Circulant Matrices, Intermediate Liapunov-Schmidt Reduction Method and Nonlinear Elliptic Equations

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Joint work with M. del Pino-W. Yao, M. Musso

My France and Taiwan collaborations

France

- H. Berestycki
- L. Dupaigne
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This talk is concerned with two well-studied nonlinear equations

••

$$(P) \qquad \Delta u - V(x)u + u^p = 0, u > 0 \text{ in } \mathbb{R}^N,$$

$$u \in H^1(\mathbb{R}^N), \quad p < \frac{N+2}{N-2}$$

AIM: existence of infinitely many positive, bound states, assuming no symmetry of V.

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AIM: establish the non-degeneracy of sign-changing solutions.
In both problems, the role of Circulant Matrices and Intermediate Liapunov-Schmidt Reduction Method will be emphasized.

Circulant Matrices

The circulant matrix $B = \mathit{circ}\{\mathbf{b}\}$ associated to the vector $\mathbf{b} = (b_1, b_2, \dots, b_K) \in \mathbb{C}^K$ is the $K \times K$ matrix:

$$B = \begin{pmatrix} b_1 & b_2 & \cdots & b_{K-1} & b_K \\ b_K & b_1 & \cdots & b_{K-2} & b_{K-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_3 & b_4 & \cdots & b_1 & b_2 \\ b_2 & b_3 & \cdots & b_K & b_1 \end{pmatrix}.$$

Denote

$$B = \mathit{circ}\{b_1, b_2, ..., b_K\}$$

Key Property of Circulant Matrices

An important property of circulant matrices is that all circulant matrices have the same ordered set of orthonormal eigenvectors $\{X_l\}$ and diagonalizable matrix P_K.
Let $q = e^{i\frac{2\pi}{K}}$ be a primitive K-th root of unity, we define

$$X_l = \frac{1}{\sqrt{K}}(1, q^{l-1}, q^{2(l-1)}, \ldots, q^{(K-1)(l-1)})^T \in \mathbb{C}^K, \text{ for } l = 1, \ldots, K,$$

and

$$P_K = \frac{1}{\sqrt{K}} \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & q & \cdots & q^{K-2} & q^{K-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & q^{K-2} & \cdots & q^{(K-2)^2} & q^{(K-2)(K-1)} \\ 1 & q^{K-1} & \cdots & q^{(K-1)(K-2)} & q^{(K-1)^2} \end{pmatrix}$$

For the circulant matrix $B = \textit{circ}\{\mathbf{b}\}$, let

$$\lambda_l = b_1 + b_2 q^{l-1} + \cdots + b_K q^{(K-1)(l-1)}, \text{ for } l = 1, \ldots, K. \quad (1)$$

$$BX_l = \lambda_l X_l, l = 1, \ldots, K$$

All circulant matrices have the same ordered set of orthonormal eigenvectors $\{X_l\}$ and diagonalizable matrix P_K. The eigenvalues of circulant matrix $B = \textit{circ}\{\mathbf{b}\}$ are given by

$$\lambda_l = b_1 + b_2 q^{l-1} + \cdots + b_K q^{(K-1)(l-1)}, \text{ for } l = 1, \ldots, K.$$

Intermediate Liapunov-Schmidt Reduction Method

Gluing Methods:

- Finite dimensional Liapunov-Schmidt reduction method
  Floer-Weinstein 1986
  Many many variations and refinements
  .....
- Infinite dimensional Liapunov-Schmidt reduction method
  del Pino, Kowalczyk, Pacard, Wei 2007, 2010
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I. Nonlinear Schrödinger Equation with Subcritical Exponent

Consider the following nonlinear Schrodinger equation

$$(P) \qquad \Delta u - V(x)u + u^p = 0, \quad u > 0, \quad \lim_{|x| \to +\infty} u(x) = 0,$$

where

$$0 < a \leq V(x) \leq b$$

.

$$\lim_{|x| \to +\infty} V(x) = V_\infty (:= 1) > 0$$

Solutions of (P) can be seen as stationary states in nonlinear equations of Klein-Gordon type

$$\frac{\partial^2\varphi}{\partial t^2} - \Delta\varphi + (V+\omega^2)\varphi - |\varphi|^{p-1}\varphi = 0$$

or Schrödinger type

$$i\frac{\partial\varphi}{\partial t} - \Delta\varphi + (V+\omega^2)\varphi - |\varphi|^{p-1}\varphi = 0$$

A solitary wave of standing wave form can be searched as solution of the form

$$\varphi = e^{i\omega t}u(x)$$

Constant Coefficient Case: $V(x) \equiv 1$

$$\Delta u - u + u^p = 0, u > 0 \quad \text{in} \quad \mathbb{R}^N \;\; u \in H^1(\mathbb{R}^N)$$

Radial symmetry

•• Gidas-Ni-Nirenberg 1981: u radially symmetric around some point and strictly decreasing.

This reduces the problem to ODE.

Existence

•• Strauss 1977; general $f(u)$ case Berestycki-Lions 1983

Uniqueness

•• Kwong 1989

General nonconstant $V(x)$ Case

$$(P) \qquad \Delta u - V(x)u + u^p = 0, \quad u > 0, u \in H^1$$

Main Problem: the map from $H^1(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$ is no longer compact due to the translation invariance of $\mathbb{R}^N$, whatever q is.
"Concentration-compactness"

Existence: different topological situations according to

1) $V(x) \to V_\infty$ from below

$$V = V_\infty - \frac{a}{|x|^m} \text{ as } |x| \to +\infty$$

2) $V(x) \to V_\infty$ from above

$$V = V_\infty + \frac{a}{|x|^m} \text{ as } |x| \to +\infty$$

When 1) is true (P) can be handled by concentration-compactness type arguments.

P.L. Lions 1984: Existence of a positive least energy solution to (P)

$$(P) \qquad \Delta u - V(x)u + u^p = 0, u > 0, u \in H^1(\mathbb{R}^N)$$

when V approaches V_∞ from below:

$$V(x) - V_\infty \in L^{\frac{N}{2}}(\mathbb{R}^N), V(x) \leq V_\infty \text{ from below}$$

The following minimization Problem is attained:

$$(*) \qquad c_V = \inf\{\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \mid \int_{\mathbb{R}^N} u^{p+1} = 1\}$$

$$c_V < c_{V_\infty}$$

When $V(x) \to V_\infty$ from above the minimization problem (*) may not have solution:

$$(*) \qquad c_V = \inf\{\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \mid \int_{\mathbb{R}^N} u^{p+1} = 1\}$$

In fact, if $V(x) \geq V_\infty$ and $V \not\equiv V_\infty$ in a positive measure, then

$$(*) \qquad c_V = \inf\{\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \mid \int_{\mathbb{R}^N} u^{p+1} = 1\} = c_{V_\infty}$$

and hence c_V is not achieved.

