Universality of random matrices, Dyson Brownian Motion and Quantum Unique Ergodicity

Horng-Tzer Yau
Harvard University

With P. Bourgade, L. Erdős, A. Knowles, B. Schlein, and J. Yin

Eugene Wigner (1956): **Successive energy levels of large nuclei have universal spacing statistics that can be described by random matrices.**

— a grand vision

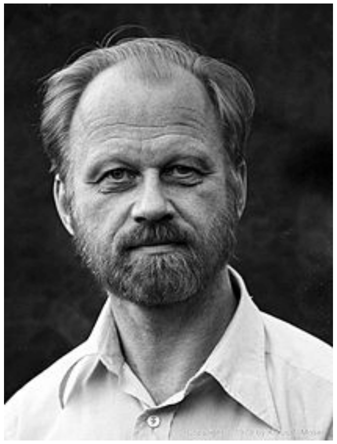
Freeman Dyson (1962): **A gas of particles with a logarithmic interaction reaches local equilibrium very fast.**

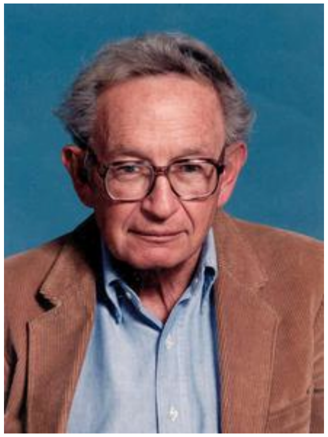
— a seminal idea

De Giorgi, Nash, Moser (1957-60):

Regularity theory for parabolic equations







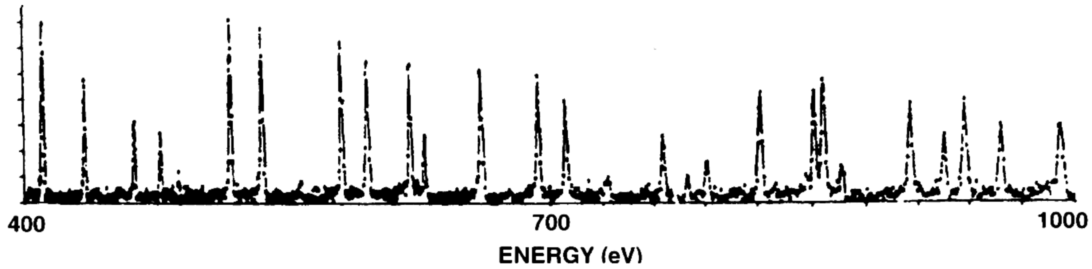
Quantum Chaos conjecture (1977-84)

and

Anderson Model (1958)

Rudnick-Sarnak (1991) **eigenfunctions on negative curved manifolds are "flat" (Quantum unique ergodicity).**

Experimental data for excitation spectra of heavy nuclei:

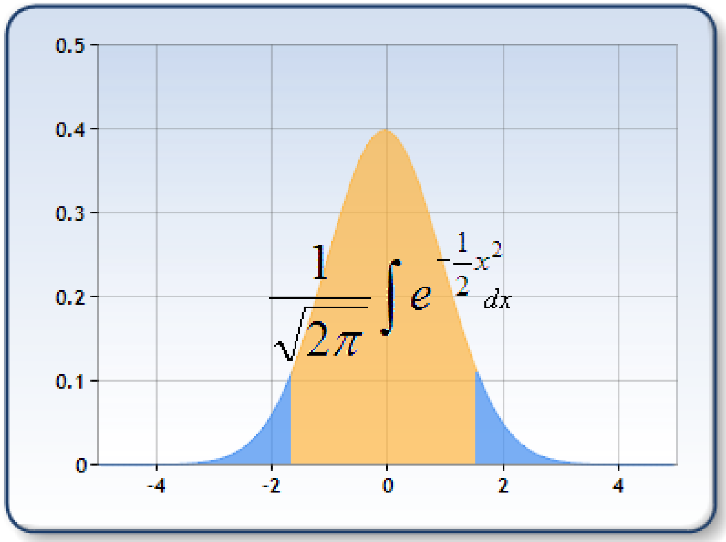


Wigner (1955): This pattern can be modeled by the spacing distribution of eigenvalues of random matrices.

