On The Derivation of Multicomponents flows Systems

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France-Taiwan joint conference on Partial Differential Equations
Thanks for the invitation to H. Berestycki, J.G. Guo,
F. Hamel, C.S. Lin, B. Perthame, H.T. Yau

Outline

1. Some multi-fluid systems;
2. Multi-fluid model as limit of mono-fluid model.

Joint works with: X. HUANG (Academia Sinica, Beijing),
M. HILLAIRET (ICJ Lyon, INRIA Rocquencourt).

A model with an algebraic closure (common pressure)

$$
\begin{aligned}
\alpha_+ + \alpha_- &= 1,\\
\partial_t(\alpha^+\rho^+) + \operatorname{div}(\alpha^+\rho^+ u) &= 0,\\
\partial_t\rho + \operatorname{div}(\rho u) &= 0,\\
\partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \nabla P &= 0,\\
P = P_-(\rho_-) = P_+(\rho_+),&
\end{aligned}
$$

with

$$0 \le \alpha_\pm \le 1$$

and

$$\rho = \alpha^+\rho^+ + \alpha^-\rho^-.$$

See for instance M. Ishii (1975), D.A. Drew and S.L. Passman (1998).
See Benoît Desjardins's talk for more general systems.

A model with a PDE closure (equation on fraction)

$$\begin{aligned}
\alpha_+ + \alpha_- &= 1, \\
\partial_t \alpha^+ + u \cdot \nabla \alpha^+ &= \frac{1}{\lambda_P}(P^+ - P^-), \\
\partial_t(\alpha^+\rho^+) + \operatorname{div}(\alpha^+\rho^+ u) &= 0, \\
\partial_t \rho + \operatorname{div}(\rho u) &= 0, \\
\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P &= 0,
\end{aligned}$$

with

$$0 \le \alpha_\pm \le 1$$

and

$$\rho = \alpha^+\rho^+ + \alpha^-\rho^-, \qquad P = \alpha^- P^- + \alpha^+ P^+$$

A model with a PDE closure (equation on fraction)

If $\lambda_P \to 0$: See for instance: R. ABGRALL, C. BERTHON, F. COQUEL, S. DELLACHERIE, D.A. DREW and S.L. PASSMAN, Th. GALLOUËT, M. ISHII, Ph. LE FLOCH, R. SAUREL and others for modeling and numerics.

For numerical simulation of wave breaking with two phase flow model: see papers by the group in Toulon 2005, 2011 (with P. FRAUNIE, F. GOLAY) .

For relaxation limit on the first model: See works by F. DIAS, D. DUTYKH and J.–M. GHIDAGLIA (2010) on a two-fluid model for violent aerated flows.

A model with a PDE closure (equation on fraction)

- Viscous multi-fluid model as limit of viscous mono-fluid model: (One-velocity field), see [1] an [2].

[1] D.B., X. HUANG. A Multi-Fluid Compressible System as the Limit of Weak-Solutions of the Isentropic Compressible Navier-Stokes Equations. *Arch. Rational Mech. Anal.* 201 (2), 647680, (2011). .

[2] D.B., M. HILLAIRET. Note on the derivation of multicomponent flow systems. Submitted (2013).

Global weak solutions for isentropic NS equations

GLOBAL WEAK SOLUTIONS FOR ISENTROPIC NS EQUATIONS: SOME RECALLS

The model

Let us consider the following barotropic compressible Navier-Stokes equations:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu \triangle u - (\mu + \lambda)\nabla(\mathrm{div} u) + \nabla P(\rho) = 0, \end{cases}$$

where ρ, u, P denote the density, velocity and pressure respectively. The pressure law is given by

$$P(\rho) = a\rho^\gamma \quad (a > 0, \quad \gamma > d/2),$$

d the space dimension, μ and λ are the shear viscosity and the bulk viscosity coefficients respectively. They satisfy the following physical restrictions:

$$\mu > 0, \qquad \lambda + \frac{2}{3}\mu \geq 0.$$

Sketch of proof

Energy estimates give

$$\|\rho_n\|_{L^\infty(0,T;L^\gamma(\Omega))} + \|u_n\|_{L^2(0,T;(H_0^1(\Omega))^3)} + \|\sqrt{\rho_n}u_n\|_{L^\infty(0,T;(L^2(\Omega))^3)} \leq C$$

therefore

$$\rho_n \to \rho \text{ in } L^\infty(0,T;L^\gamma(\Omega)) * \text{ weak}, \qquad u_n \to u \text{ in } L^2(0,T;(L^2(\Omega))^3 \text{ weak}$$

Thus convergence OK in each terms except the pressure one $p_n = a\rho_n^\gamma$.

