Classification of the Entire Radial Self-Dual Solutions to Non-Abelian Chern-Simons Systems

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Outline

Introduction

The Chern-Simons theories were developed to explain certain condensed matter phenomena, anyon physics, superconductivity, and quantum mechanics. See [Dunne, Self-Dual Chern-Simons Theory, Springer-Verlag, 1995] for more detail.

The (2+1)-dimensional relativistic Abelian Chern-Simons-Higgs model proposed by [Hong-Kim-Pac, Phys. Rev. Lett., 1990] and [Jackiw-Weinberg, Phys. Rev. Lett.,1990] was to explain high temperature superconductivity. They derived the following elliptic partial differential equation:

Abelian Chern-Simons equation

$$\Delta u + \frac{1}{\varepsilon^2} e^u (1 - e^u) = 4\pi \sum_{i=1}^{N} \delta_{p_i}$$

Here, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$, ε is a constant and δ_p is the Dirac measure in $\mathbb{R}^2$.

$$\Delta u + \frac{1}{\varepsilon^2} e^u (1 - e^u) = 4\pi \sum_{i=1}^{N} \delta_{p_i}$$

This equation has been extensively studied for the past twenty years. See the works [Wang, CMP, 1991], [Spruck-Yang, CMP, 1992], [Caffarelli-Yang, CMP ,1995], [Tarantello, JMP, 1996], [Chae-Imanuvilov, CMP ,2000], [Nolasco-Tarantello2000] [Nolasco-Tarantello, CMP ,1999], [Chan-Fu-Lin, CMP .2002], [Choe, CPDE ,2009], [Lin-Yan, CMP ,2010], [Choe-Kim-Lin, Ann. Inst. H. Poincaré Anal., 2011], [Lin-Yan, ARMA, 2012]. In these works, equations were studied either in $\mathbb{R}^2$ or flat torus in $\mathbb{R}^2$.

Some Properties for the Entire Radial Solution

Consider the entire radial solution of Abelian Chern-Simons equation with all vertex points at the origin.

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- $u(r) < 0$ on $(0, \infty)$ unless $u \equiv 0$.

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- $$\lim_{r\to\infty} u(r) = 0 \text{ or } -\infty$$

  If $\lim_{r\to\infty} u(r) = 0$, then u is called topological solution; $\lim_{r\to\infty} u(r) = -\infty$, then u is called non-topological solution.

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$$ru_r(r) = 2N - \int_0^r se^u(1 - e^u)ds.$$

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$$lim_{r \to \infty} ru_r(r) = 0 \ (lim_{r \to \infty} u(r) = 0)$$

or

$$lim_{r \to \infty} ru_r(r) = -\tilde{\beta} \ \ (u(r) = -\tilde{\beta} \log r + O(1) \text{ near } \infty).$$

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Denote $\beta = \int_0^\infty e^u(1-e^u)rdr$. Then $\tilde{\beta} = \beta - 2N$.

- $\beta > 4N + 4$ if u is a non-topological solution.
- ([Chan and Fu and Lin, CMP, 2002]) For $\beta > 4N + 4$, there exists a unique non-topological solution such that

$$\int_0^\infty re^u(1-e^u)dr = \beta.$$

Non-Abelian Chern-Simons System of Rank 2

We consider the the entire radial solutions to the Non-Abelian Chern-Simons Systems of rank 2

$$\begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} = -K \begin{pmatrix} e^u \\ e^v \end{pmatrix} + K \begin{pmatrix} e^u & 0 \\ 0 & e^v \end{pmatrix} K \begin{pmatrix} e^u \\ e^v \end{pmatrix} + \begin{pmatrix} 4\pi N_1 \delta_0 \\ 4\pi N_2 \delta_0 \end{pmatrix} \text{ in } \mathbb{R}^2, \tag{1}$$

where $N_i \geq 0,\ i = 1, 2,\ K = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}$ satisfying

$$\alpha, \beta, \gamma, \delta > 0 \text{ and } \alpha\delta - \beta\gamma > 0. \tag{2}$$

This system appears in many physical models, for example:

(1) The relativistic non-Abelian Chern-Simons model
(2) Lozano-Marqués-Moreno-Schaposnik model of bosonic sector of $\mathcal{N} = 2$ supersymmetric Chern-Simons-Higgs theory
(3) Gudnason model of $\mathcal{N} = 2$ supersymmetric Yang-Mills-Chern-Simons-Higgs theory.