> **Perhaps I am now too courageous when I try to guess the distribution of the distances between successive levels (of energies of heavy nuclei). Theoretically, the situation is quite simple if one attacks the problem in a simpleminded fashion. The question is simply what are the distances of the characteristic values of a symmetric matrix with random coefficients.** — E. Wigner

Gaussian Orthogonal Ensemble (GOE):

$$H = \begin{pmatrix} h_{11} & h_{12} & \dots & h_{1N} \\ h_{21} & h_{22} & \dots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{N1} & h_{N2} & \dots & h_{NN} \end{pmatrix}$$



$h_{jk} = h_{kj}$ (for $j < k$) are real independent normal random variables

$$\mathbb{E}h_{jk} = 0, \qquad \mathbb{E}|h_{jk}|^2 = \frac{1}{N}$$

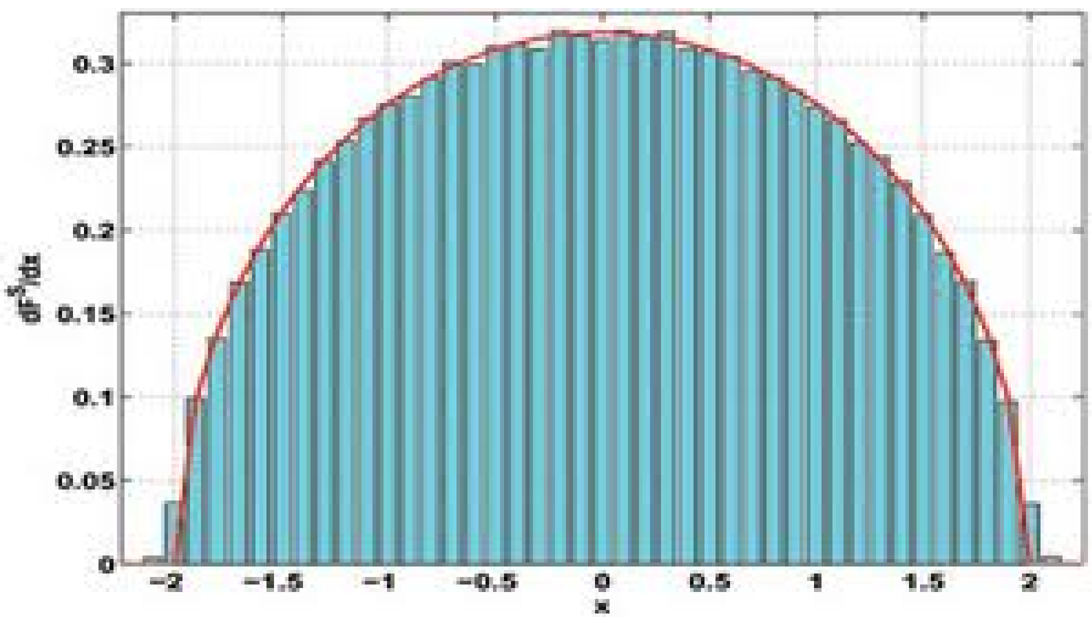
The eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ are of order one.