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and hence c_V is not achieved.

So least energy solution (or ground state) does not exist. One has to look for higher energy level solutions (bound states).

Existence of a positive, not ground state, solution to (P) has been proved by Bahri - P.L. Lions 1997

under a fast decay condition:

$$V(x) - V_\infty \geq C e^{-\sigma|x|}|x|^{-\frac{N-1}{2}}$$

Ingredients of the proof (59 pages):

•• Deep study of the compactness question;
•• Variational and topological arguments.

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•• Variational and topological arguments.

So even existence of one positive solution is already difficult.
How about the existence of infinitely many positive solutions ???

Multiplicity of positive solutions

$$(P_\varepsilon) \quad \varepsilon^2 \Delta u - V(x)u + u^p = 0, u > 0, u \in H^1(\mathbb{R}^N)$$

$$\Delta u - V(\varepsilon x)u + u^p = 0, u > 0, u \in H^1(\mathbb{R}^N)$$

Number of solutions related, as ε is small, to the number and/or the type of critical points of a(x), or to the topology of sublevel sets of a(x).

•• finite dimensional Lyapunov - Schmidt reduction

Floer-Weinstein 1986

[Ambrosetti - Badiale - Cingolani, Byeon, Cao, Dancer, Del Pino, Felmer, Floer - Weinstein, Kang, Noussair , Oh , Gui, Tanaka, Wei, Yan, Lin, Liu, Malchiodi, Pistoia, Grossi, DAprile, Musso, ...]

Typical Result

Kang-Wei 2000: If $V(x)$ has a local maximum point, then for any integer $K \geq 1$ there exists $\varepsilon_K > 0$ such that for $\varepsilon < \varepsilon_K$ there are solutions with K bumps

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However this does not answer our question:
Existence of infinitely many positive bound states for a fixed ε???

Seminar work of Coti Zelati -Rabinowitz

Coti Zelati - Rabinowiz 1992 developed variational gluing method:
Infinitely many multi-bump positive solutions when

$$V(x) \ \text{ is periodic}$$

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Non-periodic $V(x)$??

As far as we know, the first result on the existence of infinitely many positive bound states was due to Wei-Yan in the case of $V = V(r)$

Radial V

We assume that $V(x)$ is radial. That is, $V(x) = V(|x|)$. Thus, we consider the following problem

$$(P) \quad \Delta u - V(|x|)u + u^p = 0, u > 0 \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N),$$

where $1 < p < \frac{N+2}{N-2}$.

We assume that $V(r) > 0$ satisfies the following condition:
(V): There is are constants $a > 0$, $m > 1$, $\theta > 0$, and $V_\infty > 0$, such that

$$V(r) = V_\infty + \frac{a}{r^m} + O\Big(\frac{1}{r^{m+\theta}}\Big), \tag{2}$$

as $r \to +\infty$.

Results in the radial symmetry case

Theorem 1. (Wei-Yan 2008) If $V(x) = V(r)$ satisfies

$$V(r) = V_0 + \frac{a}{r^m} + O\Big(\frac{1}{r^{m+\theta}}\Big),$$

then problem (P)

$$(P) \quad \Delta u - V(|x|)u + u^p = 0, u > 0 \ \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N)$$

has infinitely many non-radial positive solutions, whose energy can be made arbitrarily large. Namely for any $M > 0$, there exists a positive solution to (P) with

$$I[u] = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + Vu^2) - \frac{1}{p+1}\int_{\mathbb{R}^N} u^{p+1} > M$$

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A new proof by variational methods:
Devillanova - Solimini 2012

Conjecture: Nonsymmetric Case

A natural question is whether or not the result remains true when the symmetry requirement is lifted:

Conjecture: Problem (P) has infinitely many positive solutions if there are constants $V_\infty > 0$, $a > 0$, $m > 1$, and $\sigma > 0$, such that the potential $V(x)$ satisfies

$$V(x) = V_\infty + \frac{a}{|x|^m} + O\Big(\frac{1}{|x|^{m+\sigma}}\Big), \text{ as } |x| \to +\infty. \tag{3}$$

Non-symmetric Case: A Perturbative Result

Results in this direction with non-symmetric potentials, as far as we know, there are only perturbative results.
Cerami, Passaseo and Solimini 2012, CPAM 2013: Assume that

- $V(x) \geq V_\infty > 0$,
- $\lim_{|x|\to\infty} \big(V(x) - V_\infty\big) e^{\bar{\eta}|x|} = +\infty$, for some $\bar{\eta} \in (0, \sqrt{V_\infty})$
- $\sup_{x\in\mathbb{R}^N} \|V(x) - V_\infty\|_{L^{N/2}(B_1(x))} < \delta$,

then there exists $\delta_0 > 0$ (with no explicit expression) such that for $\delta < \delta_0$ problem (P) has infinitely many positive solutions by purely variational methods.

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Ao-Wei 2013: gave a new proof of this result with general $f(u)$, using finite dimensional Liapunov-Schmidt reduction method
Open: General $V(x)$ without any smallness assumption

General $V(x)$ Case

Theorem 2. (del Pino-Wei-Yao 2013) Let $N = 2$. Suppose that $V(x)$ satisfies

$$V(x) = V_\infty + \frac{a}{r^m} + O\Big(\frac{1}{r^{m+\theta}}\Big),$$

and

$$\min\Big\{1, \frac{p-1}{2}\Big\} m > 2, \ \sigma > 2. \tag{4}$$

Then problem (P) has infinitely many non-radial positive solutions, whose energy can be made arbitrarily large.

1. If expansion for V holds in the C^1 sense, then "$\sigma > 2$" in (4) can be improved to be "$\sigma > 0$". The condition on p can be further relaxed if we assume more regularity of the condition or if p is an integer.