We refer to [Kao-Lee, Phys. Rev. D, 1994],[Dunne, Phys. Lett. B, 1995], [Lozano, Phys. Lett B, 2007] , [Gudnason, Nucl. Phys. B, 2009] for physical backgrounds of these models.
In the relativistic non-Abelian Chern-Simons model, K is a Cartan matrix. There are three types of Cartan matrix of rank 2, which are given by

$$A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, B_2 = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, G_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

$$\Delta u_a + \frac{1}{\varepsilon^2}\left(\sum_{b=1}^{N} K_{ab} e^{u_b} - \sum_{b=1}^{N}\sum_{c=1}^{N} e^{u_b} K_{bc} e^{u_c} K_{ac} \right) = 4\pi \sum_{j=1}^{N_a} \delta_{p_j^a}, \quad a = 1, \cdots, N$$

Let $(K^{-1})_{ab}$ be the inverse of the matrix K, and assume

$$\sum_{b=1}^{r} (K^{-1})_{ab} > 0, \quad a = 1, 2, \cdots, N. \tag{3}$$

A solution $u = (u_1, \cdots, u_N)$ is called topological solution if

$$u_a(x) \to \log\left(\sum_{b=1}^{N} (K^{-1})_{ab} \right) \text{ as } |x| \to +\infty \;\; a = 1, 2, \cdots, N;$$

is called non-topological solution if

$$u_a(x) \to -\infty \text{ as } |x| \to +\infty \;\; a = 1, 2, \cdots, N.$$

$$\Delta u_a + \frac{1}{\varepsilon^2}\Big(\sum_{b=1}^{N} K_{ab} e^{u_b} - \sum_{b=1}^{N}\sum_{c=1}^{N} e^{u_b} K_{bc} e^{u_c} K_{ac}\Big) = 4\pi \sum_{j=1}^{N_a} \delta_{p_j^a}, \, a = 1, \cdots, N$$

- existence of topological solutions in $\mathbb{R}^2$:
  - [Yang, CMP, 1997]: $\sum_{b=1}^{r} (K^{-1})_{ab} > 0$, $K = PS$
- existence of solutions on a torus:
  - [Nolosco-Tarantello, CMP, 2000]: A_2
  - [Han-Lin-Tarantello-Yang, 2013]: Gudnason model
  - [Han-Tarantello, 2013]: $K = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}$ satisfies

$$\alpha, \beta, \gamma, \delta > 0 \text{ and } \alpha\delta - \beta\gamma > 0.$$

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- existence of non-topological solutions in $\mathbb{R}^2$:
  - [Ao-Wei-Lin, 2012]: A_2 and B_2
  - [Choe-Kim-Lin, 2013]: A_2(Radial Solutions)

- existence of bubbling solution on a torus:
  - [Yan-Lin, CPAM, 2013]: A_2

Consider the entire radial solutions of

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where $N_i \geq 0$, $i = 1, 2$, $K = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}$ satisfying

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Question 1: Can we classify the entire radial solutions of the above system according to their behaviors at infinity?

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The main difficulty is the nonlinear terms in (4) may change sign, hence it is not easy to see whether the nonlinear terms $\in L^1(\mathbb{R}^2)$ or not.

By considering the transformation

$$(u,v) \to \Big(u + \log \frac{\beta+\delta}{\alpha\delta-\beta\gamma}, v + \log \frac{\alpha+\gamma}{\alpha\delta-\beta\gamma}\Big)$$

and letting

$$\Big(\frac{\beta(\alpha+\gamma)}{\alpha\delta-\beta\gamma}, \frac{\gamma(\beta+\delta)}{\alpha\delta-\beta\gamma}\Big) = (a,b).$$

Then (1) becomes

$$\begin{cases} \Delta u = -(1+a)e^u + ae^v + (1+a)^2 e^{2u} - a(1+b)e^{2v} \\ \qquad + a(b-(1+a))e^{u+v} + 4\pi N_1 \delta_0 \\ \Delta v = be^u - (1+b)e^v - b(1+a)e^{2u} + (1+b)^2 e^{2v} \\ \qquad + b(a-(1+b))e^{u+v} + 4\pi N_2 \delta_0 \end{cases} \tag{5}$$

When $u = v$, $N_1 = N_2$ in (5), then it is reduced to the Abelian Chern-Simons equation

$$\Delta u + e^u(1-e^u) = 4\pi N \delta_0$$

Theorem 1

(H. and C.S. Lin, 2013) Suppose $(u(r), v(r))$ *is an entire radial solution to* (5). *One of the following holds.*

(i) $\lim_{r\to\infty}(u, v) = (0, 0)$.