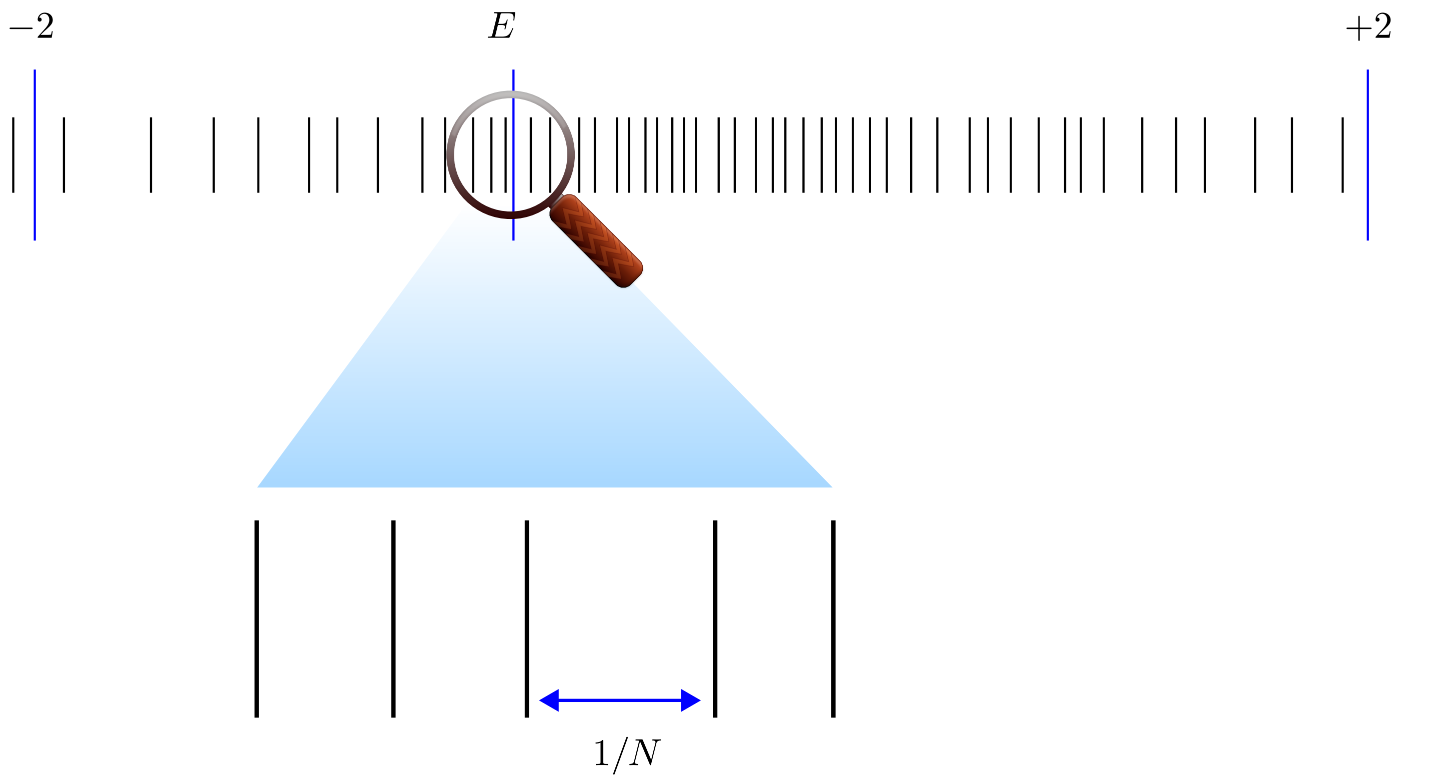
Also, Hermitian or quaternion self-dual (GUE, GSE) ensemble: classical ensembles by Dyson's classification.

Wigner ensembles: h_{ij} are just independent (not necessarily normal) distributions.

Wigner semicircle law: Density of eigenvalues

$$\rho(x) \sim \sqrt{4 - x^2}$$

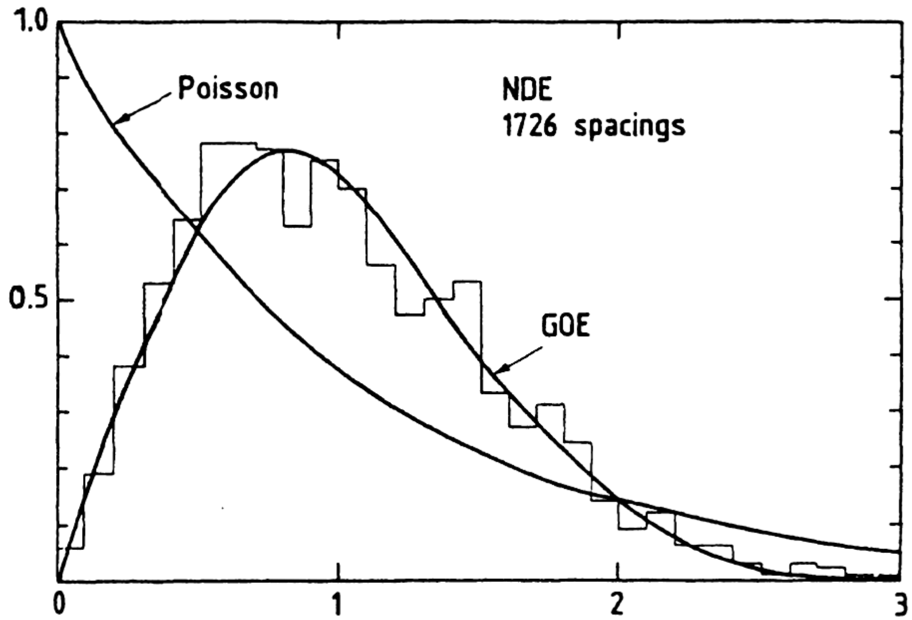




$$\mathbb{P}\left(\lambda_1 \sim E + \frac{x_1}{N}, \lambda_2 \sim E + \frac{x_2}{N}\right) \sim 1 - \left(\frac{\sin \pi(x_1 - x_2)}{\pi(x_1 - x_2)}\right)^2$$

Computation by Gaudin-Mehta and Dyson

By 1970 we had decided that random matrix theory was a beautiful piece of pure mathematics having nothing to do with physics. Random matrix theory went temporarily to sleep. — Dyson



Nearest-neighbor spacing distribution for the "Nuclear Data Ensemble" comprising 1726 spacings histogram (1983)

Is this exact computation for Gaussian models valid for general systems?

