# On a free boundary problem for the curvature flow with driving force

Jong-Shenq Guo

Tamkang University

j/w Hiroshi Matano, Masahiko Shimojo, Chang-Hong Wu

# Outline

# Introduction

Let $\Gamma^0$ be a smooth oriented curve in the upper half-plane whose endpoints lie on the $x$-axis with given contact angles $\psi_-$ on the left and $-\psi_+$ on the right:

$$\Gamma^0 := \{(x(\tau), y(\tau)) : 0 \leq \tau \leq 1\},$$
$$(x'(\tau), y'(\tau)) \neq (0,0) \text{ for } 0 \leq \tau \leq 1,$$
$$y(0) = y(1) = 0, \quad x'(0) = y'(0)\cot\psi_-, \quad x'(1) = -y'(1)\cot\psi_+.$$

- contact angle: the angle of the tangent vector *measured from the positive $x$-axis with range* $(-\pi/2, \pi/2)$.
- Thus the interior angles between $\Gamma^0$ and the horizontal line are $\psi_-$ and $\psi_+$, respectively.

Given such a curve $\Gamma^0$, we consider a problem of finding a family of oriented curves $\{\Gamma(t)\}_{t\geq 0}$, with $\Gamma(0)=\Gamma^0$, that evolve by the curvature flow equation

$$V = \kappa + c, \tag{1}$$

while keeping the endpoints on the $x$-axis with the same fixed contact angles as $\Gamma^0$.

- $V$: normal velocity, $\kappa$: the (signed) curvature, $c > 0$: a driving force.
- The signs of $V$, $\kappa$ are chosen in accordance with the orientation of the curve, in which the reference normal vector points toward the left-hand side of the tangent vector.

When the curve $\Gamma(t)$ is a graph of a function $y = u(x,t)$, $x \in [l_-(t), l_+(t)]$, where $l_\pm(t)$ denote the position of the endpoints of the curve $\Gamma(t)$, the curvature flow (1) is reduced to solving the following free boundary problem (P):

$$u_t = \frac{u_{xx}}{1+u_x^2} + c\sqrt{1+u_x^2}, \quad x \in (l_-(t), l_+(t)), \ t > 0, \tag{2}$$

$$u(l_\pm(t), t) = 0, \quad t > 0, \tag{3}$$

$$u_x(l_\pm(t), t) = \mp \tan\psi_\pm, \quad t > 0, \tag{4}$$

$$u(x,0) = u^0(x), \ x \in [l_-^0, l_+^0], \quad l_\pm(0) = l_\pm^0, \tag{5}$$

where we assume that $\psi_\pm \in (0, \pi/2)$, $c > 0$ and $-\infty < l_-^0 < l_+^0 < \infty$.

We shall focus on the free boundary problem (P).

For simplicity, we always assume that

$$\begin{cases} u^0(x) > 0 \ \text{ in } (l_-^0, l_+^0)\ ,\ u^0(l_\pm^0) = 0, \\ u_x^0(l_\pm^0) = \mp \tan\psi_\pm,\ u^0 \in C^2([l_-^0, l_+^0]). \end{cases} \tag{6}$$

## Remarks:

- To obtain a priori estimates and some main results, we need $u^0 \in C^2([l_-^0, l_+^0])$.
- However, the local existence and uniqueness of a classical solution to (P) can be derived as that in [Chang-G.-Kohsaka03] by using a fixed point argument as long as $u^0 \in C^{1+\alpha}([l_-^0, l_+^0])$, $\alpha \in (0,1)$. Indeed, the local existence time depends only on the $C^{1+\alpha}$ norm of the initial data. The solution $(u, l_\pm)$ is in the class $C^{2+\alpha,1+\alpha/2} \times [C^{1+\alpha/2}]^2$ for $t > 0$. This regularity also extends to $t = 0$ if the initial data is in $C^{2+\alpha}$ class.

- For the curvature flow (1) with $c = 0$, it appears in the study of evolution of grain domains in polycrystals, see, e.g., [Gurtin93], [Herring51,52], [Mullins63].
- The intersection of two grain domains forms a grain boundary which is usually modeled by the curvature flow (cf. [Adams et al 98,99], [Kinderlehrer-Liu01]).