(ii) $\lim_{r\to\infty}(u, v) = (-\infty, -\infty)$, *and* e^u, $e^v \in L^1(\mathbb{R}^2)$.

(iii) $\lim_{r\to\infty}(u, v) = (\log\frac{1}{1+a}, -\infty)$ *or* $(-\infty, \log\frac{1}{1+b})$, *which is called a mixed-type solution. Furthermore,* $e^u \in L^1(\mathbb{R}^2)$ *if* $u \to -\infty$ *as* $r \to \infty$; $e^v \in L^1(\mathbb{R}^2)$ *if* $v \to -\infty$ *as* $r \to \infty$

Remark 1

Consider $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *in* (1). *Then* (1) *becomes*

$$\begin{cases} \Delta u + e^v(1 - e^u) = 4\pi N_1 \delta_0 \\ \Delta v + e^u(1 - e^v) = 4\pi N_2 \delta_0 \end{cases} \quad \text{in } \mathbb{R}^2, \tag{6}$$

which is the system of Chern-Simons model with two Higgs particles. In [Chern-Chen-Lin, CMP, 2010], if $\lim_{r\to\infty}(u(r), v(r)) = (-\infty, -\infty)$, *then the decay rate of* (u, v) *may be slow so that* e^u *and* e^v *are not both in* $L^1(\mathbb{R}^2)$.

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Conjecture 1

Suppose $\lim_{r\to\infty}(u(r), v(r)) = (-\infty, -\infty)$. e^u *and* $e^v \in L^1(\mathbb{R}^2)$ *only when* a, $b > 0$,

Strategy of the Proof of Theorem 1

The strategy of the proof of Theorem 1 is to split the nonlinear terms in (5) into the linear combination of

$$f_1 = e^u - (1 + a)e^{2u} + ae^{u+v}$$

and

$$f_2 = e^v - (1 + b)e^{2v} + be^{u+v}.$$

(5) can be written as

$$\begin{cases} \Delta u = -(1+a)f_1 + af_2 + 4\pi N_1 \delta_0 \\ \Delta v = -(1+b)f_2 + bf_1 + 4\pi N_2 \delta_0 \end{cases}$$

We want to show both f_1 and f_2 are positive. But we only can show that f_1 andf_2 are positive for large r if (u, v) is not a topological solution. Then we show that f_1 and $f_2 \in L^1(\mathbb{R}^2)$ for not topological solution.

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Conjecture 2

f_1 and $f_2 > 0$ for $r > 0$.

Remark 2

Recall

$$f_1 = e^u - (1+a)e^{2u} + ae^{u+v} = e^u - e^{2u} + ae^u(e^v - e^u)$$

We note that

$$f_1(r) > 0$$

as long as $u(r) < \log \frac{1}{1+a}$ *or* $u(r) < v(r) < 0.$

Similarly,

$$f_2(r) > 0$$

as long as

$$v(r) < \log \frac{1}{1+b} \text{ or } v(r) < u(r) < 0.$$

Useful Tool: Pohozaev identity

$$\begin{aligned} & r^2\Big(\frac{b(1+b)}{2}u_r^2(r) + abu_r(r)v_r(r) + \frac{a(1+a)}{2}v_r^2(r)\Big) \\ & - 4\Big(\frac{b(1+b)}{2}N_1^2 + abN_1N_2 + \frac{a(1+a)}{2}N_2^2\Big) \\ = & -(1+a+b)r^2F(r) + 2(1+a+b)\int_0^r sF(s)ds \end{aligned} \tag{7}$$

where

$F(r) = \Big(be^{u(r)} - \frac{b(1+a)}{2}e^{2u(r)} + ae^{v(r)} - \frac{a(1+b)}{2}e^{2v(r)} + abe^{(u+v)(r)}\Big)$

Sketch of the Proof of Theorem 1

Step 1. $u < 0$ and $v < 0$ on $(0, \infty)$ unless $u \equiv v \equiv 0$ on $(0, \infty)$.
Step 2.