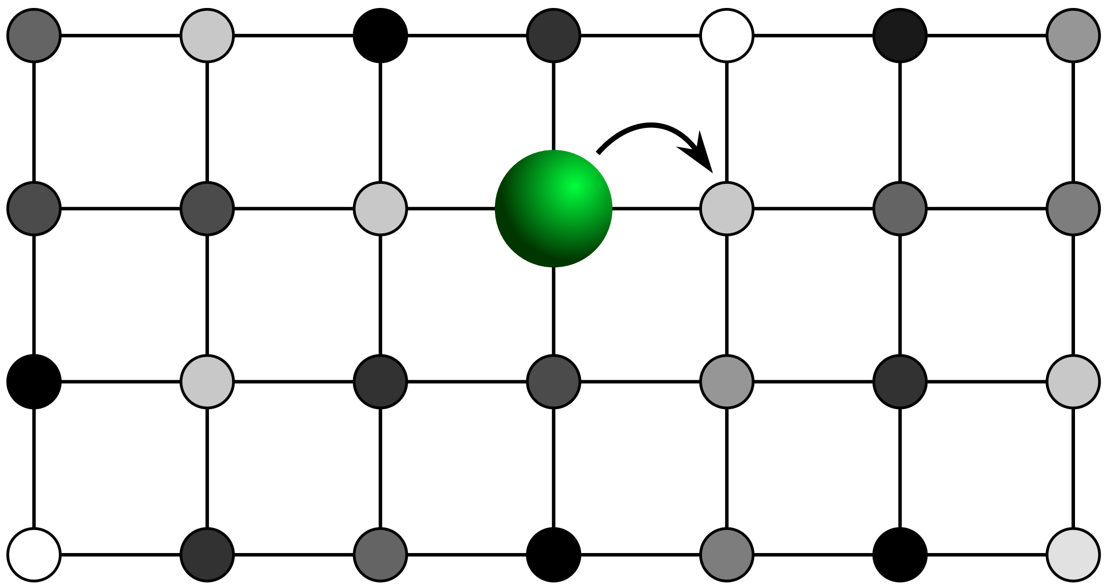
Fundamental belief of universality: Random matrix statistics are a new class of universal laws for highly correlated systems.

A simple manifestation **(Wigner-Dyson-Mehta conjecture)** : If h_{ij} are independent, then the local eigenvalue statistics are the same as those of the Gaussian ensembles.

Only symmetry types of the ensembles matter.

Quantum Chaos and Anderson Model

Anderson (1958): V_ω random potential on $\mathbb{R}^d$ or $\mathbb{Z}^d$.
$H = -\Delta + \lambda V_\omega$. Local eigenvalue statistics:



Depending on λ and d, there are two distinct regimes.

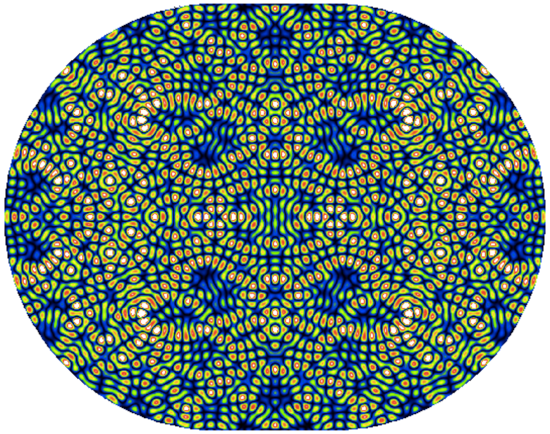
GOE in the delocalization regime (e.g., small λ in 3 dimension)

Poisson statistics (Minami) in the localization regime (Frohlich-Spencer, Aizenman).

Similarly for $-\Delta$ on a domain with chaotic classical dynamics.
Quantum Chaos conjecture: Bohigas-Giannoni-Schmit (1984)

For integrable dynamics: Berry-Tabor (1977)

Quantum unique ergodicity conjecture (QUE) [Rudnick-Sarnak]:



$(\psi_k)_{k\geq 1}$ orthogonal eigenfunctions of M with negative curvature:

$$\int a(x)|\psi_j(x)|^2 \mathrm{dVol}(x) \to \int a(x) \mathrm{dVol}(x)$$

for any large energy limit.