To prove convergence to $p(\rho) = a\rho^\gamma$,

$$\rho_n \to \rho \text{ in } L^1_{\text{loc}}(Q)?$$

More integrability on pressure

To get rid of measure, extra information on ρ_n. In fact we test against

$$\varphi = \mathcal{B}\big(\rho_n^\theta - (\int_\Omega \rho_n^\theta)\big)/|\Omega|$$

where $\mathcal{B}$ is the Bogovskii operator on Ω and $0 < \theta < \gamma$. We show

$$\int_0^T \int_\Omega \rho_n^{\gamma+\theta} \leq c(T,\Omega,E_0), \qquad \theta = \frac{2}{d}\gamma - 1.$$

Remark. Bogovski operator applied to momentum eqs formally as divergence of the momentum equation

$$\Delta(-(2\mu+\lambda)\mathrm{div}u + a\rho^\gamma) = \mathrm{div}(\rho\dot{u})$$

where $\dot{u} = \partial_t u + u\cdot\nabla u$ and then $\rho^\theta\Delta^{-1}$ applied to the resulting equation.

Convergence

We get, using similar tools than in the incompressible setting,

$$\rho_n \to \rho \text{ in } \mathcal{C}^0([0,T]; L^\gamma_{\text{weak}}(\Omega)),$$

$$\rho_n^\gamma \to \overline{\rho^\gamma} \text{ in } L^{(\gamma+\theta)/\gamma}(Q_T) \text{ weak},$$

$$\rho_n u_n \to \rho u \text{ in } \mathcal{C}^0([0,T]; (L^{2\gamma/(\gamma+1)}_{\text{weak}}(\Omega))^3)$$

$$\rho_n u_n^i u_n^j \to \rho u^i u^j \text{ in } \mathcal{D}'(Q_T).$$

Convergence

Passing to the limit, we get

$$\partial_t \rho + \mathrm{div}(\rho u) = 0 \text{ in } \mathcal{D}'(Q_T),$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu)\nabla \mathrm{div} u + a \nabla \overline{\rho^\gamma} = 0 \text{ in } (\mathcal{D}'(Q_T))^3.$$

Difficulty: $\overline{\rho^\gamma} = \rho^\gamma, \qquad a.e.$

$\Longrightarrow$ Compactness on $\{\rho_n\}_{n \in N^*}$?

Key ingredients: Effective flux G property where $G = a\rho_n^\gamma - (2\mu + \lambda)\mathrm{div} u_n$, renormalization techniques, monotonicity and strict convexity.

Idea – Some weak compactness

The sequence $\{a\rho_n^\gamma - (2\mu+\lambda)\mathrm{div} u_n\}_{n\in N^*}$ has some weak-compactness property.

P.–L. Lions: For all function $b \in \mathcal{C}^1([0,\infty))$ satisfying some conditions at infinity, we get

$$\lim_{n\to\infty}\int_0^T\int_\Omega (a\rho_n^\gamma - (2\mu+\lambda)\mathrm{div} u_n) b(\rho_n)\varphi = \int_0^T\int_\Omega (a\overline{\rho^\gamma} - (2\mu+\lambda)\mathrm{div} u)\overline{b(\rho)}\varphi$$

for all $\varphi \in \mathcal{D}(Q_T)$.

E. Feireisl: Truncation.

Defect measures equality

P.–L. Lions: $b(s) = s$

P.–L. Lions: From estimates on ρ_n, we have

$$\rho_n \in L^2(Q_T) \text{ if } \gamma \geq 9/5 \text{ for } d = 3.$$

Thus renormalized theorem (DI-PERNA-LIONS) is OK

$$\partial_t b(\rho) + \text{div}(b(\rho)u) + (\rho b'(\rho) - b(\rho))\text{div}u = 0.$$

If $b(s) = s \ln s$ then the momentum equation with weak-compactness gives

$$\partial_t(\overline{\rho \ln \rho} - \rho \ln \rho) + \text{div}((\overline{\rho \ln \rho} - \rho \ln \rho)u) = \frac{a}{2\mu + \lambda}(\overline{\rho^\gamma}\rho - \overline{\rho^{\gamma+1}}) \text{ in } \mathcal{D}'(Q_T).$$

Remark: E. Feireisl improve to $\gamma > d/2$.

