Semilinear Elliptic Equations in Convex Domains and Convex Rings

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INTRODUCTION

Semilinear elliptic equation

$$\Delta u + f(u) = 0 \ \text{ in } \Omega$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ (dimension $N = 2$ or $N \geq 2$)

Function f locally Hölder continuous

Classical solution u

- Convex domain Ω

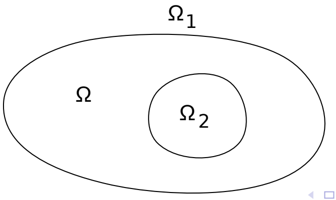
$$\begin{cases} u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega \end{cases}$$

- Convex ring

$$\Omega = \Omega_1 \backslash \overline{\Omega_2}$$

with Ω_1, Ω_2 convex and $\overline{\Omega_2} \subset \Omega_1$

$$\begin{cases} u = 0 & \text{on } \partial\Omega_1, \\ u = M > 0 & \text{on } \partial\Omega_2, \\ u > 0 & \text{in } \Omega \end{cases}$$



How do the solutions u inherit the geometrical properties of Ω ?

Concavity or convexity of u are too strong properties: not true in general

What about the convexity of the *superlevel sets* of u ?

- Convex domain Ω:

$$\Omega^\lambda = \{x \in \Omega;\ u(x) > \lambda\}$$

- Convex ring $\Omega = \Omega_1 \backslash \overline{\Omega_2}$. Extend u inside Ω_2:

$$\overline{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ M & \text{if } x \in \overline{\Omega_2} \end{cases}$$

and

$$\Omega^\lambda = \{x \in \Omega_1;\ \overline{u}(x) > \lambda\}$$

The superlevel set Ω^0 is convex by assumption: $\Omega^0 = \Omega$ if Ω is convex, and $\Omega^0 = \Omega_1$ if Ω is a convex ring.

If Ω is a convex ring and $u < M$ in Ω, then $\displaystyle\bigcap_{\lambda<M} \Omega^\lambda = \overline{\Omega_2}$ is convex.

What about the other superlevel sets Ω^λ for $\lambda > 0$?

The function u is called *quasiconcave* if the superlevel sets Ω^λ are convex for all $\lambda \geq 0$.

Question: is u always quasiconcave ?

P.-L. Lions (for convex domains Ω):

"*we believe that* [...] *for general* f,
the [super]*level sets of any solution* u *of* [(1)] *are convex*"

(*Two geometrical properties of solutions of semilinear problems*, 1981)

CONVEX DOMAINS

Many positive examples in convex domains Ω:

- Torsion problem $f(u) = 1$ $(\Delta u + 1 = 0)$: $\sqrt{u}$ is concave ($\Longrightarrow$ u is quasiconcave) [Makar-Limanov, with $N = 2$]
- Eigenvalue problem $f(u) = \lambda u$ ($\Delta u + \lambda u = 0$, principal eigenvalue $\lambda > 0$): u is log-concave, whence quasiconcave [Brascamp, Lieb]
- $f(u) = \lambda u^p$ with $\lambda > 0$ and $0 < p < 1$: $u^{(1-p)/2}$ is concave ($\Longrightarrow$ u is quasiconcave) [Keady, $N = 2$], [Kennington, $N \geq 2$]
- Many generalizations and alternate proofs:
  - $g(u)$ is concave for some increasing g (elliptic maximum principle, preservation of concavity of $g(u)$ by a parabolic equation) [Caffarelli, Spruck], [Greco, Porru], [Kawohl], [Kennington], [Korevaar], [Lions]
  - curvature of the level sets of u, rank of the Hessian matrix of $g(u)$ [Acker, Payne, Philippin], [Bian, Guan, Ma, Xu], [Caffarelli, Friedman], [Korevaar, Lewis], [Xu]
  - general overview [Kawohl]

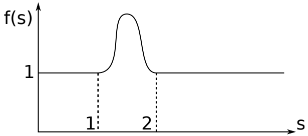
A counterexample

Theorem

Dimension $N = 2$. There are smooth convex domains Ω and C^∞ functions $f : [0, +\infty) \to \mathbb{R}$ for which the problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega \end{cases} \tag{1}$$

admits a solution u which is not quasiconcave.