Quantum Ergodicity Theorem: Averaged version of QUE [Snirelman, Colin de Verdiere, Zelditch] Anantharaman. Arithmetic QUE (Luo-Sarnak, Bourgain, Lindenstrauss, Soundararajan)

Question: Does QUE hold for Wigner matrices? Yes for GUE/GOE because the eigenvectors are uniformly distributed on $U(N)/O(N)$.

Quantum (unique) ergodicity/delocalization
$\Longleftrightarrow$ random matrix statistics ?

Two basic mathematical questions:

1. Universality conjecture for Wigner matrices

2. QUE for Wigner matrices.

Key ingredient in solving these problems: **Dyson Brownian motion**.

Generalized Wigner Ensembles

$$H = (h_{ij})_{1 \le i,j \le N}, \quad \bar{h}_{ji} = h_{ij} \quad \text{independent}$$

$$\mathbb{E} h_{ij} = 0, \quad \mathbb{E}|h_{ij}|^2 = s_{ij}, \qquad \sum_i s_{ij} = 1, \frac{c}{N} \le s_{ij} \le \frac{C}{N}$$

If h_{ij} are i.i.d. then it is called Wigner ensemble.

Solution to the Universality conjecture

Theorem [Erdos-Schlein-Y-Yin, 2009-2010] Local eigenvalue statistics are universal for generalized Wigner ensembles. in the bulk (and edge). Matrices with Bernoulli entries with varying variances are included. Assumption on the distribution of matrix elements: $4+\varepsilon$ moments are uniformly bounded.

[Tao-Vu, 2009-10] Wigner matrices with four moments matching those of $N(0,1)$. Hermitian case via combination with Johansson's result.

Extensions to sparse matrices, β-ensembles, single gap universality, the edge universality [Bourgade, Erdos, Knowles, Yin]

Theorem [Bourgade-Y, 2013] [Probabilistic local QUE]
$u_i, i = 1, \ldots, N$: normalized e-vectors of generalized Wigner matrices. $\mathscr{N}_1, \mathscr{N}_2$: independent normal distribution $N(0,1)$. Then for any j, k in the bulk and $\mathbf{q}$ a unit vector in $\mathbb{R}^N$, we have

$$\sqrt{N}\Big(|\langle \mathbf{q}, u_j \rangle|,\ |\langle \mathbf{q}, u_k \rangle| \Big) \to \Big(|\mathscr{N}_1|,\ |\mathscr{N}_2| \Big)$$

For any k in the bulk, the eigenvector u_k is flat with high probability, i.e., probabilistic local QUE holds.

There is ε such that for any $\delta > 0$ and any k in the bulk and $A \subset [\![1, \cdots, N]\!]$ with $|A| \geq N^{C\varepsilon}$ we have,

$$\mathbb{P}\left(\Big| \frac{1}{|A|} \sum_{a \in A} \Big(N|u_k(a)|^2 - 1 \Big) \Big| > \delta \right) \leq C_\delta N^{-\varepsilon}.$$

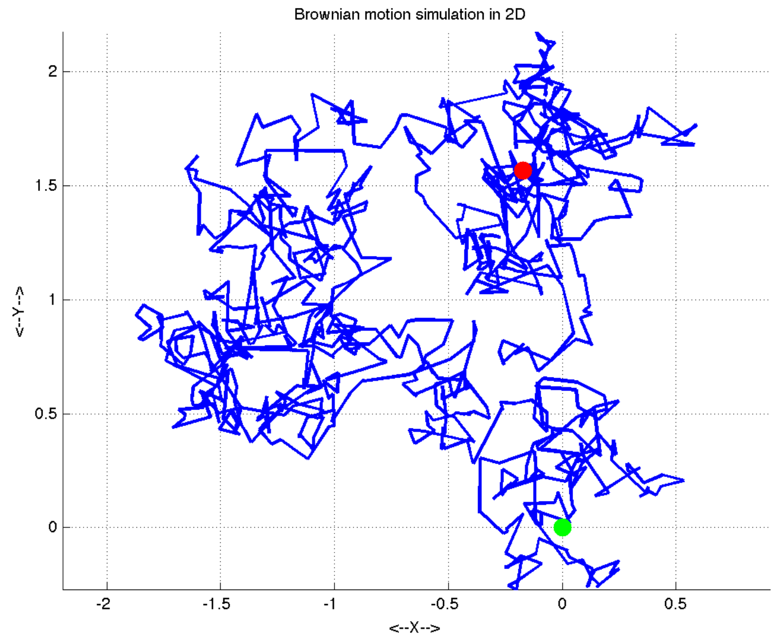
Delocalization [Erdos-Schlein-Y]
Edge QUE for Wigner ensembles [Knowles-Yin].