Use of monotonicity and strict-convexity

Integration + monotonicity of $s \mapsto as^\gamma$ and strict-convexity of $s \mapsto s \ln s$ with $s \geq 0$ implies

$$\overline{\rho \ln \rho} = \rho \ln \rho \text{ a.e. in } Q_T.$$

$\Longrightarrow$ weak convergence commutes with strictly convex function
$\Longrightarrow$ strong convergence in $L^1(Q_T)$ If strong convergence in L^1 of ρ_0^ε.

Use integrability in $L^{\gamma+\theta}$ with $\gamma > d/2$ to get strong convergence of $\{\rho_n\}_{n \in N}$ in $L^\gamma(Q_T)$.

Use of monotonicity and strict-convexity

If no strong convergence in L^1 of initial density sequence
$\Longrightarrow$ Oscillation–concentration
$\Longrightarrow$ More general system.

Weak limit and multi-fluid system justification

YOUNG MEASURE AND WEAK SOLUTIONS

Collaboration with X. HUANG: paper [1]

Collaboration with M. HILLAIRET: paper [2]

The model

Let us consider the following barotropic compressible Navier-Stokes equations:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu \triangle u - (\mu + \lambda)\nabla(\mathrm{div} u) + \nabla P(\rho) = 0, \end{cases}$$

where ρ, u, P denote the density, velocity and pressure respectively. The pressure law is given by

$$P(\rho) = a\rho^\gamma \quad (a > 0, \quad \gamma > 1),$$

μ and λ are the shear viscosity and the bulk viscosity coefficients respectively. They satisfy the following physical restrictions:

$$\mu > 0, \qquad \lambda + \frac{2}{3}\mu \geq 0.$$

Remarks and references

Question: Justification of multi-fluid system from mono-fluid one with low regularity. More precisely is there exist weak sequences corresponding to concentrating density which converge to the strong solution of a viscous multi-fluid system?
No oscillations-concentrations in velocity – concentrations in density.

As mentioned in LIONS's book (Remarks 5.8 and 5.9), weak limits of a sequence of solutions of compressible Navier–Stokes system with highly-oscillating density are not in general solutions of the compressible Navier-Stokes system.

References:

- D. SERRE (*Physica D*, (1991)) focusing on the one-dimensional case and providing a formal calculus for the multi-dimensional problem.
- To capture the effect of oscillations, using the renormalization procedure related to the mass equation, M. HILLAIRET (*J. Math Fluid Mech*, 2007) (following the formal calculus in D. SERRE) introduced Young measure as in the work by R. DI PERNA, A. MAJDA to describe a "homogenized system" satisfied in the limit.

Young Measures

Introduced by L. TARTAR (1979–1983–1986).

Let $\{u^\varepsilon\}$ be a sequence of mappings from $O \subset R^d$ to R^m such that

$$|u^\varepsilon(x)| \leq \text{ uniformly in } x \in \Omega \text{ and } \varepsilon > 0$$

Then there exists a subsequence $\{u^\varepsilon\}$ and a weakly measurable mapping, called Young measures, $x \mapsto \nu_x$ from Ω to prob$[\mathcal{M}(R^M)]$ such that

$$u^\varepsilon \to u * \text{weak in } L^\infty(\Omega)$$

for some $u \in L^\infty(\Omega)$ and

$$f(u^\varepsilon) \to < \nu_x, f >= \int_{R^M} f(\lambda) d\nu_x(\lambda) * \text{ weak in } L^\infty(\Omega)$$

$$\lim_{\varepsilon \to 0} \int \varphi(x) f(u_\varepsilon(x)) dx = \int \varphi(x) < \nu_x, f > dx \text{ for all } \varphi \in \mathcal{C}_0(\Omega).$$

Moreover $u^\varepsilon \to u$ (strongly) in $L^p(\Omega)$ $(p < +\infty)$ iff $\nu_x = \delta_{u(x)}$ almost everywhere.