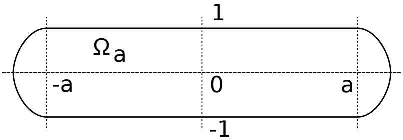
- $f(s) \geq 1$ for all $s \geq 0$
- Problem (1) also admits a quasiconcave solution v

Remark

When Ω is a ball, then u is necessarily radially symmetric and decreasing, whence quasiconcave [Gidas, Ni, Nirenberg]

The theorem cannot hold in dimension $N = 1$!

Domains Ω of the theorem:



$\Longrightarrow$ u is symmetric in x and y, and decreasing in $|x|$ and $|y|$ [G-N-N]

$\Longrightarrow$ the superlevel sets Ω^λ are convex and symmetric in x and y, and starshaped with respect to $(0,0)$, and u has only one critical point (more general results about uniqueness of critical points: [Cabré, Chanillo], [Payne], [Sperb])

$\not\Longrightarrow$ convexity of Ω^λ

Proof of the theorem

Step 1: notations

- Fixed $C^\infty(\mathbb{R})$ function g such that

$$g = 0 \text{ on } (-\infty, 1], \quad g = 1 \text{ on } [2, +\infty) \ \text{ and } \ g' \geq 0 \text{ on } \mathbb{R}$$

- Smooth convex stadium-like domains Ω_a, symmetric in x and y

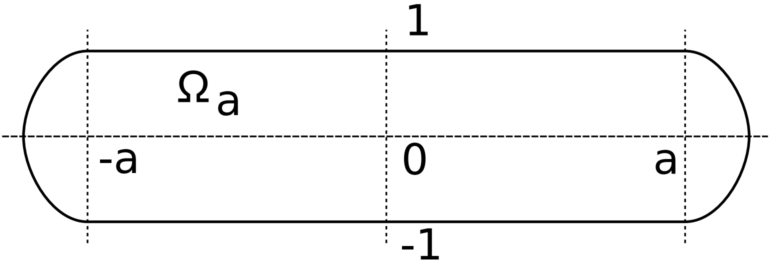


Figure: The domain Ω_a

Step 2: unconstrained and constrained variational problems

- Functional I_a defined in $H^1_0(\Omega_a)$

$$I_a(u)=\frac{1}{2}\int_{\Omega_a}|\nabla u|^2-\int_{\Omega_a}u,\quad u\in H^1_0(\Omega_a)$$

- Unique minimizer v_a:

$$\begin{cases}\Delta v_a+1 & = & 0 \text{ in } \Omega_a,\\ v_a & = & 0 \text{ on } \partial\Omega_a\end{cases}$$

The function v_a is quasiconcave [Makar-Limanov] and

$$0<v_a(x,y)<\frac{1-y^2}{2}\ \text{ in } \Omega_a$$

from the maximum principle

- Constraint

$$U_a = \left\{ u \in H^1_0(\Omega_a);\ \int_{\Omega_a} g(u) = 1 \right\}$$

The set U_a is not empty (for $a \geq 1$: $|\Omega_a| > 1$)

- Existence of a constrained minimizer $u_a \in U_a$:

$$I_a(u_a) = \min_{u \in U_a} I_a(u)$$

Euler-Lagrange equation:

$$\begin{cases} \Delta u_a + f_a(u_a) = 0 & \text{in } \Omega_a, \\ u_a = 0 & \text{on } \partial\Omega_a, \end{cases}$$

with

$$f_a(s) = 1 + \mu_a g'(s)$$

Lagrange multiplier $\mu_a \in \mathbb{R}$. The function f_a is of class C^∞.