Matrix Dyson Brownian Motion (Matrix-DBM)

Evolve the matrix with a matrix Ornstein-Uhlenbeck process :

$$\mathrm{d}H_t = \frac{1}{\sqrt{N}}\mathrm{d}B_t - \frac{1}{2}H_t\mathrm{d}t$$

The distribution of $H_t \sim e^{-t/2}H_0 + \sqrt{1-e^{-t}}\,V$ where V is a GOE.



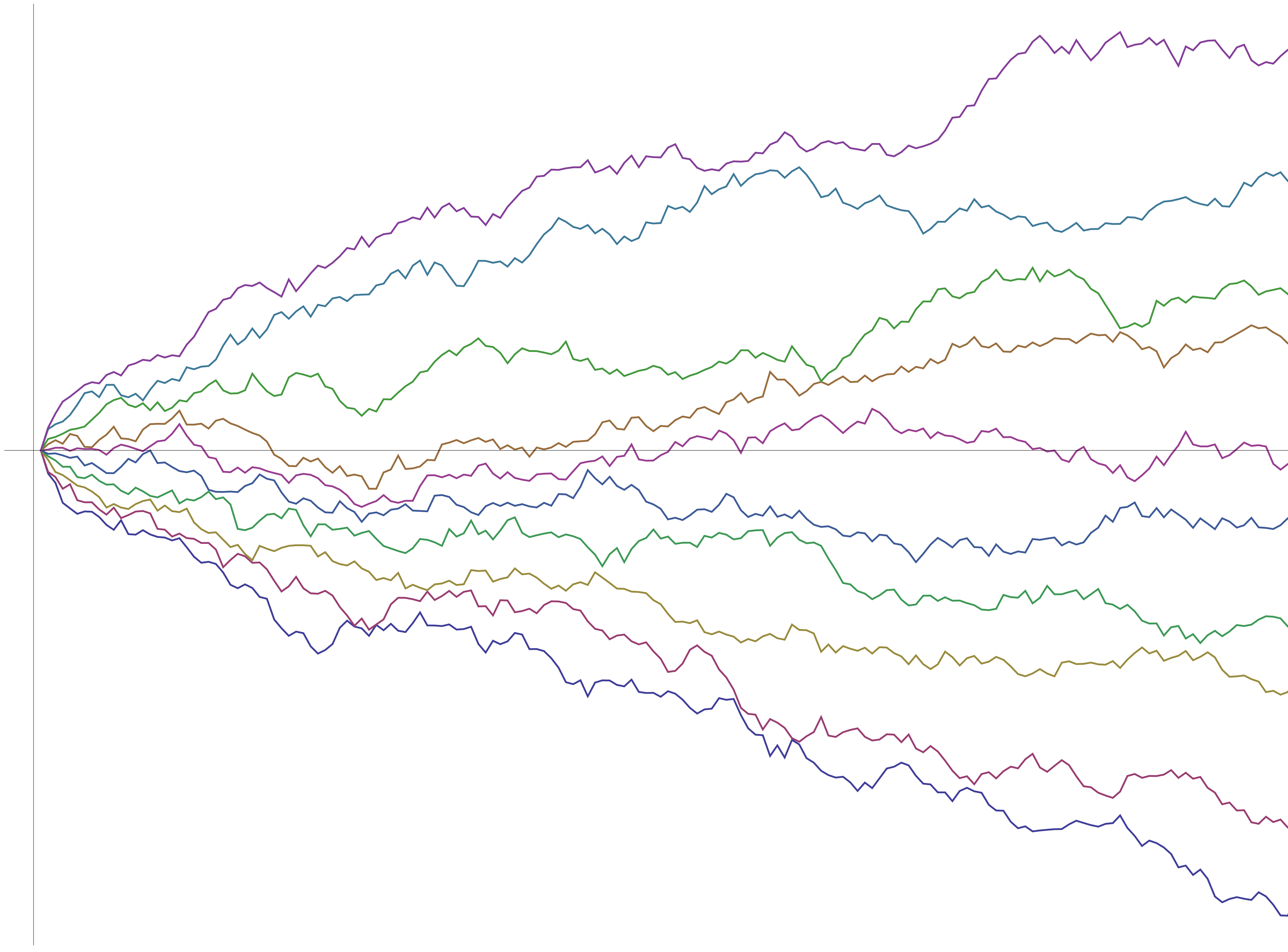
B_{ij}: symmetric independent Brownian motions

$$\mathrm{d}\lambda_k = \frac{\mathrm{d}B_{kk}}{\sqrt{N}} + \left(\frac{1}{N} \sum_{\ell \neq k} \frac{1}{\lambda_k - \lambda_\ell} - \frac{1}{2}\lambda_k \right) \mathrm{d}t$$

