On the Lane-Emden equation

Louis Dupaigne
Institut Camille Jordan, UMR CNRS 5208, Lyon, France

Universal estimates and Liouville-type theorems

Local regularity theory for semilinear elliptic equations

Consider the Lane-Emden equation

$$-\Delta u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n \tag{1}$$

and more generally any semilinear elliptic equation of the form

$$-Lu = f(x,u,\nabla u) \quad \text{in } \Omega \tag{2}$$

where Ω is any open set of euclidean space and

- L scales like a laplacian i.e.

  $$Lu = a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + c(x)u$$

  is a uniformly elliptic operator of order 2 with smooth coefficients and ellipticity constants λ, Λ.
- f scales algebraically at infinity i.e. for all $x \in \Omega$, $t \geq 0$, $\xi \in \mathbb{R}^n$,

  $$0 \leq f(x,t,\xi) \leq C(1 + t^p + |\xi|^{\frac{2p}{p+1}})$$

  and for all $x \in \overline{\Omega}$,

  $$\lim_{t\to+\infty, y\in\Omega\to x} \frac{f(y,t,t^{\frac{p+1}{2}}\xi)}{t^p} = \ell(x) \in (0,+\infty),$$

  locally uniformly in ξ.

Theorem (Polacik-Quittner-Souplet, Duke, 2007)

The following assertions are equivalent

1. *The Lane-Emden equation* (1) *has no positive solution*
2. *There exists constants* $C_i = C_i(\lambda, \Lambda, b, c, n, f) > 0$ *such that for all positive solutions of* (2)

$$u(x) \leq C_1 + C_2 dist(x, \partial\Omega)^{-\frac{2}{p-1}}$$

Furthermore, if $f(u) = u^p$, $C_1 = 0$.

$2 \implies 1$ is easy. The reverse implication uses a rescaling procedure [Gidas-Spruck, Comm. PDE, 1981] and a doubling lemma [Gromov, Geom. Funct. Anal, 1991]

Critical exponents

We say that p lies below the critical Sobolev exponent if $p < p_S(n)$, where

$$p_S(n) = \begin{cases} +\infty & \text{if } n \leq 2 \\ \dfrac{n+2}{n-2} & \text{if } n \geq 3 \end{cases}$$

and p lies below the Joseph-Lundgren exponent if $p < p_c(n)$, where

$$p_c(n) = \begin{cases} +\infty & \text{if } n \leq 10 \\ \dfrac{(n-2)^2 - 4n + 8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n \geq 11 \end{cases}$$

Critical exponents

The Lane-Emden equation (1) is scale-invariant: if u is a solution, then so is

$$u^{\lambda}(x) = \lambda^{\frac{2}{p-1}} u(\lambda x).$$

Further, it is variational, with energy functional given by

$$E_0(u; B) = \int_B \left\{ \frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |u|^{p+1} \right\} \, dx$$

- The Sobolev exponent is the unique exponent such that

$$E_0(u; B_\lambda) = E_0(u^\lambda; B_1)$$

- It is natural to consider solutions preserving the scale invariance i.e. homogeneous solutions.

In particular, there exists a singular solution of the form

$$u_s(x) = A|x|^{-\frac{2}{p-1}}.$$

Definition

A solution to the Lane-Emden equation is said to be stable if the second variation of the energy is nonnegative i.e.

$$\int_{\mathbb{R}^n} p|u|^{p-1}\varphi^2\,dx \le \int_{\mathbb{R}^n} |\nabla\varphi|^2\,dx \quad \text{for all } \varphi \in C^1_c(\mathbb{R}^n).$$

It has finite Morse index if the above inequality fails at most on a (punctured) finite dimensional subspace.

Hence, u_s is stable if

$$pA^{p-1}\int_{\mathbb{R}^n} \frac{\varphi^2}{|x|^2}\,dx \le \int_{\mathbb{R}^n} |\nabla\varphi|^2\,dx$$

which holds, in virtue of Hardy's inequality if and only if

$$pA^{p-1} \le \frac{(n-2)^2}{4} \iff p \ge p_c(n).$$

Liouville theorems

1. If $p < p_S(n)$, the Lane-Emden equation has no positive solution [Gidas-Spruck, Comm. PDE, 1981]
2. If $p = p_S(n)$, up to rescaling and translation, the positive solution is unique, thus radial [Caffarelli-Gidas-Spruck, CPAM, 1989]
3. If $p < p_c(n)$, $p \neq p_S(n)$, there is no nontrivial solution of finite Morse index [Farina, JMPA, 2007]
4. Conjecture [Wei, 2013] : if $p_c(n) \leq p < p_c(n-1)$, up to rescaling and translation, there is a unique stable solution, thus radial