Step 3: elementary properties of u_a

- Since $u_a \in U_a$ and $g'(s) = 0$ for $s \leq 1$ and $s \geq 2$:

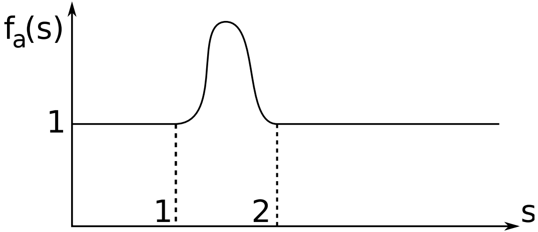
$$\max_{\Omega_a} u_a > 1 \qquad \text{and} \qquad u_a(x,y) < \frac{1-y^2}{2} + 2 \text{ in } \Omega_a$$

- $\Delta(u_a - v_a) = -\mu_a g'(u_a)$ has a sign

  Therefore, $u_a - v_a$ has a sign, from the maximum principle:

$$0 < v_a < u_a \text{ in } \Omega_a$$

Hence $\mu_a > 0$ and $f_a(s) \geq 1$ for all s:



- [Gidas, Ni, Nirenberg] $\Longrightarrow$

u_a is even in x and y and decreasing in $|x|$ and $|y|$

$\Longrightarrow$ unique critical point $(0,0)$

and the superlevel sets of u_a are symmetric and convex in x and y

($\not\Longrightarrow$ the superlevel sets are convex)

Since

$$v_a < \frac{1-y^2}{2} \leq \frac{1}{2} \text{ in } \Omega_a$$

and $f_a = 1 + \mu_a g' = 1$ on $[0,1]$, the function v_a obeys

$$\Delta v_a + f_a(v_a) = \Delta v_a + 1 = 0$$

and v_a is quasiconcave

We will show that u_a is **not** quasiconcave when a is large enough

Step 4: uniform estimates of the size of the superlevel set Ω^1 of u_a

$$\omega_a = \{(x,y) \in \Omega_a;\ u_a(x,y) > 1\}$$

Key-lemma

$$0 \leq \sup_{(x,y)\in\omega_a} |x| < C_x \quad \text{and} \quad 0 < C_y < \sup_{(x,y)\in\omega_a} |y|$$

for some positive constants C_x and C_y independent of $a \geq 1$

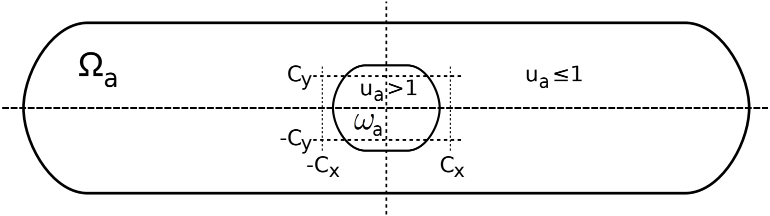


Figure: The superlevel set ω_a

Proof of the key-lemma

- The y-estimate easily follows from the x-estimate

  Denote

$$y_a = \sup_{(x,y)\in\omega_a} |y|$$

There holds

$$u_a \leq 1 \text{ and } g(u_a) = 0 \;\text{ in } \Omega_a \setminus (-C_x, C_x) \times (-y_a, y_a)$$

$\Longrightarrow$

$$1 = \int_{\Omega_a} g(u_a) = \int_{\Omega_a \cap (-C_x, C_x) \times (-y_a, y_a)} g(u_a) \;<\; 4\, C_x\, y_a$$

$\Longrightarrow$

$$y_a > (4C_x)^{-1} =: C_y > 0$$

- The x-estimate: upper bound of $I_a(u_a)$

Denote

$$\phi_0(y) = \frac{1-y^2}{2}$$

The function ϕ_0 is the unique minimizer of the energy in the section:

$$J(\phi) = \frac{1}{2}\int_{-1}^{1} \phi'(y)^2 dy - \int_{-1}^{1} \phi(y) dy, \quad \phi \in H_0^1(-1,1)$$