2. More regularity of $V \implies$ less restrictions on p.

3. We believe same result holds for $N \geq 3$. Partial progress.

4. Theorem 2 is proved by intermediate Liapunov-Schmidt reduction method

An introduction to finite dimensional Liapunov-Schmidt reduction method

Let X, Y be Banach spaces and $S(u)$ is a C^1 map from X to Y. To study the equation

$$S(u) = 0$$

a natural way is to find approximations first and then to look for genuine solutions as (small) perturbations of approximations. Assume that U_λ are the approximations such that

$$S(U_\lambda) \sim 0$$

Here $\lambda \in \Lambda$ is the parameter (we think of Λ as the configuration space). Writing $u = U_\lambda + \varphi$, then solving $S(u) = 0$ amounts to solve

$$L[\varphi] + E + N(\varphi) = 0, \tag{5}$$

where

$$L[\varphi] = S'(U_\lambda)[\varphi], \; E = S(U_\lambda)$$
$$N(\varphi) = S(U_\lambda + \varphi) - S(U_\lambda) - S'(U_\lambda)[\varphi].$$

In order to solve (5), we try to invert the linear operator L so that we can rephrase the problem as a fixed point problem. That is, when L has a uniformly bounded inverse in a suitable space, one can rewrite the equation (5) as

$$\varphi = -L^{-1}[E + N(\varphi)] = \mathcal{A}(\varphi).$$

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The Lyapunov-Schmidt reduction deals with the situation when the linear operator L is Fredholm and its eigenfunction space associated to small eigenvalues is finite dimensional.

Assuming that $\{z_1, \dots, z_n\}$ is a basis of the eigenfunction space associated to small eigenvalues of L, we can divide the procedure of solving (5) into two steps:

(i) solving the projected problem for any $\lambda \in \Lambda$,

$$\begin{cases} L[\varphi] + E + N(\varphi) = \sum_{j=1}^{n} c_j z_j, \\ \langle \varphi, z_j \rangle = 0, \ \forall\, j = 1, \dots, n, \end{cases}$$

where c_j may be constant or function depending on the form of $\langle \varphi, z_j \rangle$.

(ii) solving the reduced problem

$$c_j(\lambda) = 0, \ \forall\, j = 1, \dots, n,$$

by adjusting λ.

In the case of $V = V(x)$ and the equation (P)

$$S(u) = \Delta u - V(x)u + u^p = 0$$

with

$$V(x) = 1 - \frac{a}{|x|^m} \text{ as } |x| \to +\infty$$

Building Block: By the asymptotic behaviour of V at infinity, the basic building block is the ground state (radial) solution w of the limit problem at infinity:

$$\begin{cases} \Delta w - w + w^p = 0, \ w > 0 \text{ in } \mathbb{R}^N, \\ w = w(|x|), \quad w \in H^1(\mathbb{R}^N). \end{cases} \tag{6}$$

We choose $\lambda = (Q_1, ..., Q_K) \in \mathbb{R}^{NK}$ such that

$$|Q_j| >> 1, \min_{i \neq j} |Q_i - Q_j| >> 1$$

Approximate solution

$$U_\lambda = \sum_{j=1}^{K} w(x - Q_j)$$

Approximate Kernels

$$Z_{i,j} = \frac{\partial w(x - Q_j)}{\partial x_i}, i = 1, 2, j = 1, ..., K$$

(i) solving the projected problem for any $\lambda \in \Lambda$,

$$\begin{cases} S(U_+\varphi) = \sum_{i=1,2,j=1,\dots,K} c_{ij} Z_{i,j}, \\ \int \varphi Z_{ij} = 0, \ i = 1, 2, \forall\, j = 1, \dots, K, \end{cases}$$

(ii) solving the reduced problem

$$c_{ij}(\lambda) = 0, \ i = 1, 2, \forall\, j = 1, \dots, K$$

by adjusting $\lambda = (Q_1, ..., Q_K)$.

Reduced equation

$$c_{ij}(\lambda) = 0, i = 1, 2, j = 1, ..., K$$

There are now $2K$ number of equations!

Variational Reduction

Key variational reduction: $c_{ij}(\lambda) = 0$ if and only if $M(Q_1, ..., Q_K) := I[U_\lambda + \varphi_\lambda]$ has a critical point in the configuration space.

Even with that, it is not easy to find a critical point for a large number of points. In particular if the critical point has large number of positive and negative directions.

Variational Reduction

Key variational reduction: $c_{ij}(\lambda) = 0$ if and only if $M(Q_1, ..., Q_K) := I[U_\lambda + \varphi_\lambda]$ has a critical point in the configuration space.

Even with that, it is not easy to find a critical point for a large number of points. In particular if the critical point has large number of positive and negative directions.
However if $V(x) = V(|x|)$ problem (P) is invariant under

•• rotation by $\frac{2\pi}{k}$

•• reflection by $(x_1, x_2) \to (-x_1, x_2)$
We can reduce the problem to just adjusting one parameter–the radius R- representing the location of a single bump along a given ray.

Let

$$Q_j = \big(R\cos\frac{2(j-1)\pi}{k}, R\sin\frac{2(j-1)\pi}{k}, 0\big), \quad j = 1, \cdots, K,$$

$$R >> 1$$

These K spikes are distributed equidistance on the circle $\{|x| = R\}$.

Let

$$U_{(Q_1,\ldots,Q_K)}(y) = \sum_{j=1}^{K} w(x - Q_j),$$

Reduced Energy

The Energy

$$I[u] = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(r) u^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} u^{p+1}$$

Energy of one spike

$$I[w(x - x_1)] = A + \frac{B_1}{R^m} + O\big(\frac{1}{R^{m+\theta}}\big),$$

For multiple-spikes on a circle, we have

$$I[U] = k\Big(A + \frac{B_1}{R^m} - B_2 e^{-\frac{2\pi R}{k}} + O\big(\frac{1}{k^{m+\sigma}}\big)\Big)$$

where $B_1 > 0, B_2 > 0$ are positive constant.

Reduced Energy

Easy to see: the function

$$\frac{B_1}{R^m} - B_2 e^{-\frac{2\pi R}{k}}$$

has a maximum point

$$\bar{R}_k = \big(\frac{m}{2\pi} + o(1)\big) k \ln k.$$

Theorem 1 follows from the variational reduction method.

Main Difficulties in the Absence of Symmetry

In the above proof, the fact that V is radially symmetry allows us to build a K-bump solution for an arbitrary $K \geq 1$ with a K-dyadic symmetry, reducing the problem to just adjusting one parameter representing the location of a single bump along a given ray.

When V is non-symmetric, we cannot constrain the bump configuration to any symmetry class. We are thus forced to deal with a large number of bumps and therefore with a huge number of parameters which need to be adjusted. (In this case $2K$ number of equations.) This poses a tremendous difficulty in the construction comparatively to symmetry case.