Wei's conjecture is motivated by the fact that for $p < p_c(n-1)$, the nonradial function $\tilde{u}_s(x) = A|x'|^{-\frac{2}{p-1}}$ is unstable. For $p_S(n-1) < p < p_c(n-1)$, $\exists\, \infty'$ly many singular sol's, asymptotic to $\tilde{u}_s$, unstable as such, see [Dancer-Guo-Wei, Indiana Math J, 2013]

Partial regularity in the supercritical cases

1. If $p \geq p_S(n)$ and $u > 0$ is a (local) stationary solution, then $u \in C^2(\Omega \setminus \Sigma)$, where Σ is a closed set such that

$$\mathrm{cap}_{2,p'}(\Sigma) = 0.$$

   [Adams, EJDE, 2012]

2. If $p \geq p_c(n)$ and $u \in H^1_{loc}(\Omega)$ has finite Morse index,

$$\mathcal{H}_{dim}(\Sigma) \leq N - 2\frac{p+\gamma}{p-1},$$

   with $\gamma = 2p + 2\sqrt{p(p-1)} - 1$.
   [Dávila-D-Farina, JFA 2010]

Other nonlinearities

1. If $f(u) = e^u$, $N = 2$ and $\int_{\mathbb{R}^2} e^u < \infty$, then up to rescaling and translation, the solution is unique and radial [Chen-Li, Duke, 1981]
2. If $f(u) = e^u$, $3 \leq N \leq 9$, no u has finite Morse index [Dancer-Farina, Proc. Amer. Math. Soc., 2009]
3. If $f \geq 0$, $1 \leq N \leq 4$, every bounded stable solution is constant [D-Farina, JEMS, 2010].
4. If $1 \leq N \leq 2$, every stable solution with bounded gradient is 1D [Berestycki-Caffarelli-Nirenberg, Ann. Scuola Norm. Sup. Pisa, 1997]

The Lane-Emden system

1. Biharmonic Lane-Emden and Liouville eq. : [C.S. Lin, Comment Math. Helv., 1998], [Wei-Xu, Math. Ann., 1999], [D-Ghergu-Goubet-Warnault, ARMA 2012], [Davila-D-Wang-Wei, arxiv].
2. Lane-Emden system not yet understood : see [Mitidieri, CPDE 1993], [de Figueiredo-Felmer, Ann. Sc. Norm. Super. Pisa 1994], [Serrin-Zou, Atti Semin. Mat. Fis. Univ.Modena 1998], [Busca-Manasevich, Indiana 2002], [Polacik-Quittner-Souplet, Duke 2007], [Souplet, Adv Math 2009], [Chen-D-Ghergu, DCDS-A 2013], [Cowan, arxiv].

The fractional Lane-Emden equation

For $s \in (0,1)$ and $p > 1$, consider the equation

$$(-\Delta)^s u = |u|^{p-1} u \qquad \text{in } \mathbb{R}^n,$$

where

$$u \in C^{2\sigma}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n, (1+|x|)^{n+2s} dx),\ 0 < s < \sigma < 1$$

$$(-\Delta)^s u(x) = c_{n,s} PV \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy.$$

Warning 1: many nonlinear diffusion operators

There are many nonlocal diffusion operators and **the fractional Lane-Emden equation is not universal**, as in the local case. For example, take a Lévy symbol of the form

$$\psi(\xi) = \int_{S^{n-1}} |\xi \cdot \omega|^{2s} \mu(d\omega),$$

where μ is a positive measure on S^{n-1}.