Fixed nonnegative function $w \in C^\infty(\mathbb{R}^2)$ such that

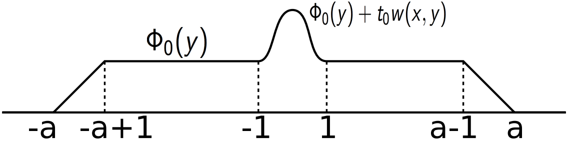
$$w = 0 \text{ in } \mathbb{R}^2 \backslash (-1,1)^2 \quad \text{and} \quad w > 0 \text{ in } [-2/3, 2/3]^2$$

There is $t_0 > 0$ such that

$$\int_{(-1,1)^2} g(\phi_0(y) + t_0 w(x,y))\, dx\, dy = 1$$

Test function $w_a \in H_0^1(\Omega_a)$:

$$w_a(x,y) = \phi_0(y)\,\chi_a(x) + t_0\,w(x,y)$$



$$\int_{\Omega_a} g(w_a) = \int_{(-1,1)^2} g(w_a) = \int_{(-1,1)^2} g(\phi_0(y) + t_0 w(x,y))\,dx\,dy = 1$$

$\Longrightarrow$ $w_a \in U_a$ $\Longrightarrow$ (u_a is a minimizer of I_a in U_a)

$$I_a(u_a) \le I_a(w_a)$$

Explicit calculation:

$$I_a(w_a) = \frac{1}{2}\int_{\Omega_a} |\nabla w_a|^2 - \int_{\Omega_a} w_a = 2\,a\,J(\phi_0) + C_1$$

$\Longrightarrow$

$$I_a(u_a) \le 2\,a\,J(\phi_0) + C_1$$

- The x-estimate: lower bound of $I_a(u_a)$

$$I_a(u_a) \geq \int_{(-a,a)\times(-1,1)} \Big(\frac{|\nabla u_a|^2}{2} - u_a\Big) - C_2 \geq \int_{-a}^{a} J(u_a(x,\cdot))\,dx - C_2$$

Denote

$$x_a = \sup_{(x,y)\in\omega_a} |x| = \sup_{(x,0)\in\omega_a} x$$

For $x \in (-x_a, x_a)$,

$$u_a(x,0) > 1 > 1/2 = \phi_0(0) \implies \|u_a(x,\cdot) - \phi_0\|_{H^1(-1,1)} \geq C_3 > 0$$

$$\implies J(u_a(x,\cdot)) \geq J(\phi_0) + C_4 \text{ with } C_4 > 0$$

Therefore

$$I_a(u_a) \geq 2\,a\,J(\phi_0) + 2\,C_4\,x_a - C_5$$

- Conclusion:

$$2\,a\,J(\phi_0) + 2\,C_4\,x_a - C_5 \leq I_a(u_a) \leq 2\,a\,J(\phi_0) + C_1 \implies x_a \leq C_x$$

Step 5: convergence to the one-dimensional profile $\phi_0(y)=(1-y^2)/2$

- $$u_a \le 1 \ \text{ in } \Omega_a \setminus (-C_x, C_x) \times (-1,1)$$

$$\Longrightarrow \quad (f_a(s) = 1 + \mu_a g'(s) = 1 \text{ for } s \le 1)$$

$$\Delta u_a + 1 = 0 \ \text{ in } \Omega_a \setminus (-C_x, C_x) \times (-1,1)$$