Dyson Brownian Motion (1962) E-value equations are autonomous!

$$\mathrm{d}u_k = \frac{1}{\sqrt{N}} \sum_{\ell \neq k} \frac{u_\ell \, \mathrm{d}B_{k\ell}}{\lambda_k - \lambda_\ell} - \frac{1}{2N} \sum_{\ell \neq k} \frac{u_k \, \mathrm{d}t}{(\lambda_k - \lambda_\ell)^2}$$

Dyson e-vector flow E-vector equations depend on the e-values.



Dyson: The classical Coulomb gas is invariant under the DBM:

$$\mu_\beta \sim e^{-\beta N \mathcal{H}(\lambda)}, \quad \mathcal{H} = \sum_i \frac{\lambda_i^2}{4} - \frac{1}{N} \sum_{i<j} \log(\lambda_j - \lambda_i)$$

prob. density for the classical ensembles with $\beta = 1$ for the GOE

Dyson's conjecture: Time to "local equilibrium" for DBM is $\gtrsim N^{-1}$.

Erdős-Schlein-Y-Yin, 2009-10: **Dyson conjecture holds for $\beta > 0$.**

The Coulomb interactions drive the system locally very fast!

$\Longrightarrow$ Wigner-Dyson-Mehta conjecture holds for H_t with $t \gtrsim N^{-1}$.

Question: How to connect to $t = 0$?

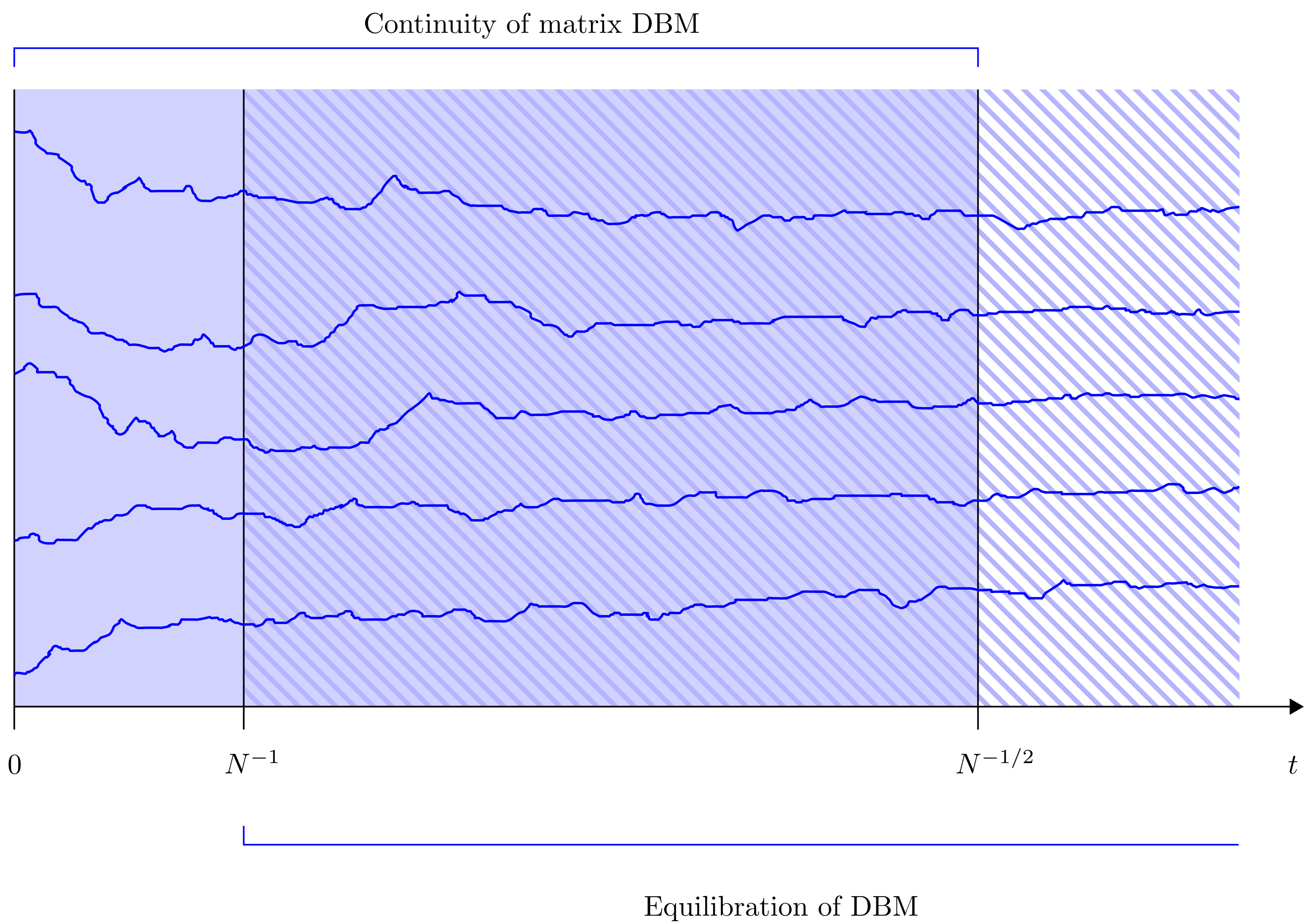
Theorem* [Continuity of matrix-DBM]