A simple example

Let u be a d-periodic piecewise continous function in R defined as follows:

Let $\{\theta_j\}_{j=0}^{L}$ be a sequence of points in $[0,d]$ with $\theta_0 = 0$, $\theta_j < \theta_{j+1}$, $\theta_L = d$ and let $\{u_j\}_{j=0}^{L}$ be a corresponding family of numbers.

Define

$$u(x) = u_j \text{ for } \theta_{j-1} \leq x < \theta_j, \qquad nj = 1, \cdots, L$$

and set

$$u^{\varepsilon} = u(x/\varepsilon)$$

Then

$$\nu_x = \sum_{j=1}^{L} \alpha_j \delta_{u_j} \text{ almost everywhere } x \in [0,d].$$

Open and known results

In M. HILLAIRET's paper, we still do not know whether the obtained solution of the multi-fluid system are weak limit to finite-energy weak solutions of compressible Navier-Stokes equations.

Two assumptions, by M. HILLAIRET (2007), have been done to formally deduce the multi-fluid system from the weak limit system:

- If the initial young measures are linear combination of m Dirac masses then it is the case for all time.
- The concentration points remains distincts (a kind of stratification): $\rho_i(t,x) \neq \rho_j(t,x)$ for all $i,j = 1,\dots,m$ with $i \neq j$.

Remark: Existence and uniqueness of local strong solution of the viscous multi-fluid system has been established by M. HILLAIRET far from vacuum.

Weak-compactness of the effective flux

Important Lemma (due to P.–L. LIONS):

Given $b \in \mathcal{C}^1(\mathbb{R}^+)$ such that $b'(z) = 0$ for z sufficiently large with compact support, let $b((\rho_n)$ converge to $\overline{b}$ in $L^\infty(Q_T)$ endowed with its weak star topology. We have:

$$\lim_{n\to+\infty} \int_0^T \int_\Omega [(p(\rho_n) - (\lambda + 2\mu)\mathrm{div}(u_n))b(\rho_n))\phi(t,x)\,dxdt$$

$$= \int_0^T \int_\Omega [(q - (\lambda + 2\mu)\mathrm{div}(u))\overline{b})\phi(t,x)\,dxdt$$

for all $\phi \in \mathcal{D}(Q_T)$.

The limit system

Assume (ρ_n, u_n) be finite-energy weak solutions to NS eqs and

$$\rho_n \rightharpoonup \rho \quad \text{in } L^\infty(0,T;L^\gamma(\mathbb{T}^3)) \star \text{ weak}, \qquad u_n \rightharpoonup u \quad \text{in } L^2(0,T;H^1(\mathbb{T}^3)),$$
$$\rho \in L^\infty(0,T;L^\gamma(\mathbb{T}^3)), \qquad u \in L^2(0,T;H^1(\mathbb{T}^3)).$$

Then there exists a measurable family of probability measures, we denote $(\nu_{(t,x)})$ such that

1. We have
$$< \nu, \mathrm{Id} >= \rho \text{ and } < \nu, p >= q, \text{ in a sense precised in next slide.}$$
2. For all $b \in C(R^+)$, smooth, with compact support,
$$(< \nu, b >)_t + div(< \nu, b > u) + < \nu, (\mathrm{Id}\, b' - b) > div(u)$$
$$= \frac{< \nu, (\mathrm{Id}\, b' - b) > q - < \nu, (\mathrm{Id}\, b' - b) p >}{\lambda + 2\mu}.$$
3. Finally,
$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu \triangle u - (\mu + \lambda)\nabla(\mathrm{div} u) + \nabla q = 0, \end{cases}$$

The multi-fluid model studied by M. Hillairet – Strong solution and uniqueness

If Young measures are assumed to be convex combinations of Dirac measures, *i.e*:

$$\nu_{(t,x)} = \sum_{i=1}^{m} \alpha_{i1}(t,x)\delta_{\rho_{i1}(t,x)}, \qquad \forall (t,x) \in (0,T) \times \Omega.$$