- For mathematically rigorous studies of problem (P) with $c = 0$, we refer to the work [Chen-G.11] and the references cited therein. See also [Chang-G.-Kohsaka03], [Chern-G.-Lo03], [G.-Hu06].
- In particular, it is shown in [Chen-G.11] that the problem (P) with $c = 0$ has a unique self-similar shrinking solution and every solution $\Gamma(t)$ shrinks to a point in finite time in an asymptotically self-similar manner.

For the problem (P) with $c > 0$, the asymptotic behavior of the solution depends on the balance between the curvature and the driving force.

- If the curvature dominates the driving force, the curve $\Gamma(t)$ shrinks to a point in finite time, as in the case $c = 0$.
- On the other hand, if the driving force dominates the curvature eventually, the curve keeps expanding for all large time.
- It can also happen that the curvature remains in delicate balance with the driving force. In that case, $\Gamma(t)$ remains bounded and converges either to a stationary solution or to a traveling wave solution of (2)-(4).

# Main results:

Here and in what follows, $[0, T)$ will denote the maximal time interval for the existence of a classical solution $(u, l_\pm)$ to the problem (P), where $T \in (0, \infty]$.
We let $A(t)$ denote the area of the domain enclosed by $\Gamma(t)$ and the $x$-axis, and $L(t)$ the length of $\Gamma(t)$, namely,

$$A(t) := \int_{l_-(t)}^{l_+(t)} u(x,t)dx, \quad L(t) := \int_{l_-(t)}^{l_+(t)} \sqrt{1 + u_x^2(x,t)}dx.$$

Our first main result gives complete classification of the behavior of solutions:

**Theorem 1 (Classification)**

*Any solution of* (P) *belongs to one of the following types:*

(A) [Expanding] $T = \infty$, *and both* $L(t)$ *and* $A(t)$ *tend to* $\infty$ *as* $t \to \infty$.

(B) [Bounded] $T = \infty$, *and both* $L(t)$ *and* $A(t)$ *remain bounded from above and below by two positive constants as* $t \to \infty$.

(C) [Shrinking] $T < \infty$, *and both* $L(t)$ *and* $A(t)$ *tend to* $0$ *as* $t \to T$.

Some criteria for the above classification:

if the initial data satisfies

$$A(0) > \frac{1}{\pi}\left(\frac{\psi_+ + \psi_-}{c}\right)^2,$$

then the solution is of type (A), while if

$$L(0) < \frac{2(1-\cos\psi_{\min})}{c}, \qquad \psi_{\min} := \min\{\psi_-, \psi_+\},$$

the solution is of type (C).

The next results are concerned with the concavity of the solution.

**Theorem 2 (Preservation of concavity)**

*Suppose that $u(x, t_0)$ is concave for some $t_0 \in [0, T)$, then it remains strictly concave for all $t \in (t_0, T)$. In particular, $u_{xx}(x, t) < 0$ for $x \in (l_-(t), l_+(t))$ for all $t \in (0, T)$, if $(u^0)_{xx} \leq 0$ on $(l_-^0, l_+^0)$.*

**Theorem 3 (Eventual concavity in the bounded case)**

*Let $(u, l_\pm)$ be a solution of type $(\mathrm{B})$. Then there exists $t^* \geq 0$ such that $u(\cdot, t)$ is strictly concave for all $t \in (t^*, \infty)$.*

**Theorem 4 (Eventual concavity in the shrinking case)**

*Let $(u, l_\pm)$ be a solution of type $(\mathrm{C})$. Then there exists $t^* \in [0, T)$ such that $u(\cdot, t)$ is strictly concave for all $t \in (t^*, T)$.*

Our final results give more precise description of the asymptotic behavior of solutions for types (A)–(C).

For the type (A), as time passes, the effect of the curvature becomes smaller and smaller compared with the constant forcing term, so one may expect that the asymptotic behavior of the solution is well approximated by the solution of $V = c$.

Indeed, the profile of the solution approaches that of a self-similar solution of $V = c$.