- The fractional Laplacian corresponds to the choice of the uniform measure
- The process $(X_t^1, \ldots, X_t^n)$, where X_t^i are independent copies of a (linear) symetric α-stable process ($\alpha = 2s$) corresponds to the choice $\mu = \sum_{i=1}^n \delta_{e_i}$. It is generated by

$$Lu(x) = \sum_{i=1}^{n} \int_{\mathbb{R}} \frac{u(x) - u(x + he_i)}{|h|^{1+2s}} \, dh.$$

Warning 2: (linear) boundary regularity is delicate

Theorem (Ros-Oton and Serra, JMPA 2013)

There exists $\alpha \in (0,1)$ *such that for* $\psi \in L^\infty(\Omega)$*, have*

$$\|\varphi/\delta^s\|_{C^\alpha(\overline{\Omega})} \le C\|\psi\|_{L^\infty(\Omega)},$$

where

$$\begin{cases} (-\Delta)^s\varphi = \psi & \text{in } \Omega, \\ \varphi = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{3}$$

$$u_s(x) = \frac{c_{n,s}}{(1-|x|^2)_+^{1-s}} \qquad \text{solves} \qquad \begin{cases} (-\Delta)^s u = 0 & \text{in } B, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \overline{B}. \end{cases}$$

Theorem (Abatangelo, arxiv 2013)

The problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \\ \delta^{1-s} u = h & \text{on } \partial\Omega. \end{cases}$$

is uniquely solvable. Given φ *solving* (3) *for some* $\psi \in C_c^\infty(\Omega)$*, Green's indentity is*

$$\int_\Omega u(-\Delta)^s\varphi = \int_\Omega \varphi(-\Delta)^s u - \int_{\Omega^c} u(-\Delta)^s\phi + \int_{\partial\Omega} a(x)(\delta^{1-s}u)\left(\frac{\varphi}{\delta^s}\right).$$

Basic properties of the fractional Lane-Emden eq.

The equation

$$(-\Delta)^s u = |u|^{p-1} u \qquad \text{in } \mathbb{R}^n$$

- is scale invariant: if u is a solution, then so is

$$u_\lambda(x) = \lambda^{\frac{2s}{p-1}} u(\lambda x), \quad x \in \mathbb{R}^N, \lambda > 0,$$

- is variational with energy functional given by

$$\int_{\mathbb{R}^n} \left\{ \frac{1}{2} |(-\Delta)^{s/2} u|^2 - \frac{1}{p+1} |u|^{p+1} \right\} dx$$

Critical exponents

We say that p lies below the critical Sobolev exponent if $p < p_S(n)$, where

$$p_S(n) = \begin{cases} +\infty & \text{if } n \leq 2s \\ \dfrac{n+2s}{n-2s} & \text{if } n > 2s \end{cases}$$

and p lies below the Joseph-Lundgren exponent if

$$p \frac{\Gamma(\frac{n}{2} - \frac{s}{p-1})\Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{s}{p-1})\Gamma(\frac{n-2s}{2} - \frac{s}{p-1})} > \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2}.$$

For p supercritical, the above condition fails if and only if the singular solution

$$u_s(x) = A|x|^{-\frac{2s}{p-1}} \text{ is stable .}$$

Known Liouville theorems

1. If $p < p_S(n)$, the Lane-Emden equation has no positive solution [Chen-Li-Ou, CPAM 2006 and Y.Y. Li, JEMS 2004]
2. If $p = p_S(n)$, up to rescaling and translation, the positive solution is unique, thus radial [Chen-Li-Ou, CPAM 2006 and Y.Y. Li, JEMS 2004]

Our Liouville theorem

Theorem ([Dávila-D-Wei])

Let u be a solution with finite Morse index.

- *If p lies below the Joseph-Lundgren exponent, $p \neq p_S(n)$, then $u \equiv 0$;*
- *If $p = p_S(n)$, then u has finite energy i.e.*

$$\int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 \, dx = \int_{\mathbb{R}^n} |u|^{p+1} \, dx < +\infty.$$

If in addition u is stable, then in fact $u \equiv 0$.

The proof

[Bernstein, Comm. Soc. Math. de Kharkov, 1915]

Theorem

Let $N \leq 7$. Assume $u \in C^2(\mathbb{R}^N; \mathbb{R})$ is a solution of the minimal surface equation in $\mathbb{R}^N$. Then, the graph of u is a hyperplane.