Using such hypothesis, the homogeneized compressible Navier-Stokes system reads:

$$\begin{cases} (\alpha_{i1})_t + u_1 \cdot \nabla \alpha_{i1} = f_{\alpha_{i1}}, \qquad i = 1, \ldots, m \\ \alpha_{i1}\left((\rho_{i1})_t + div(\rho_{i1} u_1)\right) = \alpha_{i1} f_{\rho_{i1}}, \qquad i = 1, \ldots, m \\ \partial_t \rho + \mathrm{div}(\rho u_1) = 0, \\ \partial_t(\rho u_1) + \mathrm{div}(\rho u_1 \otimes u_1) + \nabla q = \mu \triangle u_1 + (\mu + \lambda)\nabla(\mathrm{div} u_1), \end{cases}$$

$$f_{\alpha_{i1}} = \frac{\alpha_{i1}(a\rho_{i1}^\gamma - q)}{\lambda + 2\mu}, \quad f_{\rho_{i1}} = \frac{\rho_{i1}(q - a\rho_{i1}^\gamma)}{\lambda + 2\mu}$$

$$0 \le \alpha_{i1}, \qquad \sum_{i=1}^{m} \alpha_{i1} = 1$$

$$\rho = \sum_{i=1}^{m} \alpha_{i1}\rho_{i1}, \qquad q = a\sum_{i=1}^{m} \alpha_{i1}\rho_{i1}^\gamma,$$

where ρ_{i1}, u_1 denotes the density, velocity respectively and α_{i1} is the coefficients.

The multi-fluid model studied by M. Hillairet – Strong solution and uniqueness

Far from vacuum, kill the red terms:

$$\begin{cases} (\alpha_{i1})_t + u_1 \cdot \nabla \alpha_{i1} = f_{\alpha_{i1}}, & i = 1, \ldots, m \\ (\rho_{i1})_t + div(\rho_{i1} u_1) = f_{\rho_{i1}}, & i = 1, \ldots, m \\ \partial_t \rho + \mathrm{div}(\rho u_1) = 0, \\ \partial_t(\rho u_1) + \mathrm{div}(\rho u_1 \otimes u_1) + \nabla q = \mu \triangle u_1 + (\mu + \lambda)\nabla(\mathrm{div} u_1), \end{cases}$$

$$f_{\alpha_{i1}} = \frac{\alpha_{i1}(a\rho_{i1}^\gamma - q)}{\lambda + 2\mu}, \quad f_{\rho_{i1}} = \frac{\rho_{i1}(q - a\rho_{i1}^\gamma)}{\lambda + 2\mu}$$

$$0 \le \alpha_{i1}, \qquad \sum_{i=1}^m \alpha_{i1} = 1$$

$$\rho = \sum_{i=1}^m \alpha_{i1}\rho_{i1}, \qquad q = a\sum_{i=1}^m \alpha_{i1}\rho_{i1}^\gamma,$$

Sketch of proof

First Result: Solution of the multi-fluid system obtained as weak limit to finite-energy weak solutions of compressible Navier-Stokes equations: Stratification assumption seems to be necessary (improved recently with M. HILLAIRET). Model is linked to the BAER-NUNZIATO model.

Weak sequence related to the existence result by B. DESJARDINS. Given initial data

$$\rho_0 \in L^\infty(\mathbb{T}^3), \qquad \rho_0 \geq 0, \qquad u_0 \in H^1(\mathbb{T}^3).$$

There exists $T_0 \in (0,\infty)$ and a weak solution (ρ, u) to the compressible Naiver-Stokes equations with $(\rho, \rho u)|_{t=0} = (\rho_0, \rho_0 u_0)$. For all $0 < T < T_0$,

$$\rho \in L^\infty((0,T)\times\mathbb{T}^3) \cap C([0,T]; L^q(\mathbb{T}^3)), \qquad \text{for all q} \in [1,\infty)$$

$$\nabla u \in L^\infty(0,T; (L^2(\mathbb{T}^3))^9).$$

$$\sqrt{\rho}\partial_t u \in (L^2((0,T)\times\mathbb{T}^3))^3, \quad Pu \in L^2(0,T; H^2(\mathbb{T}^3)),$$

$$G = (\lambda + 2\mu)\mathrm{div} u - p(\rho) \in L^2((0,T); H^1(\mathbb{T}^3)),$$

where P denotes the projection on the space of divergence-free vector fields.

