critical points. Find formal limit
* In the spirit of Bethuel, Brezis, Hélein 94, limit should be of the form

$$u_0(z) = \prod_{j=1}^{d} \left(\frac{z - a_j}{|z - a_j|} \right) e^{\imath H(z)}$$

with unknown $a_1, \dots, a_d \in \Omega$ and H harmonic

* There is a formal relation between $a = (a_1, \dots, a_d)$ and $g := \operatorname{tr} u_0$: the configuration a is a critical point of some appropriate renormalized energy (intuitively not so clear)

Strategy of the proof -ctd

- Next step consists in constructing critical points of E_ε with Dirichlet boundary condition $g_\varepsilon \approx g$ and "emanating from" a "singular" configuration $a_\varepsilon \approx a$
- This can be performed by either variational methods (Fang Hua Lin, Tai-Chia Lin 97, del Pino, Felmer 97) or gluing methods (Pacard, Rivière 00 , del Pino, Kowalczyk, Musso 06)
- This requires a (first) nondegeneracy assumption
- Next find g_ε s.t. the solution with boundary value g_ε is a critical point with prescribed degrees
- This requires a (second) nondegeneracy assumption
- Existence of g_ε is not obtained by inverse functions, but by Leray-Schauder degree theory

Strategy of the proof -ctd

* Up to now, everything adapts to arbitrary domains and degrees
* But we were able to prove genericity only when Ω is simply connected and $d = 1$
* Even when $d = 1$, nondegeneracy assumptions do not look generic if taken separately. But their couple is generically satisfied
* The last part relies on transversality results (à la Quinn 70)

Thank you for your attention!