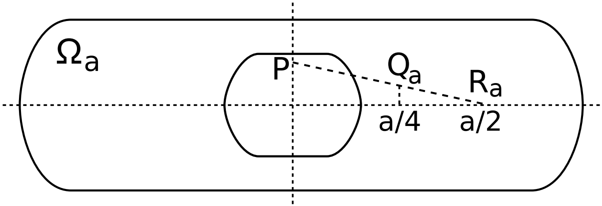
- For all $\varepsilon > 0$, there exist $A \ge 1$ and $M \in [0, A/2]$ s.t. for all $a \ge A$,

$$|u_a(x,y) - \phi_0(y)| \le \varepsilon \text{ in } \big([-a+M, -M] \cup [M, a-M]\big) \times [-1,1]$$

Proof: by contradiction and Liouville-type result for the solutions U of

$$\left\{\begin{array}{rcll} \Delta U + 1 & = & 0 & \text{in } \ \mathbb{R} \times [-1,1] \\ U & = & 0 & \text{on } \ \mathbb{R} \times \{\pm 1\} \end{array}\right. \quad \Longrightarrow \quad U(x,y) = \phi_0(y) = \frac{1-y^2}{2}$$

Step 6: conclusion



$$P = (0, C_y) \quad \Longrightarrow \quad u_a(P) > 1$$

$$Q_a = \left(\frac{a}{4}, \frac{C_y}{2}\right) \quad \Longrightarrow \quad u_a(Q_a) \simeq \frac{1-(C_y/2)^2}{2} = \frac{1}{2} - \frac{C_y^2}{8} \quad \text{for } a \gg 1$$

$$R_a = \left(\frac{a}{2}, 0\right) \quad \Longrightarrow \quad u_a(R_a) \simeq \frac{1}{2} \quad \text{for } a \gg 1$$

Therefore,

$$u_a(Q_a) < \min\big(u_a(P), u_a(R_a)\big) \quad \text{for } a \gg 1$$

For large a, the function u_a is **not** quasiconcave

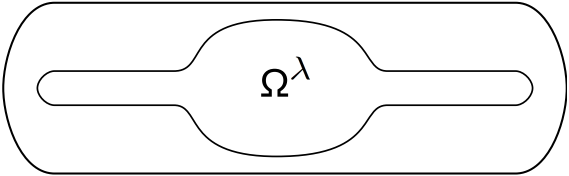
For any fixed λ such that

$$\frac{1}{2} - \frac{C_y^2}{8} < \lambda < \frac{1}{2}$$

the superlevel set

$$\Omega^\lambda = \{(x, y) \in \Omega_a;\ u_a > \lambda\}$$

is **not** convex for large a



Stability of u_a ?

Semi-stability:

$$\forall\,\phi\in C_c^\infty(\Omega),\quad \int_\Omega |\nabla\phi|^2-\int_\Omega f'(u)\phi^2\geq 0$$

[Cabré, Chanillo]: if u is a semi-stable solution of (1) in a strictly convex domain Ω, then it has a unique critical point (its maximum) and the superlevel sets Ω^λ are convex for $\lambda\simeq\max u$

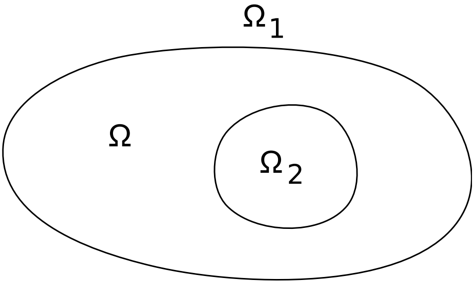
If $f'\leq 0$ on the range of u, then u is semi-stable

If semi-stability implies quasiconcavity, then our solutions u_a would be unstable

For our problem in Ω_a:

$$\left\{\begin{array}{l} f_a'(u_a)=0\ \text{ in the (large) set } \Omega_a\backslash(-C_x,C_x)\times(-1,1),\\ \big\{(x,y)\in\Omega_a;\ f_a'(u_a(x,y))>0\big\}\ \text{ is never empty},\\ \big\{(x,y)\in\Omega_a;\ f_a'(u_a(x,y))<0\big\}\ \text{ is not empty in general}\end{array}\right.$$