F: function of matrix elements with uniformly bounded 3rd derivatives. If $t \ll N^{-1/2}$ then

$$\mathbb{E}F\Big(H_t\Big) - \mathbb{E}F\Big(H_0\Big) \to 0$$

Proof: By Ito's formula and local semicircle laws.

* [Bourgade-Y 2013 (also a special case of Erdos-Knowles-Y-Yin)]



No contradiction since we use the matrix structure.

[Bourgade-Y 2013] For any unit vector $\mathbf{q}$ fixed, define

$$f(t,j) = E\Big[|\mathbf{q}\cdot u_j(t)|^2\Big|\boldsymbol{\lambda}(\cdot)\Big].$$

The **eigenvector moment flow** (in the rescaled time $t \to Nt$)

$$\partial_t f(t,j) = \sum_{k\neq j} \frac{f(t,k) - f(t,j)}{(\lambda_j - \lambda_k)^2(t)}$$

(non-local) Random walk in random environments with singular coefficients.

A very simple equation in the indices j.

There are analogues for higher moments.

Theorem [Bourgade-Erdos-Y 2012-13] With high probability, for any $|i-j| \ll t$,

$$|f(t,k) - f(t,j)| \leq C\|f(t)\|_\infty \, t^{-\varepsilon}$$

ε is a Hölder regularity exponent from De Giorgi-Nash-Moser (Caffarelli-Chan-Vasseur) theory!

$\Longrightarrow$ local QUE holds for H_t; $\quad |\mathbf{q} \cdot u_j(t)|^2 \to N(0,1)^2$.

How can one get Gaussian by “regularity"?

By continuity of matrix-DBM $\Longrightarrow$ local QUE holds for H_0.

The mechanism underlying of the universality and QUE for random matrices: **Dyson Brownian motion (A dynamical idea)**

Universality: *Fast relaxation of DBM*

+ continuity of matrix-DBM

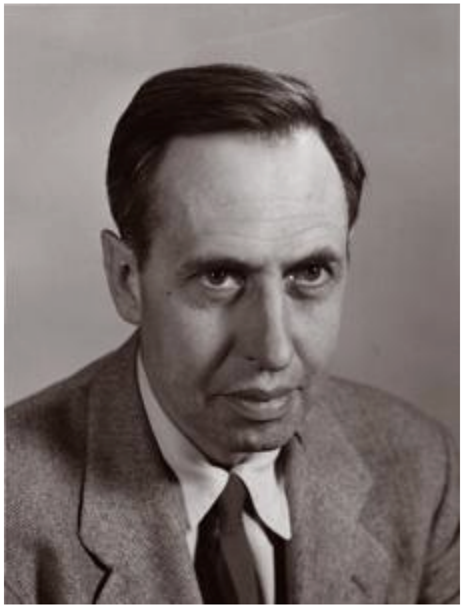
QUE: *Holder regularity of e-vector moment flow*

+ continuity of matrix-DBM

- Dynamical idea to solve time independent problems: *study the limits of the flow as* $t \to \infty$.

- Dyson Brownian Motion: *study the initial layer to understand* $t = 0$.

If you admit that the Wigner ensemble gives a completely wrong answer for the level density, why do you believe any of the other predictions of random-matrix theory?



George Uhlenbeck

- Local theory for universality was developed. A general theory ...