To explain this result, we first note that the solution of $V = c$ is expressed by the graph of a function $y = g(x,t)$ satisfying

$$g_t = c\sqrt{1 + g_x^2}. \tag{7}$$

Under the boundary conditions corresponding to (3) and (4), the above equation has a unique self-similar solution of the form $g(x,t) = tG(x/t)$, where $G(\zeta)$ is a function that is defined on some interval $\hat{p} \le \zeta \le \hat{q}$ and satisfies

$$G(\zeta) > 0, \qquad G(\zeta) - \zeta G'(\zeta) = c\sqrt{1 + (G')^2(\zeta)}, \quad \hat{p} < \zeta < \hat{q}, \tag{8}$$

along with the boundary conditions

$$G(\hat{p}) = G(\hat{q}) = 0, \quad G'(\hat{p}) = \tan\psi_-, \;\; G'(\hat{q}) = -\tan\psi_+. \tag{9}$$

The constants $\hat{p}, \hat{q} \in \mathbb{R}$ with $\hat{p} < \hat{q}$ are determined uniquely by the condition

$$-\hat{p}\sin\psi_- = \hat{q}\sin\psi_+ = c. \tag{10}$$

For any given $\psi_\pm \in (0, \pi/2)$ and $c > 0$, the problem (8)-(9) is solvable if and only if (10) holds, and the solution is given by

$$G(\zeta;\psi_\pm) := \begin{cases} (\tan\psi_-)(\zeta - \hat{p}), & \hat{p} \le \zeta \le -c\,\sin\psi_-, \\ \sqrt{c^2 - \zeta^2}, & -c\,\sin\psi_- \le \zeta \le c\,\sin\psi_+, \\ -(\tan\psi_+)(\zeta - \hat{q}), & c\,\sin\psi_+ \le \zeta \le \hat{q} \end{cases} \tag{11}$$

with $\hat{p}, \hat{q}$ as in (10). From (10) one sees that $\hat{p} < -c < 0 < c < \hat{q}$. Geometrically, the graph of $G$ consists of a part of the circle and two line segments in the upper half plane.

## Theorem 5 (Asymptotics for the expanding case)

*Let* $(u, l_\pm)$ *be a type* $(\mathrm{A})$ *solution of* (2)-(5) *and let* $G = G(\zeta; \psi_\pm)$ *be defined by* (11). *Then there exist a function* $\rho(t)$ *satisfying* $\lim_{t\to\infty}[\rho(t)/t] = 1$ *and a constant* $t_0 > 0$ *such that*

$$\rho(t)G\Big(\frac{x}{\rho(t)}\Big) \le u(x,t) \le (t+t_0)G\Big(\frac{x}{t+t_0}\Big) \qquad (12)$$

*for all* $x \in [l_-(t), l_+(t)]$ *and* $t > 0$. *Consequently*

$$\lim_{t\to\infty} \frac{\pm l_\pm(t)}{t} = \frac{c}{\sin\psi_\pm}, \qquad (13)$$

$$\lim_{t\to\infty} \frac{u(x,t)}{tG(x/t)} = 1 \quad \textit{uniformly on cpt subsets of } \mathbb{R}. \qquad (14)$$

In (12), $G := 0$ outside the interval $[\hat{p}, \hat{q}]$.

Next, in the case of type (B), one may expect that the solution converges to a stationary solution if a stationary solution exists.

Indeed, when $\psi_+ = \psi_-$, the problem (P) admits a stationary solution whose shape is a portion of a circle with radius $1/c$.

However, in the case $\psi_+ \neq \psi_-$, there is no positive stationary solution. Actually, in this case the solution converges to a traveling wave in the form $u(x,t) = \Phi(x - \nu t - a)$, where $\nu$ denotes the wave speed, $\Phi(\xi)$ is a function that defines the profile of the wave and $a$ is an arbitrary constant that adjusts the phase.

Substituting this form into (2)-(4) yields the following, where $\beta > 0$ is some constant and $\Phi$ is normalized in such a way that the center of its support comes to the origin:

$$\begin{cases} \dfrac{\Phi_{\xi\xi}}{1+\Phi_\xi^2} + \nu\Phi_\xi + c\sqrt{1+\Phi_\xi^2} = 0 \quad \text{in } (-\beta,\beta), \\ \Phi(\pm\beta) = 0, \quad \Phi_\xi(\pm\beta) = \mp\tan\psi_\pm. \end{cases} \tag{15}$$

Multiplying (15) by $\Phi_\xi/\sqrt{1+\Phi_\xi^2}$ and integrating it over $[-\beta,\beta]$, we easily see that $\nu > 0$ (resp. $=0,\ <0$) if and only if $\psi_- - \psi_+ > 0$ (resp. $=0,\ <0$).