Remark

The original proof of Bernstein, in dimension $N = 2$, contained a gap, discovered and fixed by [Hopf, Proc. Amer. Math. Soc., 1950]. The case $N = 3$ is due to [De Giorgi, Ann. Scuola Norm. Sup. Pisa, 1965], $N = 4$ to [Almgren, Ann. of Math., 1966], $N \leq 7$ to [Simon, Ann. of Math.,1968]. A counter-example was found by [Bombieri-De Giorgi-Giusti, Invent. Math., 1969] for $N \geq 8$. An important step in the proofs is the following result due to Fleming:

Theorem ([Fleming, Rend. Circ. Mat. Palermo, 1962])

If there exists a nonplanar entire minimal graph, then there exists a singular area-minimizing hypercone.

sketch of the proof of our theorem

- I will discuss only the case where p supercritical and u is stable, i.e. for all $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$p \int_{\mathbb{R}^n} |u|^{p-1} \varphi^2 \, dx \leq \|\varphi\|^2_{\dot{H}^s(\mathbb{R}^n)}$$

- Estimate solutions in the $L^{p+1} \cap H^s$ norm (Cacciopoli or energy method)
- Localize the problem by extension in the half-space $\mathbb{R}^{n+1}_+$.
- Derive a monotonicity formula $E = E(r)$.
- Consider the blow-down (weak) limit

$$\bar{u}^\infty(x) = \lim_{\lambda \to \infty} \lambda^{\frac{2s}{p-1}} \bar{u}(\lambda x)$$

- $\bar{u}^\infty$ satisfies $E(r) \equiv const$. Hence, $\bar{u}^\infty$ is a homogeneous stable solution
- Prove that such solutions are trivial if p is below $p_c(n)$, by analyzing the equation on the half-sphere of $\mathbb{R}^{n+1}_+$.
- Using the monotonicity formula again, prove that in fact $\bar{u}$ is trivial.

Step 1: energy estimate

Lemma

For $m > n/2$ and $x \in \mathbb{R}^n$, let

$$\eta(x) = (1 + |x|^2)^{-m/2} \qquad \text{and} \quad \rho(x) = \int_{\mathbb{R}^n} \frac{(\eta(x) - \eta(y))^2}{|x - y|^{n+2s}} \, dy$$

Then, there exists a constant $C = C(n, s, m) > 0$ such that

$$\rho(x) \leq C \left(1 + |x|^2\right)^{-\frac{n}{2} - s}.$$

Lemma

Let u be a stable solution i.e. for all φ,

$$p \int_{\mathbb{R}^n} |u|^{p-1} \varphi^2 \, dx \leq \|\varphi\|^2_{\dot{H}^s(\mathbb{R}^n)}$$

Assume that $m \in (\frac{n}{2}, \frac{n}{2} + \frac{s}{2}(p + 1))$. Take η as above. Then, there exists a constant $C = C(n, p, s, m) > 0$ such that

$$\int_{\mathbb{R}^n} |u|^{p+1} \eta^2 \, dx + \frac{1}{p} \|u\eta\|^2_{\dot{H}^s(\mathbb{R}^n)} \leq C.$$

$$(-\Delta)^s u = |u|^{p-1} u \qquad \times \; u\eta^2$$

Then,

$$\begin{aligned}
\int_{\mathbb{R}^n} |u|^{p+1}\eta^2 \; dx &= \int_{\mathbb{R}^n} (-\Delta)^s u \, u\eta^2 \; dx \\
&= \int_{\mathbb{R}^n} (-\Delta)^{s/2} u \, (-\Delta)^{s/2} (u\eta^2) \; dx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(u(x)\eta(x)^2 - u(y)\eta(y)^2)}{|x-y|^{n+2s}} \; dx\, dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u^2(x)\eta^2(x) - u(x)u(y)(\eta^2(x)+\eta^2(y)) + u^2(y)\eta^2(y)}{|x-y|^{n+2s}} \; dx\, dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)\eta(x) - u(y)\eta(y))^2 - (\eta(x)-\eta(y))^2 u(x)u(y)}{|x-y|^{n+2s}} \; dx\, dy \\
&= \|u\eta\|^2_{\dot{H}^s(\mathbb{R}^n)} - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\eta(x)-\eta(y))^2 u(x)u(y)}{|x-y|^{n+2s}} \; dx\, dy
\end{aligned}$$

Using the inequality $2ab \le a^2 + b^2$, we deduce that

$$\|u\eta\|^2_{\dot{H}^s(\mathbb{R}^n)} - \int_{\mathbb{R}^n} |u|^{p+1}\eta^2 \; dx \le \int_{\mathbb{R}^n} u^2 \rho \; dx$$