Theorem 2
If (u, v) is not a topological solution, then there exists $R_0 > 0$ such that

$$f_i > 0, \ \ i = 1, 2 \ \text{for} \ r > R_0.$$

Step 3.

Theorem 3
(u, v) is a topological solution if and only if

$$(1 + 2b)u_r(r) + (1 + 2a)v_r(r) > 0 \ \text{on} \ (0, \infty).$$

Sketch of the Proof of Theorem 1

Step 4 Using the Pohozaev identity, we show that f_1 and $f_2 \in L^1(R^2)$ for not topological solution (u, v). Thus, $\lim_{r\to\infty}(u(r), v(r))$ must be one of

$$(-\infty, -\infty), \ (\log \frac{1}{1+a}, -\infty), \ (-\infty, \log \frac{1}{1+b}),$$

and

$$ru(r)_r = 2N_1 + \int_0^r \Big(-(1+a)f_1(s) + af_2(s)\Big) sds$$

and

$$rv(r)_r = 2N_2 + \int_0^r \Big(-(1+b)f_2(s) + bf_1(s)\Big) sds$$

have limit as $r \to \infty$.

Sketch of the Proof of Theorem 2

Theorem 2 If (u, v) is not a topological solution, then there exists $R_0 > 0$ such that

$$f_i > 0, \; i = 1, 2 \text{ for } r > R_0.$$

In this theorem, we establish the apriori bound for not topological

solutions:

$$\begin{cases} u(r) < \log \frac{1}{1+a} & \text{if} \quad v(r) \leq u(r) \\ v(r) < \log \frac{1}{1+b} & \text{if} \quad u(r) \leq v(r) \end{cases} \tag{8}$$

for r large.
If these hold, then

$$f_1 = e^u - (1+a)e^{2u} + e^{u+v} > 0$$

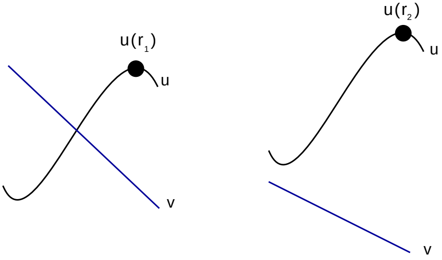
and

$$f_2 = e^v - (1+a)e^{2v} + e^{u+v} > 0$$

for r large.

Sketch of the Proof of Theorem 2

Step 1. We have the following local estimate.



Then $u(r_1),\ \ u(r_2) < \log \frac{1}{1+a}$.

Sketch of the Proof of Theorem 2

Step 2 Suppose $(u(r), v(r))$ satisfies either

$$u(r_0) \geq v(r_0),\ \ u_r(r_0) \geq v_r(r_0) \text{ and } (bu + (1+a)v)_r(r_0) \leq 0, \quad (9)$$

or

$$v(r_0) \geq u(r_0),\ \ v_r(r_0) \geq u_r(r_0) \text{ and } (av + (1+b)u)_r(r_0) \leq 0, \quad (10)$$

then there exists $R_0 > r_0$, such that

$$\begin{cases} u(r) < \log \frac{1}{1+a} & \text{if} \quad v(r) \leq u(r) \\ v(r) < \log \frac{1}{1+b} & \text{if} \quad u(r) \leq v(r) \end{cases}$$

Sketch of the Proof of Theorem 2

Step 3 Consider

$$r((1+2b)u+(1+2a)v)_r(r) \tag{11}$$

$$=2((1+2a)N_1+(1+2b)N_2)-(1+a+b)\int_0^r s(f_1+f_2)ds, \tag{12}$$

we know that either

$$r((1+2b)u+(1+2a)v)_r(r)>0 \text{ for } r\in(0,\infty),$$

or there is r_1 such that

$$r_1((1+2b)u+(1+2a)v)_r(r_1)=0 \text{ and } ((1+2b)u+(1+2a)v)_r>0 \text{ on } [0,r_1)$$

Sketch of the Proof of Theorem 2

Step 4 For the second case, there are three possibilities on the derivative of (u, v) at r_1. (Here, we assume that $u(r) > v(r)$ on some interval (r_1, r_1^*)):

(A) $u_r(r_1) = v_r(r_1) = 0$.