Some remarks

The L^∞ bound on ρ_0^n is also assumed in D. SERRE (one-dimensional case). This is also required in weak-strong uniqueness by B. DESJARDINS, P. GERMAIN. We will also consider $\rho_0^n \geq C > 0$ as in D. SERRE's paper for the full mathematical justification the multi-fluid system.

Note that steps div$u \in L^1(0, T; L^\infty(\mathbb{T}^3))$ and defect measure characterization do not assume to be far from vacuum.

Initial density sequence $(\rho_0^n)_{n\in N}$ far from vacuum is necessary for justification of the multi-fluid system.

Sketch of proof

Prove that weak sequence based on B. DESJARDINS's lemma has extra-regularity. Namely,

$$\mathrm{div} u \in L^1(0,T;L^\infty(\mathbb{T}^3)).$$

B. Desjardins's weak sequence satisfies:

$$\sup_{0<t\leq T} \|\nabla u\|_2 + \int_0^T \int_{\mathbb{T}^3} \rho|\dot{u}|^2 \leq C, \tag{1}$$

with $\dot{u}$ the total time derivative. Write now extra estimates following D. HOFF's Ideas.

First step :

$$\sup_{0<t\leq T} (\|G\|_2 + \|\omega\|_2) \leq C,$$

$$\|\nabla G\|_6 + \|\nabla \omega\|_6 \leq C(\|\rho^{\frac{1}{2}}\dot{u}\|_2 + \|\nabla \dot{u}\|_2),$$

$$G = (2\mu + \lambda) divu - P, \quad \omega = \nabla \times u$$

where G and ω denote the effective viscous flux and vorticity, respectively.

Sketch of proof

This step uses the equation

$$\nabla G - \mu \nabla \times \omega = \rho \dot{u}$$

and standard L^p elliptic estimates since

$$\Delta G = \mathrm{div}(\rho \dot{u}), \qquad \mu \Delta \omega = \nabla \times (\rho \dot{u}).$$

We use also the following inequality

$$\|v\|_{(L^6(\mathbb{T}^3))^3} \leq C\Big(\int_{\mathbb{T}^3} \rho v + \|\nabla v\|_{(L^2(\mathbb{T}^3))^9}\Big)$$

such that $v \in (H^1(\mathbb{T}^3))^3$ and ρ a non negative function such that $\int_{\mathbb{T}^3} \rho \geq C > 0$.

Sketch of proof

Second step : Deduce the following estimate

$$\sup_{0<t\leq T} \int_{\mathbb{T}^3} \sigma\rho|\dot{u}|^2 + \int_0^T \int_{\mathbb{T}^3} \sigma|\nabla \dot{u}|^2 \leq C$$

where $\sigma = \min(1,t)$.

This estimate is deduced operating $\sigma(\partial_t + \mathrm{div}(u\cdot))$ on the momentum equation and taking the scalar product with $\dot{u}$ and summing. We control high power norm using the link between ∇u and G, ω and p.

Sketch of proof

Use this estimate, the ones in previous slide and the expression of G, to deduce the result that means:

$$\mathrm{div} u \in L^1(0,T;L^\infty(\mathbb{T}^3)).$$

The key part is that we can use interpolation on G and the bound

$$\int_0^T \|\nabla \dot{u}\|_{L^2}^{3/4} \le C\Big(\int_0^T \sigma \|\nabla \dot{u}\|_{L^2}^2\Big)^{3/8}\Big(\int_0^T \sigma^{-3/5}\Big)^{5/8}.$$

Remark: In Recent HOFF-SANTOS's paper, $\rho_0 \in L^\infty$ and $u_0 \in H^s$ + smallness assumption and relation betweeg λ and μ is considered. Propagation of singularity result when $s > 1/2$: such regularity implies also div$u \in L^1 L^\infty$. If local existence with same kind of estimates thus OK. No direct interpolation easily.