Furthermore the critical point is in fact a saddle-point type. There are large number of both positive and negative small eigenvalues.

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Furthermore the critical point is in fact a saddle-point type. There are large number of both positive and negative small eigenvalues. To overcome this difficulty we develop an intermediate Liapunov-Schmidt reduction method

Intermediate Liapunov-Schmidt Reduction Method

In finite-dimensional reduction method, one moves the points (which is a finite-dimensional space) in order to find the true solution.

In infinite-dimensional Liapunov-Schmidt reduction method, one moves the entire curves or surfaces (which are infinite-dimensional space) in order to find a true interface.

In intermediate Liapunov-Schmidt reduction method, we move large number of points (finite dimensional space) along curves/surfaces (infinite dimensional space) in order to find a true equilibrium.

In intermediate Liapunov-Schmidt reduction method, after finite dimensional procedure, the large number of reduced equations, in the limit, become an ODE/PDE system of limiting Jacobi-type operators. In some sense this can be considered as discretized version of infinite dimensional reduction method.

The main difference between the intermediate and infinite dimensional reduction, is that in the latter procedure only the variations in the normal direction are needed so the usual Jacobi operator for a curve/surface appears

$$\text{Fermi Coordinates} : x = y + (t + h(y))\nu_y, y \in \Gamma$$

$$J[h] = \Delta_\Gamma(h) + |A|^2 h$$

In the former procedure we also need to take into account variations in the tangential direction of points, which in the limit may be interpreted as a reparametrization of the curve.

The main difference between the intermediate and infinite dimensional reduction, is that in the latter procedure only the variations in the normal direction are needed so the usual Jacobi operator for a curve/surface appears

$$\text{Fermi Coordinates} : x = y + (t + h(y))\nu_y, y \in \Gamma$$

$$J[h] = \Delta_\Gamma(h) + |A|^2 h$$

In the former procedure we also need to take into account variations in the tangential direction of points, which in the limit may be interpreted as a reparametrization of the curve.

Infinite dimensional Liapunov-Schmidt reduction method: variations in normal direction only

Intermediate Liapunov-Schmidt reduction method: variations in both normal and tangential directions

Description of the construction

Let $K \in \mathbb{N}_+$ be the number of spikes, whose locations are given by $Q_j \in \mathbb{R}^N$, $j = 1, \ldots, K$. We define

$$w_{Q_j}(x) = w(x - Q_j) \text{ and } U(x) = \sum_{j=1}^{K} w_{Q_j}(x), \text{ for } x \in \mathbb{R}^N \quad (7)$$

where

$$\begin{cases} -\Delta w + w - w^p = 0, \ w > 0 \text{ in } \mathbb{R}^N, \\ w = w(|x|), \quad w \in H^1(\mathbb{R}^N). \end{cases}$$

A natural and central question is how to choose Q_j's such that a small perturbation of U will be a genuine solution.

Assuming that

$$\inf_{1\leq j\leq K}|Q_j|\to\infty \text{ and } \inf_{j\neq l}|Q_j-Q_l|\to\infty,$$

by the asymptotic behaviour of V at infinity and the property of w, one can get (at least formally) the following energy expansion

$$I[U]=\underbrace{KI_0+a_0\sum_{j=1}^{K}|Q_j|^{-m}-\frac{1}{2}\gamma_0\sum_{j\neq l}w(|Q_j-Q_l|)}_{J(Q_1,...,Q_K)}+\text{other terms}, \tag{8}$$

where I_0, a_0 and γ_0 are positive constants. Here we denote the leading order expansion as $J(Q_1,...,Q_K)$.

Observe that for any rotation R_θ around the origin in $\mathbb{R}^N$, there holds

$$J(R_\theta Q_1, \ldots, R_\theta Q_K) = J(Q_1, \ldots, Q_K).$$

Hence any critical point of $J(Q_1, \ldots, Q_K)$ is degenerate.
Therefore, except in the symmetric class, it is not easy to find critical points of small perturbations of $J(Q_1, \ldots, Q_K)$.

Initial Configuration

We choose initial configuration as follows

$$Q_j^0(\alpha) = (R\cos\theta_j, R\sin\theta_j, 0) \in \mathbb{R}^2 \times \{0\}, \text{ for } j = 1, \dots, K,$$

where

$$\theta_j = \alpha + (j-1)\frac{2\pi}{K} \in \mathbb{R}.$$

- $R = \bar{R}_k = \left(\frac{m}{2\pi} + o(1)\right) K \ln K$ is the radius in the radial trapping potential case,
- α is the starting point on the curve, to be determined later. Observe that each point Q_j^0 depends on α. Thus we write $Q_j^0 = Q_j^0(\alpha)$. If $V(x)$ is radially symmetric, it is obvious that the parameter α plays no role in the construction. But it is very important in our construction as we will see later.

Perturbed Configuration

Let f_j, $g_j \in \mathbb{R}$, $j = 1, \dots, K$, we define

$$Q_j = Q_j^0 + f_j \vec{n}_j + g_j \vec{t}_j = (R + f_j)\vec{n}_j + g_j \vec{t}_j, \tag{9}$$

where

$$\vec{n}_j = (\cos\theta_j, \sin\theta_j, 0)\,, \text{ and } \vec{t}_j = (-\sin\theta_j, \cos\theta_j, 0)\,.$$

$\vec{n}_j$–normal direction
$\vec{t}_j$–tangential direction

- f_j and g_j measure the displacement in the normal and tangential directions respectively. Define

$$\mathbf{q} = (f_1, \cdots, f_K, g_1, \cdots, g_K)^T \in \mathbb{R}^{2K}.$$

- together with α there are now $2K + 1$ free parameters

$\dot{\mathbf{q}} = (\dot{f}_1, \cdots, \dot{f}_K, \dot{g}_1, \cdots, \dot{g}_K)^T,$ and $\ddot{\mathbf{q}} = (\ddot{f}_1, \cdots, \ddot{f}_K, \ddot{g}_1, \cdots, \ddot{g}_K)^T,$

$$\begin{aligned}
&\dot{f}_j = (f_{j+1} - f_j)\frac{K}{2\pi},\ \ddot{f}_j = (f_{j+1} - 2f_j + f_{j-1})\frac{K^2}{4\pi^2},\\
&\dot{g}_j = (g_{j+1} - g_j)\frac{K}{2\pi},\ \ddot{g}_j = (g_{j+1} - 2g_j + g_{j-1})\frac{K^2}{4\pi^2},\\
&f_{K+1} = f_1,\ f_0 = f_K,\ g_{K+1} = g_1,\ g_0 = g_K.
\end{aligned}$$

Observe that if $f_j = f(\theta_j)$ for some 2π periodic smooth function f, then $\dot{f}_j$ is the forward difference of f and $\ddot{f}_j$ is the 2nd order central difference of f.