Since u is stable, we deduce that

$$(p-1)\int_{\mathbb{R}^n} |u|^{p+1}\eta^2 \; dx \le \int_{\mathbb{R}^n} u^2 \rho \; dx$$

Going back, it follows that

$$\frac{1}{p}\|u\eta\|^2_{\dot{H}^s(\mathbb{R}^n)} + \int_{\mathbb{R}^n} |u|^{p+1}\eta^2 \; dx \le \frac{2}{p-1}\int_{\mathbb{R}^n} u^2 \rho \; dx$$

Now,

$$\int_{\mathbb{R}^n} u^2 \rho \, dx = \int_{\mathbb{R}^n} u^2 \rho \, \eta^{-\frac{4}{p+1}} \, \eta^{\frac{4}{p+1}} \, dx$$
$$\leq \left(\int_{\mathbb{R}^n} |u|^{p+1} \eta^2 \, dx \right)^{\frac{2}{p+1}} \left(\int_{\mathbb{R}^n} \rho^{\frac{p+1}{p-1}} \eta^{-\frac{4}{p-1}} \, dx \right)^{\frac{p-1}{p+1}}$$

By technical lemma,

$$\int_{\mathbb{R}^n} \rho^{\frac{p+1}{p-1}} \eta^{-\frac{4}{p-1}} \, dx \leq C \int_{\mathbb{R}^n} (1 + |x|^2)^{-\left(\frac{n}{2}+s\right)\frac{p+1}{p-1} + \frac{2m}{p-1}} \, dx.$$

The integral is finite since $m < \frac{n}{2} + \frac{s}{2}(p+1)$ and the lemma follows easily.

□

Step 2: localizing the problem

Theorem (Spitzer (Trans. AMS, 1958), Molcanov-Ostrovskii (Theory Probab. Appl., 1969), Caffarelli-Silvestre (Comm. in Part. Diff. Eq., 2007))

Take $0 < s < \sigma < 1$ *and* $u \in C^{\sigma}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n, (1+|x|)^{n+2s}dx)$. *For* $X = (x,t) \in \mathbb{R}^{n+1}_+$, *let*

$$\bar{u}(X) = \int_{\mathbb{R}^n} P(X,y)u(y)\,dy.$$

Then,

$$\begin{cases} \nabla \cdot (t^{1-2s}\nabla \bar{u}) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \bar{u} = u & \text{on } \partial\mathbb{R}^{n+1}_+, \\ -t^{1-2s}\partial_t \bar{u} = \kappa_s(-\Delta)^s u & \text{on } \partial\mathbb{R}^{n+1}_+, \end{cases}$$

where

$$P(X,y) = c_{n,s}t^{2s}|X-y|^{-(n+2s)}, \quad \kappa_s = \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}.$$

In particular, if u solves the fractional Lane-Emden equation, its extension $\bar{u}$ solves

$$\begin{cases} \nabla \cdot (t^{1-2s} \nabla \bar{u}) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -t^{1-2s} \partial_t \bar{u} = \kappa_s |\bar{u}|^{p-1} \bar{u} & \text{on } \partial \mathbb{R}^{n+1}_+ \end{cases}$$

Note that

- the energy estimate is transferred on $\bar{u}$
- the equation is still scale-invariant, i.e. if $\bar{u}$ is a solution then so is

$$\bar{u}^{\lambda}(X) = \lambda^{\frac{2s}{p-1}} \bar{u}(\lambda X).$$

- and variational: the energy on a ball $B(x, r)$ is given by

$$E(\bar{u}, r) = \frac{1}{2} \int_{\mathbb{R}^{n+1}_+ \cap B(x,r)} t^{1-2s} |\nabla \bar{u}|^2 dxdt - \frac{\kappa_s}{p+1} \int_{\partial \mathbb{R}^{n+1}_+ \cap B(x,r)} |\bar{u}|^{p+1} dx$$

Compute now the energy of the rescaled solution:

$$E(\bar{u}^{\lambda}, 1) = \lambda^{2s\frac{p+1}{p-1} - n} E(\bar{u}, \lambda) =: E_1(\bar{u}, \lambda)$$

Then

$$E_1(\bar{u}^{\lambda}, 1) = E_1(\bar{u}, \lambda).$$

Step 3: the monotonicity formula

Theorem (Dávila-D-Wei)