CONVEX RINGS



$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega_1, \\ u = M & \text{on } \partial\Omega_2, \\ u > 0 & \text{in } \Omega \end{cases} \tag{2}$$

Many positive examples in convex rings Ω:

- Harmonic functions $f(u)=0$ ($\Delta u=0$) [Gabriel, with $N=3$]
- p-capacitary functions ($\Delta_p u=0$) [Lewis]
- $f(0)=0$ and f nonincreasing [Caffarelli, Spruck]
- Further results and generalizations using properties of the level sets of u or the rank of $D^2 g(u)$ for some increasing g [Caffarelli, Friedman], [Korevaar], [Korevaar, Lewis]
- Minimal points of $(x,y)\mapsto u((x+y)/2)-\min(u(x),u(y))$ [Caffarelli, Spruck], [Diaz, Kawohl], [Greco], [Kawohl]
- Comparison principle for the quasiconcave envelope of u [Bianchini, Longinetti, Salani], [Colesanti, Salani], [Cuoghi, Salani]
- General overview [Kawohl]

Starshaped ring: $\overline{\Omega_2} \subset \Omega_1$ are starshaped with respect to 0

Assume $f(0) = 0$ and f is nonincreasing ($f' \leq 0$)

Then $0 < u < M$ in Ω (max. principle)
and $x \cdot \nabla u(x) \leq 0$ on $\partial\Omega$ (max. principle and starshapedness of Ω)

Call $v(x) = x \cdot \nabla u(x)$

$$\Delta v = x \cdot \nabla \Delta u + 2\Delta u = -f'(u)v - 2f(u)$$

that is

$$\Delta v + f'(u)v = -2f(u) \geq 0$$

Then $v \leq 0$ in Ω and even $v < 0$ in Ω (max. principle)

Then the superlevel sets Ω^λ are starshaped with respect to 0

A counterexample in some convex rings [Monneau, Shahgholian]

In dimension $N = 2$, there are some convex rings and some functions $f \geq 0$ such that any solution u of (2) is not quasiconcave

The functions f are close to a Dirac mass concentrated at some real number $\lambda \in (0, M)$

The construction uses the existence of non-convex domains for some approximated free boundary problems [Acker]

Other counterexamples in arbitrary dimension $N \geq 2$

Functions $f : [0, +\infty) \to \mathbb{R}$ such that

$$\begin{cases} f \text{ is bounded from above: } f(s) \leq C, \\ s \mapsto \dfrac{f(s)}{s} \text{ is decreasing over } (0, +\infty), \\ \text{either } f(0) > 0, \quad \text{or} \quad f(0) = 0 \text{ and } \displaystyle\lim_{s \to 0^+} \frac{f(s)}{s} > \lambda_1(-\Delta, \Omega_1) \end{cases} \tag{3}$$

where $\lambda_1(-\Delta, \Omega_1)$ is the smallest eigenvalue of $-\Delta$ in Ω_1 with Dirichlet boundary conditions on $\partial\Omega_1$

Theorem

Dimension $N \geq 2$. Let Ω_1 be any smooth convex domain of $\mathbb{R}^N$ and f any function satisfying (3). Then there is $M_0 \geq 0$ such that, for all $M \geq M_0$, there are smooth convex rings $\Omega = \Omega_1 \backslash \overline{\Omega_2}$ for which the problem

$$\left\{\begin{array}{rcll} \Delta u + f(u) & = & 0 & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega_1, \\ u & = & M & \text{on } \partial\Omega_2, \\ u & > & 0 & \text{in } \Omega \end{array}\right.$$

has a unique solution u, and u is not quasiconcave.