Norm for $\mathbf{q}$:

$$\|\mathbf{q}\|_* = \|\mathbf{q}\|_\infty + \|\dot{\mathbf{q}}\|_\infty + \|\ddot{\mathbf{q}}\|_\infty \le 1.$$

To prove Theorem 2, it is sufficient to show that for K sufficiently large there are parameters α and $\mathbf{q}$ such that $U + \varphi$ is a genuine solution for a small perturbation φ. To achieve this goal, we will use finite dimensional Lyapunov-Schmidt reduction.

Step 1: Solving the projected problem.

Let $\alpha \in \mathbb{R}$ and $\mathbf{q}$ be defined as before. We look for a function φ and some multiplier $\widehat{\beta} \in \mathbb{R}^{2K}$ such that

$$\begin{cases} L[\varphi] + E + N(\varphi) = \widehat{\beta} \cdot \frac{\partial U}{\partial \mathbf{q}}, \\ \int_{\mathbb{R}^N} \varphi \, \mathcal{Z}_{Q_j} \, dx = 0, \ \forall \, j = 1, \ldots, K, \end{cases} \tag{10}$$

where the vector field $\mathcal{Z}_{Q_j}$ is defined by

$$\mathcal{Z}_{Q_j}(x) = \nabla w(x - Q_j). \tag{11}$$

By direct computation, we have

$$\frac{\partial U}{\partial \mathbf{q}} = -\big(\mathcal{Z}_{Q_1} \cdot \vec{n}_1, \cdots, \mathcal{Z}_{Q_K} \cdot \vec{n}_K, \mathcal{Z}_{Q_1} \cdot \vec{t}_1, \cdots, \mathcal{Z}_{Q_K} \cdot \vec{t}_K\big)^T.$$

This is the first step in the Lyapunov-Schmidt reduction. Hence we write $\varphi = \varphi(x; \alpha, \mathbf{q})$ and $\widehat{\beta} = \widehat{\beta}(\alpha, \mathbf{q})$.

Step 2: Solving the reduced problem
Reduced Problem:

$$\widehat{\beta}(\alpha, \mathbf{q}) = 0$$

This can not be solved directly since the linear part of the expansion of $\widehat{\beta}$ in $\mathbf{q}$ is degenerate (due to the invariance of $J(Q_1, \ldots, Q_K)$ under rotations).

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$$\begin{aligned}
&\widehat{\beta}(\alpha, \mathbf{q}) \\
&= a_0 R^{-m-2} T\mathbf{q} + R^{-m-\sigma} \Pi_1(\alpha, \mathbf{q}) + R^{-m-3} \Pi_2(\alpha, \mathbf{q}) + R^{-2m} \Pi_3(\alpha, \mathbf{q}) \\
&\quad + R^{-\min\{2-\eta, \frac{p+1-\eta}{2}\}m} \Pi_4(\alpha, \mathbf{q}) + R^{-m-3} (\ln K)^2 \Pi_5(\alpha, \mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})
\end{aligned}$$

where $\Pi_1(\alpha, \mathbf{q}), \ldots, \Pi_4(\alpha, \mathbf{q}), \Pi_5(\alpha, \mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})$ are uniformly bounded smooth vector valued functions with $\Pi_5(\alpha, 0, 0, 0) = 0$, More precisely, let us write

$$R^{m+2}\widehat{\beta}(\alpha, \mathbf{q}) = T\mathbf{q} + \Phi(\alpha, \mathbf{q}),$$

where $T\mathbf{q}$ is the linear part and $\Phi(\alpha, \mathbf{q})$ denotes the remaining term. $T\mathbf{q}$ does not depend on α and there is a unique vector (up to a scalar)

$$\mathbf{q}_0 = (\underbrace{0, \ldots, 0}_{K}, \underbrace{1, \ldots, 1}_{K})^T \in \mathbb{R}^{2K}$$

such that $T\mathbf{q}_0 = 0$.

T is an $2K \times 2K$ circulant matrix defined by

$$T = \begin{pmatrix} c_1\, A_1 + c_4\, I & c_2\, A_2 \\ \\ -c_2\, A_2 & c_3\, A_1 \end{pmatrix}, \tag{12}$$

Both A_1 and A_2 are $K \times K$ circulant matrices given by

$$A_1 = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 1 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & \cdots \\ 0 & -1 & 0 & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & -1 & 0 \\ 1 & 0 & \cdots & 0 & - \end{pmatrix}$$

A_1, A_2 are circulant matrices are $K \times K$ circulant matrices. In fact,

$A_1 = \textit{circ}\big\{(-2, 1, 0, \ldots, 0, 1)\big\}$ and $A_2 = \textit{circ}\big\{(0, 1, 0, \ldots, 0, -1)\big\}$.

whose eigenvalues can be computed.

Important: 0 is always an eigenvalue with eigenvector $\mathbf{q}_0$.

An important observation is that the system $T\mathbf{q} = \mathbf{b}$ can be seen as the discretization of the following continuous system:

$$\left\{\begin{array}{l} -(m+1)f(\theta) + (f'' - g')(\theta) + \widehat{d}(f + g')(\theta) = \varphi(\theta), \quad \theta \in (0, 2\pi), \\ g(\theta) + (f' - g)(\theta) - \widehat{d}(f' + g'')(\theta) = \varphi(\theta), \quad \theta \in (0, 2\pi), \\ f(0) = f(2\pi),\ f'(0) = f'(2\pi),\ g(0) = g(2\pi),\ g'(0) = g'(2\pi). \end{array}\right. \tag{13}$$

Jacobi-like operators

Lemma

Given φ, φ *satisfying* $\int_0^{2\pi} \varphi = 0$*, the system* (13) *has a unique solution* (f, g) *satisfying* $\int_0^{2\pi} g = 0$*. Moreover, there exists a constant* $C > 0$ *such that*

$$\|f\|_{C^2([0,2\pi])} + \|g\|_{C^2([0,2\pi])} \leq C \left(\|\varphi\|_{C^0([0,2\pi])} + \|\varphi\|_{C^0([0,2\pi])} \right). \tag{14}$$

Continuous version of solvability of Jacobi operators.

The discretized version of the above lemma gives the invertibility of T.