For $\lambda > 0$, let

$$E(\bar{u};\lambda) \;=\; E_1(\bar{u};\lambda) + \lambda^{\frac{4s}{p-1}-n}\frac{s}{p+1}\int_{\partial B_\lambda\cap\mathbb{R}^{n+1}_+} t^{1-2s}\bar{u}^2 \quad d\sigma.$$

Then, E is a nondecreasing function of λ. Furthermore,

$$\frac{dE}{d\lambda} = \lambda^{-n+2+\frac{4s}{p-1}}\int_{\partial B(x,\lambda)\cap\mathbb{R}^{n+1}_+} t^{1-2s}\left(\frac{\partial\bar{u}}{\partial r} + \frac{2s}{p-1}\frac{\bar{u}}{r}\right)^2 \; d\sigma$$

Proof of the monotonicity formula

Recall that if

$$U(X;\lambda) = \bar{u}^{\lambda}(X) = \lambda^{\frac{2s}{p-1}}\bar{u}(\lambda X)$$

then,

- U solves the equation,
- $$\begin{aligned} E_1(\bar{u};\lambda) &= E_1(U;1) \\ &= \frac{1}{2}\int_{\mathbb{R}^{n+1}_+\cap B_1} t^{1-2s}|\nabla U|^2 dxdt - \frac{\kappa_s}{p+1}\int_{\partial\mathbb{R}^{n+1}_+\cap B_1} |U|^{p+1} dx, \end{aligned}$$
- and, using subscripts to denote partial derivatives,

$$\lambda U_\lambda = \frac{2s}{p-1}U + rU_r,$$

So,

$$\begin{aligned} \frac{dE_1}{d\lambda}(\bar{u};\lambda) &= \int_{\mathbb{R}^{n+1}_+\cap B_1} t^{1-2s}\nabla U\cdot\nabla U_\lambda\ dx\,dt - \kappa_s\int_{\partial\mathbb{R}^{n+1}_+} |U|^{p-1}UU_\lambda\ dx \\ &= \int_{\partial B_1\cap\mathbb{R}^{n+1}_+} t^{1-2s}U_r U_\lambda\ d\sigma \\ &= \lambda\int_{\partial B_1\cap\mathbb{R}^{n+1}_+} t^{1-2s}U_\lambda^2\ d\sigma - \frac{2s}{p-1}\int_{\partial B_1\cap\mathbb{R}^{n+1}_+} t^{1-2s}UU_\lambda\ d\sigma \\ &= \lambda\int_{\partial B_1\cap\mathbb{R}^{n+1}_+} t^{1-2s}U_\lambda^2\ d\sigma - \frac{s}{p-1}\frac{d}{d\lambda}\int_{\partial B_1\cap\mathbb{R}^{n+1}_+} t^{1-2s}U^2\ d\sigma \end{aligned}$$

Step 4: the blow-down limit is homogeneous

Lemma

$\bar{u}^\infty$ is homogeneous.

Proof:

- We know that $(\bar{u}^\lambda)$ is bounded in the energy space, so it has a weak limit.
- Since $r \mapsto E(\bar{u}, r)$ is increasing, its limit at infinity exists. This limit is finite. Indeed, take $0 < r < R < +\infty$. Write $E = E_1 + E_2$, where E_1 is bounded thanks to the energy estimate and

$$E_2 = \lambda^{\frac{4s}{p-1}-n} \frac{s}{p+1} \int_{\partial B(0,\lambda)\cap\mathbb{R}^{n+1}_+} t^{1-2s} \bar{u}^2 \, d\sigma$$

  Since E is nondecreasing,

$$E(\bar{u}, \lambda) = E(U, 1) \leq \int_1^2 E(U, t) \, dt \leq C + \int_{B_2\cap\mathbb{R}^{n+1}_+} t^{1-2s} U^2 \leq C.$$