From the physical point of view, the condition $\psi_+ \neq \psi_-$ means that the surface tension on the floor that pulls the curve is different between the left and right endpoints, since the contact angle is determined by the relation between the surface tension on the floor and that on the curve. This explains intuitively why a traveling wave appears when $\psi_+ \neq \psi_-$.

Thus, if $\psi_- = \psi_+ =: \psi$, we have $\nu = 0$, in which case $\Phi$ is a stationary solution.
We shall distinguish this case by using the notation $\varphi$ instead of $\Phi$:

$$\begin{cases} \dfrac{\varphi_{xx}}{1+\varphi_x^2} + c\sqrt{1+\varphi_x^2} = 0 \quad \text{in } (-\beta, \beta), \\ \varphi(\pm\beta) = 0, \quad \varphi_x(\pm\beta) = \mp \tan\psi. \end{cases}$$

The solution $\varphi$ represents a portion of a circle with radius $1/c$.

In the following theorem, we understand that $\Phi = 0$ outside the interval $[-\beta, \beta]$.

**Theorem 6 (Asymptotics for the bounded case)**

(i) *There exist unique constants* $\beta > 0$ *and* $\nu \in \mathbb{R}$ *and a unique function* $\Phi(\xi)$ *that satisfy* (15). *Furthermore, the sign of* $\nu$ *coincides with the sign of* $\psi_- - \psi_+$.

(ii) *Let* $(u, l_\pm)$ *be a solution of type* $(\mathrm{B})$. *Then there exists a constant* $a \in \mathbb{R}$ *such that* $u(x, t) \to \Phi(x - \nu t - a)$ *uniformly for* $x \in [l_-(t), l_+(t)]$, *and that* $l_\pm(t) - \nu t \to a \pm \beta$ *as* $t \to +\infty$, *where* $\Phi, \nu, \beta$ *are as in* (15). *In the special case where* $\psi_- = \psi_+ =: \psi$, *this means that* $u$ *converges to a stationary solution* $\varphi(x - a)$ *as* $t \to +\infty$.

Lastly, as for type (C), we shall show that the curve $\Gamma(t)$ shrinks to a point as $t \to T$ in a self-similar manner.

In this case, as $t$ approaches $T$, the solution behaves like a solution of $V = \kappa$ under the same boundary conditions.

We introduce the following similarity transformation:

$$
\begin{aligned}
& z = \frac{x}{\sqrt{2(T-t)}}, \quad s = -\frac{1}{2}\ln(T-t)\,, \\
& u(x,t) = \sqrt{2(T-t)}\, w(z,s), \\
& l_-(t) = \sqrt{2(T-t)}\, p(s), \;\; l_+(t) = \sqrt{2(T-t)}\, q(s).
\end{aligned}
$$

- By spatial translation, we may assume without loss of generality that $x = 0$ is the limit point of the shrinking curve.

Then $u$ satisfies (2)-(5) if and only if $w$ satisfies

$$w_s = \frac{w_{zz}}{1+w_z^2} - zw_z + w + \sqrt{2}ce^{-s}\sqrt{1+w_z^2},$$
$$z \in (p(s), q(s)),\ s > s_0, \tag{16}$$
$$w(p(s), s) = w(q(s), s) = 0, \quad s > s_0, \tag{17}$$
$$w_z(p(s), s) = \tan\psi_-,\ w_z(q(s), s) = -\tan\psi_+,\ s > s_0, \tag{18}$$
$$w(z, s_0) = w_0(z) := (2T)^{-1/2}u^0(z\sqrt{2T}\,),$$
$$z \in [l_-^0/\sqrt{2T}, l_+^0/\sqrt{2T}\,], \tag{19}$$

where $s_0 := -\frac{1}{2}\ln T$.

If $c = 0$, the equation (16) is autonomous, and the stationary problem for (16)-(18) is given in the form

$$\frac{\varphi_{zz}}{1 + (\varphi_z)^2} - z\varphi_z + \varphi = 0, \quad z \in (\bar{p}, \bar{q}), \tag{20}$$

$$\varphi(\bar{p}) = \varphi(\bar{q}) = 0, \tag{21}$$

$$\varphi_z(\bar{p}) = \tan\psi_-, \quad \varphi_z(\bar{q}) = -\tan\psi_+ \tag{22}$$

for some $\bar{p} < \bar{q}$.