(B) $u_r(r_1) = -\left(\frac{1+2a}{1+2b}\right) v_r(r_1) > 0$.

(C) $u_r(r_1) = -\left(\frac{1+2a}{1+2b}\right) v_r(r_1) < 0$.

Sketch of the Proof of Theorem 2

Step 4 For the second case, there are three possibilities on the derivative of (u, v) at r_1. (Here, we assume that $u(r) > v(r)$ on some interval (r_1, r_1^*)):

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(C) $u_r(r_1) = -\left(\frac{1+2a}{1+2b}\right)v_r(r_1) < 0.$

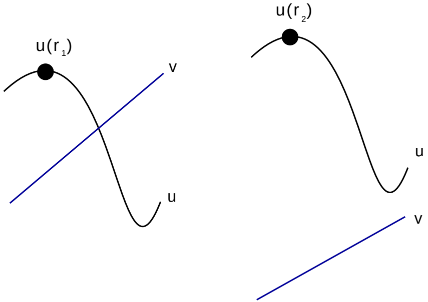
Recall the condition (9)

$$u(r_0) \geq v(r_0),\ \ u_r(r_0) \geq v_r(r_0) \text{ and } (bu + (1+a)v)_r(r_0) \leq 0$$

Sketch of the Proof of Theorem 3

Suppose $(1+2b)u_r(r) + (1+2a)v_r(r) > 0$ for $r > 0$

Step 1. We have the following local estimate



$$u(r_1), \;\; u(r_2) < \log \frac{1}{1+a}.$$

Sketch of the Proof of Theorem 3

Step 2. If $u(r)$ and $v(r)$ have infinitely many intersection points on $(0, \infty)$. Then we have either

$$v(r) \leq u(r) < \log \frac{1}{1+a}$$

or

$$u(r) \leq v(r) < \log \frac{1}{1+b}$$

for r large. But it will get a contradiction.

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for r large. But it will get a contradiction.

For convenience, let $a = b = 1$, integrating $\Delta(u+v) = -(f_1 + f_2)$,

$$\begin{aligned} r_0(u+v)_r(r_0) &> \int_{r_0}^{r} s(f_1 + f_2)ds \\ &= \int_{r_0}^{r} s(e^u - 2e^{2u} + 2e^{u+v} + e^v - 2e^{2v})ds \end{aligned}$$

Sketch of the Proof of Theorem 3

Step 2. If $u(r)$ and $v(r)$ have infinitely many intersection points on $(0, \infty)$. Then we have either

$$v(r) \leq u(r) < \log \frac{1}{1+a}$$

or

$$u(r) \leq v(r) < \log \frac{1}{1+b}$$

for r large. But it will get a contradiction.
For convenience, let $a = b = 1$, integrating $\Delta(u+v) = -(f_1 + f_2)$,

$$\begin{aligned} r_0(u+v)_r(r_0) &> \int_{r_0}^{r} s(f_1 + f_2) ds \\ &= \int_{r_0}^{r} s(e^u - 2e^{2u} + 2e^{u+v} + e^v - 2e^{2v}) ds \\ &> e^{(u+v)(r_0)} \int_{r_0}^{r} s ds \end{aligned}$$

Sketch of the Proof of Theorem 3

Step 3. Suppose $u > v$ for $r > r_0$. We consider the following possible cases:

- u oscillates on (r_0, ∞)
- u is decreasing for r large, which implies v is decreasing for r large.
- u is increasing for r large.

Asymptotic Behaviors

Corollary 4

(1) *If* (u, v) *is a topological solution, then* $(u, v) \to (0,)$ *exponentially fast.*

Asymptotic Behaviors

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(1) *If (u, v) is a topological solution, then $(u, v) \to (0,)$ exponentially fast.*

Any topological solution (u, v) near infinity satisfies

$$\begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} + M \begin{pmatrix} u \\ v \end{pmatrix} + \text{ higher order terms of } (u, v) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (13)$$

where $M = \begin{pmatrix} -(1+a)^2 - ab & a(2+a+b) \\ b(2+a+b) & -(1+b)^2 - ab \end{pmatrix}$. Let $-\lambda_1 < -\lambda_2 < 0$ be the eigenvalues of M. Then u and v decay as fast as $-r^{-\frac{1}{2}} e^{-\sqrt{\lambda_2} r}$.