Examples of such functions f:

- $f(s) = \beta > 0$ constant
- $f(s) = \gamma s - \delta s^p$ with $\gamma > \lambda_1(-\Delta, \Omega_1)$, $\delta > 0$ and $p > 1$

  For any fixed $\gamma > 0$, the condition $\gamma > \lambda_1(-\Delta, \Omega_1)$ is satisfied if Ω_1 contains a ball with a large enough radius

  For these functions f, there is a unique solution u in the convex domain Ω_1 [Berestycki] and u is log-concave [Lions]

Assume, in addition to (3), that there is $\mu > 0$ such that

$$f(s) \leq 0 \quad \text{for all } s \geq \mu$$

Then one can take $M_0 = \mu$ in the theorem and the solutions u satisfy

$$0 < u < M \quad \text{in } \Omega$$

The boundary condition $u = M$ on $\partial\Omega_2$ can be replaced by $u = 1$:

$$\begin{cases} \Delta\tilde{u} + \tilde{f}(\tilde{u}) & = & 0 & \text{in } \Omega, \\ \tilde{u} & = & 0 & \text{on } \partial\Omega_1, \\ \tilde{u} & = & 1 & \text{on } \partial\Omega_2, \\ \tilde{u} & > & 0 & \text{in } \Omega, \end{cases}$$

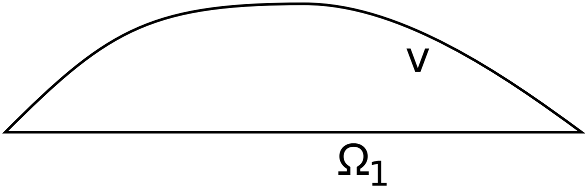
with

$$\tilde{u} = \frac{u}{M} \quad \text{and} \quad \tilde{f}(s) = \frac{f(Ms)}{M}$$

Proof of the theorem

Step 1: unique solution v in Ω_1

$$\begin{cases} \Delta v + f(v) = 0 & \text{in } \Omega_1, \\ v = 0 & \text{on } \partial\Omega_1, \\ v > 0 & \text{in } \Omega_1 \end{cases}$$



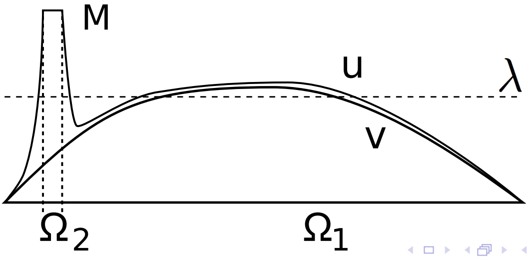
Step 2: unique solution u in $\Omega = \Omega_1 \backslash \overline{\Omega_2}$

Choose

$\Omega_2 = B(x_0, \varepsilon)$ with $0 < v(x_0) < \max_{\overline{\Omega_1}} v = M_0, \quad \varepsilon \ll 1,$ and $M \geq M_0$

There exists a unique solution u of

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega = \Omega_1 \backslash \overline{\Omega_2}, \\ u = 0 & \text{on } \partial\Omega_1, \\ u = M & \text{on } \partial\Omega_2, \\ u > 0 & \text{in } \Omega \end{cases}$$



Step 3: passage to the limit as $\varepsilon \to 0^+$

Denote $\Omega_2 = \Omega_2^\varepsilon$ and $u = u^\varepsilon$

$$u^\varepsilon \to v \ \text{ in } C^2_{loc}\big(\overline{\Omega_1}\backslash\{x_0\}\big) \ \text{ as } \varepsilon \to 0^+$$

Since $v(x_0) < \max_{\overline{\Omega_1}} v$ and $M \geq \max_{\overline{\Omega_1}} v$, it follows that

the function u^ε is **not** quasiconcave for all $\varepsilon > 0$ small enough

Similar result for more general equations

$$\begin{cases} \nabla\cdot(A(x)\nabla u)+b(x)\cdot\nabla u+f(x,u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega_1, \\ u = M & \text{on } \partial\Omega_2, \\ u > 0 & \text{in } \Omega \end{cases}$$

under similar assumptions on $f(x,s)$