- Fix $R_2 > R_1 > 0$. Then,

$$\begin{aligned}
0 &= \lim_{n\to+\infty} E(\bar{u}, \lambda_n R_2) - E(\bar{u}, \lambda_n R_1) \\
&= \lim_{n\to+\infty} E(\bar{u}^{\lambda_n}, R_2) - E(\bar{u}^{\lambda_n}, R_1) \\
&= \lim_{n\to+\infty} \int_{R_1}^{R_2} \frac{dE}{dt}(\bar{u}^{\lambda_n}, t) \, dt \\
&\geq \liminf_{n\to+\infty} \int_{(B_{R_2}\setminus B_{R_1})\cap\mathbb{R}^{n+1}_+} t^{1-2s} r^{2-n+\frac{4s}{p-1}} \left(\frac{2s}{p-1} \frac{\bar{u}^{\lambda_n}}{r} + \frac{\partial \bar{u}^{\lambda_n}}{\partial r} \right)^2 dx\, dt \\
&\geq \int_{(B_{R_2}\setminus B_{R_1})\cap\mathbb{R}^{n+1}_+} t^{1-2s} r^{2-n+\frac{4s}{p-1}} \left(\frac{2s}{p-1} \frac{\bar{u}^{\infty}}{r} + \frac{\partial \bar{u}^{\infty}}{\partial r} \right)^2 dx\, dt
\end{aligned}$$

Step 5: Liouville for homogeneous stable solutions

Write

$$\bar{u}^{\infty}(r,\theta) = r^{-\frac{2s}{p-1}}\psi(\theta).$$

Then,

$$\begin{cases} -\nabla \cdot (\theta_1^{1-2s}\nabla\psi) + \lambda\theta_1^{1-2s}\psi = 0 & \text{on } S_+^n, \\ -\theta_1^{1-2s}\partial_{\theta_1}\psi = \kappa_s|\psi|^{p-1}\psi & \text{on } \partial S_+^n \end{cases}$$

where $\lambda = \frac{2s}{p-1}\left(n - 2s - \frac{2s}{p-1}\right)$.
Multiply the equation by ψ

$$\int_{S_+^n} \theta_1^{1-2s}|\nabla\psi|^2 + \lambda\int_{S_+^n}\theta_1^{1-2s}\psi^2 = \kappa\int_{\partial S_+^n}|\psi|^{p+1} \qquad (4)$$

Just proved

$$\kappa_s \int_{\partial S^n_+} |\psi|^{p+1} = \int_{S^n_+} \theta_1^{1-2s} |\nabla \psi|^2 + \lambda \int_{S^n_+} \theta_1^{1-2s} \psi^2$$

Stability

$$\kappa_s p \int_{\mathbb{R}^n} |\bar{u}^\infty|^{p-1} \varphi^2 \leq \int_{\mathbb{R}^{n+1}_+} t^{1-2s} |\nabla \varphi|^2$$

+ test functions optimizing the corresponding Hardy inequality $\varphi = r^{-\frac{n-2s}{2}} \eta(r) w(\theta)$:

$$\kappa_s p \int_{\partial S^n_+} |\psi|^{p-1} w^2 \leq \int_{S^n_+} \theta_1^{1-2s} |\nabla w|^2 + \left(\frac{n-2s}{2} \right)^2 \int_{S^n_+} \theta_1^{1-2s} w^2$$

Does not suffice to take $w = \psi$ to conclude!

Let ϕ_α be the solution of

$$\begin{cases} div(\theta_1^{1-2s}\nabla\phi_\alpha) - ((\frac{n-2s}{2})^2 - \alpha^2)\theta_1^{1-2s}\phi_\alpha = 0 & \text{on } S_+^n \\ \phi_\alpha = 1 & \text{on } \partial S_+^n. \end{cases} \tag{5}$$

Multiplying by φ^2/ϕ_α, get

$$\int_{S_+^n} \theta_1^{1-2s}|\nabla\varphi|^2 + ((\frac{n-2s}{2})^2 - \alpha^2)\int_{S_+^n} \theta_1^{1-2s}\varphi^2 = \kappa\lambda(\alpha)\int_{\partial S_+^n} \varphi^2 + \int_{S_+^n} \theta_1^{1-2s}\phi_\alpha^2|\nabla(\frac{\varphi}{\phi_\alpha})|^2 \quad \text{for all } \varphi \tag{6}$$

A particular case is

$$\int_{S_+^n} \theta_1^{1-2s}|\nabla\varphi|^2 + (\frac{n-2s}{2})^2\int_{S_+^n} \theta_1^{1-2s}\varphi^2 = \kappa\Lambda_{n,s}\int_{\partial S_+^n} \varphi^2 + \int_{S_+^n} \theta_1^{1-2s}\phi_0^2|\nabla(\frac{\varphi}{\phi_0})|^2 \quad \text{for all } \varphi \tag{7}$$

We note that for $\alpha \in (0, \frac{n-2s}{2})$

$$\phi_0 \le \phi_\alpha \quad \text{on } S_+^n. \tag{8}$$

Indeed, on S_+^n

$$div(\theta_1^{1-2s}\nabla\phi_0) = (\frac{n-2s}{2})^2\theta_1^{1-2s}\phi_0 \ge ((\frac{n-2s}{2})^2 - \alpha^2)\theta_1^{1-2s}\phi_0$$

so ϕ_0 is a sub-solution of (5). Then we can conclude by the maximum principle.