Asymptotic Behaviors

(2) If $(u(r), v(r))$ is a non-topological solution, then

$$u(r) = -2\beta_1 \log r + O(1)$$

$$v(r) = -2\beta_2 \log r + O(1)$$

at ∞ for some $\beta_1 > 1$ and $\beta_2 > 1$. Thus,

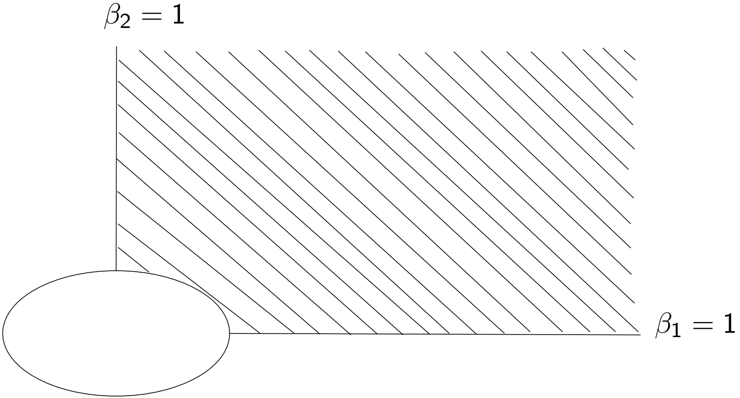
$$e^u, e^v \in L^1(\mathbb{R}^2).$$

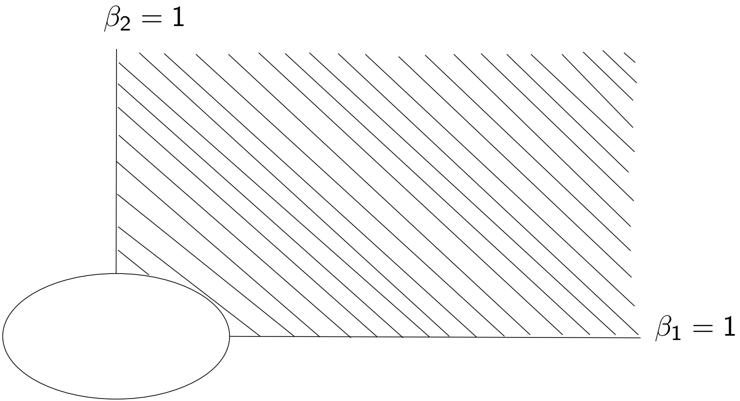
Furthermore,

$$\begin{aligned} & J(\beta_1 - 1, \beta_2 - 1) - J(N_1 + 1, N_2 + 1) \\ =& (1 + a + b) \int_0^\infty s\Big(\frac{(1+a)b}{2} e^{2u} + \frac{(1+b)a}{2} e^{2v} - abe^{(u+v)}\Big) ds > 0 \end{aligned}$$

where

$$J(x, y) = \frac{b(1+b)}{2} x^2 + abxy + \frac{a(1+a)}{2} y^2.$$


$\beta_2 = 1$
$\beta_1 = 1$

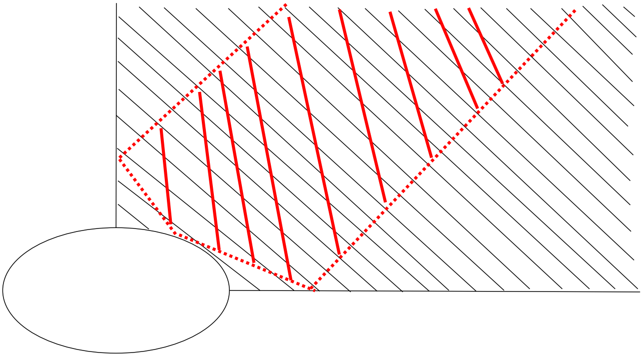


Question 2: Is this a sufficient condition for the existence of non-topological solutions subject to the boundary condition

$$u(r) = -2\beta_1 \log r + O(1),$$

$$v(r) = -2\beta_2 \log r + O(1),$$

as $r \to \infty$?