Lemma

There is an $K_0 \in \mathbb{N}$ such that for all $K \geq K_0$ and every $\mathbf{b} \in \mathbb{R}^{2K}$, there exists a unique vector $\mathbf{q} \in \mathbb{R}^{2K}$ and a unique constant $\gamma \in \mathbb{R}$ such that

$$T\mathbf{q} = \mathbf{b} + \gamma\,\mathbf{q}_0,\ \mathbf{q} \perp \mathbf{q}_0. \tag{15}$$

Moreover, there is a positive constant C which is independent of K such that

$$\|\mathbf{q}\|_2 \leq C\|\mathbf{b}\|_2,\ \|\dot{\mathbf{q}}\|_2 \leq C(\ln K)^{1/2}\|\mathbf{b}\|_2,\ \textit{and}\ \|\ddot{\mathbf{q}}\|_2 \leq C(\ln K)^{3/2}\|\mathbf{b}\|_2. \tag{16}$$

Furthermore, the number of zero (negative, positive) eigenvalues of T is 1 ($K-1$, K), respectively.

By the Lyapunov-Schmidt reduction again, the step of solving the reduced problem $\widehat{\beta}(\alpha, \mathbf{q}) = 0$ is reduced to

$$\widehat{\beta}(\alpha, \mathbf{q}) \frac{\partial U}{\partial q} = \gamma(\alpha) \frac{\partial U}{\partial \alpha}$$

Then the original problem (P) is reduced to the problem $\gamma(\alpha) = 0$ of one dimension.

Step 3. Solving $\gamma(\alpha) = 0$ by choosing α.

At the last step, we want to prove that there exists an α such that $\gamma(\alpha) = 0$. As a result, the function $u = U + \varphi$ is a genuine solution of problem (P).

To achieve this step, by Step 2, the function $\varphi = \varphi(x; \alpha, \mathbf{q}(\alpha))$ found in Step 1 solves the following problem:

$$\begin{cases} L[\varphi] + E + N(\varphi) = \gamma(\alpha)\frac{\partial U}{\partial \alpha}, \\ \int_{\mathbb{R}^N} \varphi \, \mathcal{Z}_{Q_j} \, dx = 0, \ \forall \, j = 1, \ldots, K, \end{cases} \tag{17}$$

where all of the quantities depending implicitly on $(\alpha, \mathbf{q})$ are taken values at $(\alpha, \mathbf{q}(\alpha))$.

To solve $\gamma(\alpha) = 0$, we apply the so-called variational reduction to show that equation $\gamma(\alpha) = 0$ has a solution if the reduced energy function

$$M(\alpha) = I[U + \varphi]$$

has a critical point.

Since $M(\alpha)$ is 2π periodic in α, it has at least **two** critical points, either maximum or minimum points.

As a result, for each $K >> 1$, we obtain at least TWO solutions to Theorem 2.

II.Sign-changing Solutions to Yamabe Problem

$$(II) \qquad \Delta u + |u|^{\frac{4}{N-2}} u = 0, \quad \text{in } \mathbb{R}^N, \quad N \geq 2$$

Classical Known Results on Positive Solutions

$$\Delta u + u^{\frac{N+2}{N-2}} = 0, u > 0 \text{ in } \mathbb{R}^N$$

- (Cafferalli-Gidas-Spruck 1989; Chen-Li 1993) All solutions are given by
$$U_{\varepsilon,\xi}(x) = C_N \Big(\frac{\varepsilon}{\varepsilon^2 + |x-\xi|^2} \Big)^{\frac{N-2}{2}}.$$
- (nondegeneracy) The linearized opertaor
$$\Delta \varphi + \frac{N+2}{N-2} U_{\varepsilon,\xi}^{\frac{4}{N-2}} \varphi = 0, \ \|\varphi\|_{L^\infty(\mathbb{R}^N)} < +\infty$$
consists of exactly $N+1$ dimensional kernels:
$$\frac{\partial U}{\partial \varepsilon}, \frac{\partial U}{\partial \xi_j}, j = 1, ..., N$$

These $N+1$ dimensional kernels corresponds to exactly the following invariances of Yamabe problem

$$\Delta u + u^{\frac{N+2}{N-2}} = 0, u > 0 \text{ in } \mathbb{R}^N$$

- (scaling) $\lambda^{\frac{N-2}{2}} u(\lambda \cdot)$ is also a solution
- (translation) $u(x-\xi)$ is also a solution

Sign-Changing solutions

$$\Delta u + |u|^{\frac{4}{N-2}} u = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{18}$$

Existence of infinitely many sign-changing solutions (non-radial)

- Ding 1986: assume partial symmetry; $u(x', x'') = -u(x'', x')$
- del Pino-Musso-Pacard-Pistoia 2012: For $K >> 1$, found a solution to (18)

$$U_K(x) \sim U(x) - \sum_{j=1}^{K} U_{\mu_j}(x - \xi_j) \tag{19}$$

where

$$U(x) = C_N \left(\frac{2}{1+|x|^2} \right)^{\frac{N-2}{2}}, \quad U_\mu(x) = \mu^{-\frac{N-2}{2}} U(\mu^{-1} x) \tag{20}$$

where

$$\mu = \mu(K), \quad \xi_l = \sqrt{1-\mu^2}\,(1,0) e^{i\frac{2\pi(l-1)}{K}}$$

Symmetries of Yamabe Problem

$$\Delta u + |u|^{\frac{4}{N-2}} u = 0$$

The proof of del Pino-Musso-Pistoia-Pacard uses the following invariances of the equation:

•• Rotation Invariance : for $(\bar{y}, y') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$,

$$u(\bar{y}, y') = u\left(e^{\frac{2\pi}{K}\sqrt{-1}}\bar{y}, y'\right)$$

•• Reflection Invariance:

$$u(-y_1, y_2, y^{'}) = u(y_1, -y_2, y_3) = u(y_1, y_2, |y^{'}|).$$

•• Kelvin Transform Invariance:

$$u(y) = u|y|^{-(N+2)} u\left(\frac{y}{|y|^2}\right).$$

•• Scaling Invariance:

$$u(y) = \lambda^{\frac{N-2}{2}} u(\lambda y)$$

These invariances reduce the problem to one parameter problem: adjusting the scaling parameter of negative bumps

Let U_K be the solution constructed by del Pino-Musso-Pacard-Pistoia

$$\Delta U_K + |U_K|^{\frac{4}{N-2}} U_K = 0$$

Question: Is U_K non-degenerate ?