From now on we fix $\alpha \in (0, \frac{n-2s}{2})$ given by

$$\alpha = \frac{n-2s}{2} - \frac{2s}{p-1}$$

so that

$$(\frac{n-2s}{2})^2 - \alpha^2 = \frac{2s}{p-1}(n-2s-\frac{2s}{p-1}) = \lambda.$$

Use the stability inequality with $\varphi = \frac{\psi\phi_0}{\phi_\alpha}$:

$$\kappa p \int_{\partial S^n_+} \psi^{p+1} \leq \int_{S^n_+} \theta_1^{1-2s}|\nabla(\frac{\psi\phi_0}{\phi_\alpha})|^2 + (\frac{n-2s}{2})^2 \int_{S^n_+} \theta_1^{1-2s}(\frac{\psi\phi_0}{\phi_\alpha})^2$$

Combining with (7) (used with $\varphi = \frac{\psi\phi_0}{\phi_\alpha}$):

$$\kappa p \int_{\partial S^n_+} \psi^{p+1} \leq \kappa\Lambda_{n,s} \int_{\partial S^n_+} \psi^2 + \int_{S^n_+} \theta_1^{1-2s}\phi_0^2|\nabla(\frac{\psi}{\phi_\alpha})|^2.$$

Since $\phi_0 \leq \phi_\alpha$,

$$\kappa p \int_{\partial S^n_+} \psi^{p+1} \leq \kappa\Lambda_{n,s} \int_{\partial S^n_+} \psi^2 + \int_{S^n_+} \theta_1^{1-2s}\phi_\alpha^2|\nabla(\frac{\psi}{\phi_\alpha})|^2.$$

and using (6)

$$\kappa p \int_{\partial S^n_+} \psi^{p+1} \leq \kappa\Lambda_{n,s} \int_{\partial S^n_+} \psi^2 + \int_{S^n_+} \theta_1^{1-2s}|\nabla\psi|^2 + ((\frac{n-2s}{2})^2 - \alpha^2) \int_{S^n_+} \theta_1^{1-2s}\psi^2 - \kappa\lambda(\alpha) \int_{\partial S^n_+} \psi^2$$

Recall the energy identity

$$\int_{S^n_+} \theta_1^{1-2s}|\nabla\psi|^2 + ((\frac{n-2s}{2})^2 - \alpha^2) \int_{S^n_+} \theta_1^{1-2s}\psi^2 = \kappa \int_{\partial S^n_+} \psi^{p+1}$$

so that

$$\kappa(p-1) \int_{\partial S^n_+} \psi^{p+1} \leq \kappa(\Lambda_{n,s} - \lambda(\alpha)) \int_{\partial S^n_+} \psi^2.$$

Step 6: The solution itself is trivial

We have (almost) proved that, as $\lambda \to +\infty$,

$$E(\bar{u}, \lambda) = E(\bar{u}^{\lambda}, 1) \to E(\bar{u}^{\infty}, 1) = 0$$

Now, as $\lambda \to 0$,

$$E(\lambda; 0, \bar{u}) = \lambda^{2s\frac{p+1}{p-1}-n} \cdot \left(\frac{1}{2} \int_{\mathbb{R}^{n+1}_{+} \cap B_{\lambda}} t^{1-2s} |\nabla \bar{u}|^2 dxdt \right.$$
$$\left. - \frac{\kappa_s}{p+1} \int_{\partial \mathbb{R}^{n+1}_{+} \cap B_{\lambda}} |\bar{u}|^{p+1} dx \right)$$
$$+ \lambda^{\frac{4s}{p-1}-n} \frac{s}{p+1} \int_{\partial B_{\lambda} \cap \mathbb{R}^{n+1}_{+}} t^{1-2s} \bar{u}^2 \, d\sigma \to 0$$

So, $\bar{u}$ itself has constant zero energy and so it must be homogeneous.