For the case of A_2, [Choe-Kim-Lin] use degree theory to show that for (β_1, β_2) in the red region: S, there exists radial solutions subject to

$$u(r) = -2\beta_1 \log r + O(1); \;\; v(r) = -2\beta_2 \log r + O(1) \text{ as } r \to \infty.$$

$$S \equiv \{(\alpha_1, \alpha_2) \mid -2N_1 - N_2 - 3 < \alpha_2 - \alpha_1 < 2N_2 + N_1 + 3, \quad 2\alpha_1 + \alpha_2 > N_1 + 2N_2 + 6, \;\; \alpha_1 + 2\alpha_2 > 2N_1 + N_2 + 6\}.$$

Asymptotic Behaviors

(3) $(u(r), v(r))$ is a mixed-type solution, then either

$$u(r) \to \log \frac{1}{1+a} \text{ and } v(r) = -2\beta \log r + O(1) \text{ for some } \beta > 1,$$

or

$$v(r) \to \log \frac{1}{1+b} \text{ and } u(r) = -2\beta \log r + O(1) \text{ for some } \beta > 1,$$

as $r \to \infty$.

Corollary 5

Suppose $(u(r), v(r))$ *be an entire radial solution. Then* u *and* v *have intersection finite times.*

Existence of Mixed-type Solution and Uniqueness of Topological Solution

We denote $\Big(u(r;\alpha_1,\alpha_2), v(r;\alpha_1,\alpha_2)\Big)$ be a radial solution of (5) with the initial value

$$\begin{cases} u(r) = 2N_1 \log r + \alpha_1 + o(1) \\ v(r) = 2N_2 \log r + \alpha_2 + o(1) \end{cases} \quad \text{as } r \to 0^+. \tag{14}$$

The region of initial data of the non-topological solutions of (5).

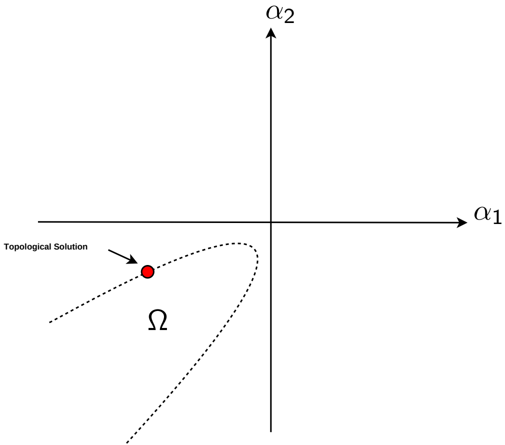
$$\Omega = \{(\alpha_1,\alpha_2) | (u(r;\alpha_1,\alpha_2), v(r;\alpha_1,\alpha_2)) \text{ is a non-topological solution of (5)}\}. \tag{15}$$

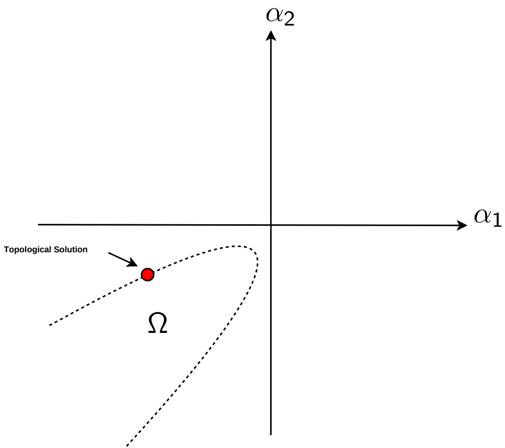
Theorem 6

Ω is an open set. Furthermore, if $\alpha = (\alpha_1, \alpha_2) \in \partial\Omega$, then $(u(r;\alpha), v(r;\alpha))$ is either a topological solution or a mixed-type solution.

Remark 3

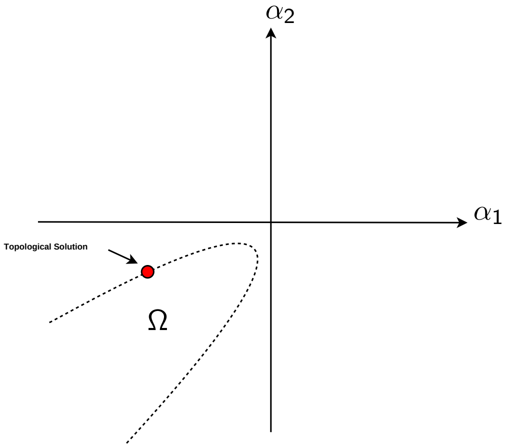
- $\Omega \neq \mathbb{R}^2$
- *For $N_1 = N_2$ and $u = v$, we know that $\Omega \neq \emptyset$.*
- *By the existence result of [Choe-Kim-Lin, 2013], we know $\Omega \neq \emptyset$ for the case of A_2. Hence, $\partial\Omega \neq \emptyset$.*


α_2
α_1
Topological Solution
Ω



Remark 4

When $N_1 = N_2 = 0$, we have the uniqueness of topological solutions.