It is shown in [Chen-G.11] that, for any $\psi_\pm \in (0, \pi/2)$, the problem (20)– (22) has a unique solution $\varphi(z)$; the constants $\bar{p}, \bar{q}$ are also uniquely determined by $\psi_\pm$.

Note that, when $c \neq 0$, the equation (16) is no longer autonomous but is asymptotically autonomous as $s \to +\infty$.

The following theorem gives asymptotics for type (C) in terms of the rescaled solution $w$.

**Theorem 7 (Asymptotics for the shrinking case)**

*Let* $(u, l_{\pm})$ *be a solution of type* $(\mathrm{C})$ *and* $(w, p, q)$ *be the corresponding solution of* (16)-(19)*. Then* $(w(z, s), p(s), q(s))$ *converges to the unique solution of* (20)-(22) *as* $s \to +\infty$ *uniformly for* $y \in [p(s), q(s)]$*.*

- The solution of (20)-(22) corresponds to a self-similar shrinking solution of the form

$$u(x,t) = \sqrt{2(T-t)}\,\varphi\Big(\frac{x}{\sqrt{2(T-t)}}\Big)$$

to the problem $(\mathrm{P}_0)$, that is, the problem (P) with $c = 0$.

- Theorem 7 asserts that any shrinking solution of (P) behaves like the unique self-similar solution of $(\mathrm{P}_0)$ as $t$ approaches $T$.
- Intuitively this sounds reasonable as the curvature tends to infinity in the case (C) and therefore the driving force $c$ should become negligible.

# Some key ideas

- One of the main tools in this paper is the intersection number argument.
- However, as we are dealing with a free boundary problem in which the endpoints of the curve can slide freely along the $x$-axis, the standard intersection number principle (of [Angenent88] and [Matano82]) dose not work.
- Indeed, the number of intersections between two solutions may increase in time.

- To overcome this difficulty, we introduce the notion of extended intersection number by extending the solutions linearly below the $x$-axis outside their domain of definition, and counting the number of intersections between the extended solutions.
- It turns out that this extended intersection number does not increase in time; moreover, it drops strictly each time a multiple zero occurs. We call this property the extended intersection number principle, which turns out to be exceedingly useful in analyzing the problem (P) such as the concavity properties.

- For a strictly concave solution, the problem (P) can be reduced to the following problem for the curvature:

$$\begin{aligned} &\kappa_t = \kappa^2(\kappa_{\theta\theta} + \kappa + c), \quad -\psi_+ < \theta < \psi_+, \quad t > 0, \\ &\kappa_\theta = (\kappa + c)\cot\theta, \quad \theta = \mp\psi_\pm, \quad t > 0, \\ &\kappa(\theta, 0) = \kappa_0(\theta), \quad -\psi_+ < \theta < \psi_-, \end{aligned}$$

  where $\theta(x,t) := \arctan u_x(x,t)$ and the initial data $\kappa_0$ satisfies

$$\int_{-\psi_+}^{\psi_-} \frac{\sin\theta}{\kappa_0(\theta)}\, d\theta = 0.$$

- The stationary solutions are given by $\kappa = -\nu\sin\theta - c$ for $\nu \in \mathbb{R}$. This gives $u_t = -\nu u_x$, a traveling wave solution.

- The methods for proving the theorems on asymptotics for types (A)-(C) are all different.
- For the expanding case, we use the method of super-sub-solutions.
- For the area bounded case, we apply the extended intersection number principle to show the convergence of the solution to a traveling wave or a steady-state.
- It is somewhat surprising that such a convergence result follows simply by counting the intersection numbers, without constructing a Lyapunov functional.

- As for the shrinking case, the proof of Theorem 7 goes in two steps:
- We first show that the aspect ratio of the curve remains bounded by using an idea similar to [Grayson87]. The boundedness of the aspect ratio implies that the rescaled solution $w$ possesses certain compactness properties.
- We then use a Lyapunov functional borrowed from [Huisken90]. Here, as equation (16) is non-autonomous since $c \neq 0$, the Lyapunov functional is not necessarily decreasing. However, since the perturbation term decays exponentially as $s \to +\infty$, it creates no problem in proving the convergence.