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Namely, what are possible kernels of

$$\Delta \varphi + \frac{N+2}{N-2} |U_k|^{\frac{4}{N-2}} \varphi = 0, \qquad \|\varphi\|_{L^\infty} < +\infty?$$

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Understanding the non-degeneracy (and the kernels) is one of the most important steps in the study of bubbling behaviors or soliton dynamices in nonlinear Schrodinger (wave) equation (work of Kenig-Merle)

Possible Kernels

Let

$$L(\varphi) = \Delta\varphi + \frac{N+2}{N-2}|U_K|^{\frac{4}{N-2}}\varphi \tag{21}$$

We have

$$L(Z_j) = 0, \quad j = 0, \ldots, N+1. \tag{22}$$

where

- $Z_0(x) = \frac{\partial}{\partial\Lambda}[\Lambda^{-\frac{N-2}{2}}u(\Lambda^{-1}x)]_{|\Lambda=1}$ (scaling invariance)
- $Z_j(x) = \frac{\partial}{\partial x_j}u(x), j = 1, ..., K$ (translation invariance)
- $Z_{N+1}(x) = \frac{\partial}{\partial\theta}[u(R_\theta x)]_{|\theta=0}$ (rotation invariance)

where R_θ is the rotation in the x_1, x_2 plane of angle θ.

Theorem 3. (Musso-Wei 2013) Assume that $N \neq 2m^2$, for any integer m. Then there exists a sequence $K_n \to \infty$ such that all bounded solutions to the equation

$$\Delta\varphi + \frac{N+2}{N-2}|U_K|^{\frac{4}{N-2}}\varphi = 0$$

are a linear combination of the functions $Z_j(x)$, for $j = 0, 1, \ldots, N+1$.

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$$\Delta\varphi + \frac{N+2}{N-2}|U_K|^{\frac{4}{N-2}}\varphi = 0$$

are a linear combination of the functions $Z_j(x)$, for $j = 0, 1, \ldots, N+1$.

•• Unlike the positive solution case, the dimension of the kernels is $N+2$. ($Z_{N+1} = 0$ in the positive solution case.)

•• resonance dimensions:

$$N \neq 8, 18, 32, ..., 2m^2, ...$$

•• resonance condition in K:

$$|\frac{m}{K} - \tau_0| \geq \frac{C}{K^2}$$

τ_0 is an irrational number (a root of a polynomial).

Scheme of the proof

Let φ a bounded function with $L(\varphi) = 0$. Write

$$\varphi = \sum_{j=0}^{N+1} a_j Z_j(x) + \tilde{\varphi}^{\perp}$$

with

$$\int U_K^{\frac{4}{N-2}} Z_j \tilde{\varphi}^{\perp} = 0, \quad j = 0, \ldots, N+1$$

We want to show that $\tilde{\varphi}^{\perp} = 0$.

There are $K+1$ bubbles: each bubble contributes to $N+1$ dimensional approximate kernels:
Approximate Kernels at the center bubble

$$z_0(y) = \frac{N-2}{2}U(y) + \nabla U(y)\cdot y, \quad z_i(y) = \frac{\partial}{\partial x_i}U(y)$$

Approximate Kernels at the circle bubble: For any $l = 1, \ldots, K$, we define

$$z_{\alpha l}(x) = \mu_l^{-\frac{N-2}{2}} z_\alpha(\frac{x-\xi_l}{\mu_l}), \quad \alpha = 0, \ldots, N$$

$$\mathsf{Z}_\alpha(x) = \begin{bmatrix} z_{\alpha 1}(x) \\ z_{\alpha 2}(x) \\ .. \\ z_{\alpha K}(x) \end{bmatrix}$$

Write $\varphi = \sum_{j=0}^{N+1} a_j Z_j(x) + \tilde{\varphi}^{\perp}$ with $L(\varphi) = 0$ and

$$\tilde{\varphi}^{\perp} = \sum_{\alpha=0}^{N} \mathsf{c}_\alpha \cdot \mathsf{Z}_\alpha(x) + \varphi^{\perp}$$

with

$$\int U_{\mu_l}^{\frac{4}{N-2}}(x-\xi_l) Z_{\alpha l}(x) \varphi^{\perp} = 0, \quad l = 1, \ldots, K, \quad \alpha = 0, \ldots, N.$$

Thus

$$\tilde{\varphi}^{\perp} \equiv 0 \Longleftrightarrow \mathsf{c}_\alpha = 0 \quad \text{for all} \quad \alpha \quad \text{and} \quad \varphi^{\perp} \equiv 0.$$

Now

$$L(\varphi) = 0 \Longrightarrow L(\sum_{\alpha=0}^{N} \mathsf{c}_\alpha \cdot \mathsf{Z}_\alpha(x) + \varphi^{\perp}) = 0$$

since

$$L(Z_i) = 0 \quad i = 0, \ldots, N+1.$$

Take

$$L(\sum_{\alpha=0}^{N} \mathsf{c}_\alpha \cdot \mathsf{Z}_\alpha(x) + \varphi^{\perp}) = 0 \tag{23}$$

We multiply (23) against $Z_{\beta l}$, for $\beta = 0, \dots, N$ and $l = 1, \dots, K$, we integrate in $\mathbb{R}^n$ and we get a linear system in the constants $c_{\alpha j}$ of the form

$$M \begin{bmatrix} \mathsf{c}_0 \\ \mathsf{c}_1 \\ .. \\ \mathsf{c}_n \end{bmatrix} = \begin{bmatrix} \mathsf{b}_0 \\ \mathsf{b}_1 \\ .. \\ \mathsf{b}_n \end{bmatrix}$$

where M is a square matrix of dimension $[(N+1) \times K]^2$

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$$

where M_1 is a square matrix of dimension $(3 \times K)^2$ and M_2 is a square matrix of dimension $[(N-2) \times K]^2$.

$$M_1 = \begin{bmatrix} A & B & C \\ B & F & D \\ C & D & G \end{bmatrix}$$

$$M_2 = \begin{bmatrix} H_3 & 0 & 0 & 0 & 0 \\ 0 & H_4 & 0 & 0 & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & H_{N-1} & 0 \\ 0 & 0 & 0 & 0 & H_N \end{bmatrix}$$

All the matrices

$$\color{red}{A}\color{black}{, B, C, D,} \color{red}{F}\color{black}{, G, H_3, ..., H_N}$$

are $K \times K$ circulant matrices

A circulant matrix, $K \times K$:

$$A_1 = C_N \bar{\mu}^{N-4} \textbf{circ}[0, \frac{1}{(1-\cos\theta_2)^{\frac{N-2}{2}}}, \frac{1}{(1-\cos\theta_3)^{\frac{N-2}{2}}}, .., \frac{1}{(1-\cos\theta_k)^{\frac{N-2}{2}}}$$