Remark 4

When $N_1 = N_2 = 0$, we have the uniqueness of topological solutions.

Question 4: The structure of Ω: simply connected? $\partial\Omega$?

Discussion

1. The existence results for this system:

Topological	Non-Topological	Mixed-type
[Yang,CMP,1997]	1. [Ao-Wei-Lin, preprint]:A_2 and B_2 2. [Choe-Kim -Lin,preprint] A_2: for centain range of (β_1, β_2)	$N_1 = N_2 = 0$

Discussion

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2. The uniquness result for the system: When $N_1 = N_2 = 0$, there is unique topological solution $u(r) = v(r) = 0$ for $r \in [0, \infty)$

3. Classification of radial solutions of these cases:

- $a, b > 0$ doesn't hold.
-
$$K = \begin{pmatrix} 2 & -1 & & & & \mathbf{0} & \\ -1 & 2 & -1 & & & & \\ 0 & -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & & \\ & & & & -1 & 2 & -1 \\ & \mathbf{0} & & & & -1 & 2 \end{pmatrix}?$$

  Here, K is $SU(N+1)$ Cartan matrix.

Thank you!

Lozano-Marqués-Moreno-Schaposnik Model

$$
\begin{aligned}
&r\partial_r\phi = \frac{\epsilon}{N}\left(f - f^{N^2-1}\right)\phi,\ \ r\partial_r\phi_N = \frac{\epsilon}{N}\left(f + (N-1)f^{N^2-1}\right)\phi_N,\\
&r\partial_r f = \frac{1}{4N\kappa_1}\left[f_0\left((N-1)\phi^2 + \phi_N^2\right) + f_0^{N^2-1}(N-1)(\phi^2 - \phi_N^2)\right]\\
&r\partial_r f^{N^2-1} = \frac{1}{4N\kappa_2}\left[f_0(\phi^2 - \phi_N^2) + f_0^{N^2-1}(N-1)(\phi^2 + (N-1)\phi_N^2)\right]\\
&f_0 = \frac{\epsilon}{2\kappa_1}\left((N-1)\phi^2 + \phi_N^2 - N\right),\ \ f_0^{N^2-1} = \frac{\epsilon}{2\kappa_2}(\phi^2 - \phi_N^2),
\end{aligned}
\tag{16}
$$

where N is positive integer, $\epsilon = \pm 1$, and κ_1, $\kappa_2 > 0$.

Gudnason Model

$$
\begin{aligned}
\Delta U =& \frac{\alpha_*}{M^2}\left(\sum_{i=1}^{M}[e^{U+u_i}+e^{U-u_i}-2]\right)\left(\sum_{j=1}^{M}[e^{U+u_j}+e^{U-u_j}]\right)\\
&+\frac{\alpha_*\beta_*}{M}\sum_{i=1}^{M}\left(e^{U+u_i}-e^{U-u_i}\right)+4\pi\sum_{i=1}^{M}\sum_{s=1}^{n_i}\delta_{p_i,s}(x),\\
\Delta u_j =& \frac{\alpha_*\beta_*}{M}\left(\sum_{i=1}^{M}[e^{U+u_i}+e^{U-u_i}-2]\right)\left(e^{U+u_j}-e^{U-u_j}\right)\\
&+\beta_*^2(e^{2U+2u_j}-e^{2U-2u_j})+4\pi\sum_{s=1}^{n_j}\delta_{p_j,s}(x),\ j=1,\cdots,M,
\end{aligned}
\tag{17}
$$

where $\alpha_* > 0$ and $\beta_* > 0$ are constant and $\{p_j, s\}_{j=1,\cdots,M}^{s=1,\cdots,n_j} \in \mathbb{R}^2$.