Important remarks:

- Assumptions on ρ_0 same than D. SERRE (1991) in the one-dimensional case.
- Note that Young measures characterization is needed looking at three moments since $(\theta_0, \theta_1) = (0, 1)$ to prove that $\nu_{(t,x)} = \delta_{(\rho(t,x), m(t,x))}$ in vanishing viscosity for compressible Euler flow, see G.Q. CHEN, M. PEREPELITSA, (2009) (Physical viscosity limit of solutions of the Navier-Stokes equations to a finite-energy entropy solution of the isentropic Euler equations with finite-energy initial data, 1D case linked to adequate energy estimates and reduction of measure-valued solutions with unbounded support).
- Y. BRENIER, C. DE LELLIS, L. SZÉKELYHIDI JR. (2010): Weak strong uniqueness for measure valued solutions. Argument based on admissible solution of Euler such that $\int_0^T \|D(u)\|_\infty < +\infty$. Linked to the Blow-up criteria for Euler system (see G. PONCE (1985)). For compressible Navier-Stokes equations: Blow-up criteria linked to $\int_0^T \|\mathrm{div} u\|_\infty < +\infty$ or $\int_0^T \|\rho\|_\infty < +\infty$ (see recent papers by X.HUANG, J. LI and Z.P. XIN, B. HASPOT, Z. ZHANG ....).

Third and last step

Recently improved with M. HILLAIRET.

Let $u \in L^\infty(0,T;L^2(T^3)) \cap L^2(0,T;H^1(T))$ and ν a family of Young measures solution to (HCNS). We assume that $\mathrm{div} u \in L^1(0,T;L^\infty(T^3))$ and there exists $M > 0$ such that

$$\mathrm{Supp}\nu_{t,x} \subset [\frac{1}{M}, M] \text{ for } a.a.(t,x) \in (0,T) \times T^3.$$

Assume there exists $(\alpha_i^0, \rho_i^0) \in [L^\infty(\mathbb{T}^3)]^2$ (for $i = 1, \dots, k$), satisfying $\alpha_i^0 \geq 0, \sum_{i=1}^k \alpha_i^0 = 1$ and $\rho_i^0 \geq c > 0$, so that

$$\nu_{0,x} = \sum_{i=1}^k \alpha_i^0(x)\delta_{\rho_i^0(x)}, \qquad \text{for a.a. } x \in \mathbb{T}^3.$$

Then, there exists $0 < T_0 \leq T$ and a solution $((\alpha_i, \rho_i)_{i=1,\dots,k}, u)$ to *(PHCNS)* with initial data $((\alpha_i^0, \rho_i^0)_{i=1,\dots,k}, u^0)$ on $(0, T_0)$, such that

$$\nu_{t,x} = \sum_{i=1}^k \alpha_i(t,x)\delta_{\rho_i(t,x)}, \quad \text{for a.a. } (t,x) \in (0,T_0) \times \mathbb{T}^3.$$

Recent improvement - Easiest proof

Two steps:

1– Construct α_i and ρ_i solution for u and q given.

2– Define $\overline{\nu} = \sum_{i=1}^m \alpha_i \delta_{\rho_i}$

2– Show uniqueness on the measure ν.

Prove by induction that

$$< \nu, \rho^{1-k\gamma} >=< \overline{\nu}, \rho^{1-k\gamma} >, \qquad \forall k \in N \; a.e.$$

Give $< \nu, b >=< \overline{\nu}, b >$ for all b of the form $b(s) = k + \beta(s^{-k})$ with $k \in R$ and β polynomial. Compact supports of ν and $\overline{\nu}$, density argument and $s \mapsto s^\gamma$ realizes homeomorphism of $(0, +\infty)$.

$\Longrightarrow$ Result !

Remark

The vanishing viscosities limit on the homogenized system

gives

same kind of limit system than the relaxation limit on the model with PDEs closure.

$\Longrightarrow$ Original physical relaxation model for the model with algebraic closure

We find

$$P_+(\rho^+) = P_-(\rho_-)$$

and also

$$\partial_t \alpha^+ + u \cdot \nabla \alpha^+ + \frac{\alpha_+ \alpha_- (a_+^2 \rho^+ - a_-^2 \rho^-)}{\alpha_- a_+^2 \rho^+ + \alpha_+ a_-^2 \rho^-} \mathrm{div} u = 0$$

They are the same using the mass equation because pressure depends only on densities.