Eigenvalues: for $m = 0, \ldots, K-1$

$$\lambda_m = C \sum_{l \neq 1} \frac{\cos(m\theta_l)}{(1-\cos\theta_l)^{\frac{N-2}{2}}} = C_1 \sum_{l \neq 1} \frac{\cos(m\theta_l)}{(l-1)^{N-2}}$$

$$= C_1 \sum_{l \neq 1} \frac{\cos((l-1)\theta_{m+1})}{(l-1)^{N-2}} = \mathcal{A}_{N-2}(\theta_{m+1})$$

Now: $\mathcal{A}_{N-2}$ is decreasing in $(0, \pi)$,

$$\mathcal{A}_{N-2}(\pi) = (-1 + \frac{1}{2^{N-3}})\mathcal{A}_{N-2}(0) < \mathcal{A}_{N-2}(\theta) < \mathcal{A}_{N-2}(0)$$

We have $\mathcal{A}_{N-2}(0) > 0$, while $(-1 + \frac{1}{2^{N-3}})\mathcal{A}_{N-2}(0) < 0$. So for some θ $\mathcal{A}_{N-2}(\theta) = 0$

Resonance on K !!!

F circulant matrix, $K \times K$. First row of F is $C_N \bar{\mu}^{N-2} \times$

$$\Big[\; (N-2)\Lambda_{N-2,k} + \sum_{l \neq 1} \frac{N\cos(\theta_l)-(N-2)}{(1-\cos\theta_l)^{\frac{N}{2}}} \quad \frac{(N-2)\cos\theta_2 - N}{(1-\cos\theta_2)^{\frac{N}{2}}} \quad \ldots \quad \frac{(N-2)\cos\theta}{(1-\cos\theta_k}$$

Eigenvalues: for $m = 0, \ldots, K-1$

$$\lambda_m = \frac{N-2}{2^{\frac{N}{2}}} \mu^{\frac{N-2}{2}} \left[m^2 - \frac{N}{2} + l.o.t \right]$$

Resonance when $N = 2m^2$

C circulant matrix, $K \times K$. First row of C is $-\frac{(N-2)^2}{2^{\frac{N}{2}}}\bar{\mu}^{N-3}\times$

$$\left[\begin{array}{ccccc} 0 & \frac{\sin\theta_2}{(1-\cos\theta_2)^{\frac{N}{2}}} & \frac{\sin\theta_3}{(1-\cos\theta_3)^{\frac{N}{2}}} & .. & \frac{\sin\theta_K}{(1-\cos\theta_K)^{\frac{N}{2}}} \end{array}\right]$$

Eigenvalues: for $m = 0, \ldots, K-1$

$$\lambda_m = C \sum_{l \neq 1} \frac{\sin(\theta_l)\cos(m\theta_l)}{(1-\cos\theta_l)^{\frac{N-2}{2}}} = 0$$

by symmetry.

D circulant matrix, $K \times K$. First row of D is $\frac{(N-2)^2}{2^{\frac{N}{2}}}\bar{\mu}^{N-3}\times$

$$\begin{bmatrix} 0 & \frac{\sin\theta_2}{(1-\cos\theta_2)^{\frac{N}{2}}} & \frac{\sin\theta_3}{(1-\cos\theta_3)^{\frac{N}{2}}} & .. & \frac{\sin\theta_k}{(1-\cos\theta_K)^{\frac{N}{2}}} \end{bmatrix}$$

Eigenvalues: for $m = 0, \ldots, K-1$

$$\lambda_m = C \sum_{l \neq 1} \frac{\sin(\theta_l)\cos(m\theta_l)}{(1-\cos\theta_l)^{\frac{N-2}{2}}} = 0$$

by symmetry.

G circulant matrix, $k \times k$. First row of G is $-\frac{(N-2)}{2^{\frac{N+2}{2}}} \bar{\mu}^{N-2} \times$

$$\left[\begin{array}{cccc} 2\Lambda_{N-2,k} + \sum_{l \neq 1} \frac{N \cos(\theta_l) - (N-2)}{(1-\cos\theta_l)^{\frac{N}{2}}} & -\frac{(N-2)\cos\theta_2 - N}{(1-\cos\theta_2)^{\frac{N}{2}}} & \cdots & -\frac{(N-2)\cos\theta_k - N}{(1-\cos\theta_k)^{\frac{N}{2}}} \end{array} \right]$$

Eigenvalues: for $m = 0, \ldots, k-1$

$$\lambda_m = \frac{N-2}{2^{\frac{N}{2}}} \mu^{\frac{N-2}{2}} \left[m^2 N - m^2 + l.o.t \right]$$

H_j are $K \times K$ circulant matrices whose first row is given by

$$-\frac{(N-2)}{2^{\frac{N}{2}}} \bar{\mu}^{N-2} \left[\begin{array}{cccc} 1 + \Lambda_{N,K} & -\frac{1}{(1-\cos\theta_2)^{\frac{N}{2}}} & \cdots & -\frac{1}{(1-\cos\theta_K)^{\frac{N}{2}}} \end{array} \right]$$

Applications to Prescribed Scalar Curvature Problem

Consider the third problem which is well-known

$$\begin{cases} -\Delta u = K(y)u^{\frac{N+2}{N-2}},\ u > 0 \quad \text{in } \mathbb{R}^N \\ u \in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{24}$$

where $0 < C_1 \leq K(y) \leq C_2$.

Theorem (Wei-Yan 2010): Suppose that $N \geq 5$. If K is radially symmetric and there is a constant $r_0 > 0$, such that

$$K(r) = K(r_0) - c_0|r - r_0|^m + O(|r - r_0|^{m+\theta}), \quad r \in (r_0 - \delta, r_0 + \delta)$$

where $c_0 > 0$, $\theta > 0$ are some constants, and the constant m satisfies $m \in [2, N-2)$. Then problem (24) has infinitely many non-radial positive solutions

Theorem (Musso-Wei 2013): Same result holds for nonradial

$$K(x) = K_0 - c_0 d(x, \Gamma)^m + O(d(x, \Gamma)^{m+\theta}),\ \text{near } \Gamma$$

where $\Gamma = \{|(x_1, x_2) = R, x^{'} = 0\}$ is a circle.

Thank You