This is not the case for pressure depending on densities and entropies. See for instance such model in the paper by R. ABGRALL and R. SAUREL.

What was done in the previous paper with X. HUANG: sketch of 3 and 4 step

Third step : Introduce an adequate defect measure (discussions and properties on such kind of defect measures in M. HILLAIRET's PhD Thesis) to prove that young measures are in fact linear combination of dirac measures.

$$M_\alpha[\Theta,\nu] \text{ the determinant of the } (m+1)\times(m+1) \text{ matrix } \widetilde{M}_\alpha[\Theta,\nu]$$

with elements

$$\left(\widetilde{M}_\alpha[\Theta,\nu]\right)_{i,j} = < \nu, \rho^{(\theta_i+\theta_j)\alpha} >$$

with $\Theta = (\theta_0,\cdots,\theta_m) \in \mathbb{N}^{m+1}$ a weight vector composed with two by two distinct coefficients and α a coefficient.

In the sequel: Choice: $\Theta = (0,\cdots,m)$ and α chosen later on !!

Sketch of proof

Using that $\rho \in L^\infty$, we write, using renormalization procedure:

$$\partial_t(M_\alpha[\Theta,\nu]) + \mathrm{div}(M_\alpha[\Theta,\nu]) + \kappa M_\alpha[\Theta,\nu]\mathrm{div} u + \frac{\mathcal{Q}(\nu)}{\lambda+2\mu} = 0$$

with

$$\kappa = 2\alpha\sum_{i=0}^{m}\theta_i - 1,$$

and

$$\mathcal{Q}(\nu) = \sum_{i,j=0}^{m}(2\alpha\theta_i - 1)\big(\overline{\rho^{(\theta_i+\theta_j)\alpha+\gamma}} - \overline{\rho^{(\theta_i+\theta_j)\alpha}}\,\overline{\rho^\gamma}\big)M_\alpha^{(i,j)}[\Theta,\nu].$$

Note that there exists α , for instance $\alpha = \gamma/(E[m\gamma]+1)$, such that $\mathcal{Q}(\nu) \geq 0$, thus integrating in space, we get

$$\frac{d}{dt}\Big[\int_{\mathbb{T}^3} M_\alpha[\Theta,\nu]\Big] \leq |\kappa||\mathrm{div} u|_\infty \int_{\mathbb{T}^3} M_\alpha[\Theta,\nu].$$

Using that $M_\alpha[\Theta,\nu_0] = 0$ and integrating in time, we get that $M_\alpha[\Theta,\nu] = 0$.

Sketch of proof

This implies that, using characterization given in M. HILLAIRET's PhD thesis

$$\nu = \sum_{i=1}^{m} \alpha_i \delta_{\rho_i}.$$

Assumption: A kind of stratification assumption, namely denoting level sets interval

$$L(f) = \big[\inf_{z\in Q_T} f(z), \sup_{z\in Q_T} f(z)\big]$$

we assume

$$\rho_i \in L^\infty, \qquad L(\rho_i) \cap L(\rho_j) = \emptyset \text{ for } i,j = 1,\cdots,m \text{ with } i \neq j$$

then weak limit satisfies a multi-fluid system using expression of ν and then adequate b compactly supported in the limit equation (2).

Note that $\rho \in L^\infty \implies \alpha_i \geq c > 0$ if initially. We then find the multi-fluid system written previously mixing the equation on α_i and $\alpha_i\rho_i$.

Sketch of proof

Fourth step: Use a weak-strong procedure to prove that the strong solution built by M. HILLAIRET corresponds to the weak limit. This use that $\mathrm{div} u \in L^1 L^\infty$ and $\rho \in L^\infty$. If initially the case, we prove that $(\alpha_i, \rho_i, u) = (\alpha_{i1}, \rho_{i1}, u_1)$.

Remark: In fact the strong solution has only to satisfy

$$\alpha_1 \in L^\infty, \rho_1 \in L^\infty, q_1 \in L^\infty, \nabla\alpha_1 \in L^\infty L^3, \nabla\rho_1 \in L^\infty L^3, \nabla u_1 \in L^1 L^\infty, \dot{u}_1 \in L^2 L^3.$$

A kind of generalization of B. DESJARDINS's (1997) and P. GERMAIN's (2009) results.