Weak solutions for degenerate compressible fluid models

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Coworkers

A series of joint works with:

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Outline

The full Navier–Stokes equations

The full Navier–Stokes equations for compressible viscous and heat conducting fluids writes as

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p = \mathrm{div}\,(S + K) + \rho f, \\ \partial_t \left(\rho E + \dfrac{\sigma}{2} |\nabla \rho|^2 \right) + \mathrm{div}\,(\rho u H) = \mathrm{div}((S + K) \cdot u) + \mathrm{div}(\kappa \nabla \theta) + \rho f \cdot u, \\ E = e + \dfrac{|u|^2}{2}, \quad H = h + \dfrac{|u|^2}{2}, \quad h = e + \dfrac{p}{\rho}, \end{cases}$$

where

- $u \in \mathbb{R}^3$ fluid velocity, ρ density, p pressure, θ temperature.
- $\kappa(\rho, \theta)$ thermal conductivity coefficient, $\Sigma = S - pI$ stress tensor.
- e specific internal energy, h specific enthalpy.
- E specific total energy, H associated specific total enthalpy.
- K capillary stress tensor and f external bulk forces.

Newtonian fluids

Only Newtonian fluids are considered here:

$$S = 2\mu D(u) + \mathrm{Id}\lambda \mathrm{div} u \quad \text{where} \quad D(u)_{ij} = (\partial_i u_j + \partial_j u_i)/2 \quad \text{strain rate.}$$

Thermodynamical closure condition writes as:

$$p = \mathcal{P}(\rho, \theta), \qquad e = \mathcal{E}(\rho, \theta).$$

with compatibility condition induced by the second Principle of Thermodynamics

$$\mathcal{P}(\rho, \theta) = \rho^2 \frac{\partial \mathcal{E}}{\partial \rho}|_\theta + \theta \frac{\partial \mathcal{P}}{\partial \theta_\rho}. \quad \text{(Maxwell relation)}$$

Capillary stress tensor (Korteweg model) writes as

$$K_{ij} = \frac{\sigma}{2} \left(\Delta \rho^2 - |\nabla \rho|^2 \right) \delta_{ij} - \sigma \partial_i \rho \partial_j \rho. \qquad (\mathrm{div}\, K = \sigma \rho \nabla \Delta \rho)$$

Initial conditions

The system of PDE's is supplemented with initial conditions

$$\begin{cases} \rho|_{t=0} = \rho_0 \geq 0, \\ \rho u|_{t=0} = m_0, \\ \rho E|_{t=0} = G_0 = \dfrac{|m_0|^2}{2\rho_0} + \rho_0 e_0 \geq 0. \end{cases}$$

Boundary conditions

$$\Omega = T^d \quad \text{(periodic box) or} \quad R^d, \quad d = 2, 3.$$

The barotropic case

Barotropic Navier–Stokes equations with no capillary effects corresponds to

$$\mathcal{P}(\rho,\theta) = p(\rho) \quad \text{and} \quad \sigma = 0,$$

so that:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) = \operatorname{div}(2\mu D(u)) + \nabla(\lambda \operatorname{div} u) - \nabla p + \rho f, \\ p = p(\rho) \\ \rho_{|t=0} = \rho_0, \quad \rho u_{|t=0} = m_0. \end{cases}$$

Results for barotropic case with constant viscosity coefficients

- Solutions close to equilibrium, or local in time solutions ($d \geq 2$):
  - Matsumura, Nishida (1980)
  - Hoff (1994)
  - Danchin (2000)

  Initial data are assumed to satisfy: $\|\rho_0 - \bar{\rho}\| \leq \varepsilon$ and $\|u_0\| \leq \varepsilon'$ for small enough parameters $(\varepsilon, \varepsilon')$. Then, global in time solutions exist and are unique.
- Global in time weak solutions of

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho u) &= 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + A \nabla \rho^\gamma &= \mu \Delta u + (\lambda + \mu) \nabla \operatorname{div} u, \end{aligned}$$

P.-L. Lions (1993), E. Feireisl (1998) proved when $\gamma > d/2$ the global existence of weak solutions satisfying

$$\int \left(\rho \frac{|u|^2}{2} + A \frac{\rho^\gamma}{\gamma - 1} \right)(t,x)dx + \int_0^t \int \left(\mu |\nabla u|^2 + (\lambda + \mu) |\operatorname{div} u|^2 \right)(s,x)dxds$$

$$\leq \int \left(\rho_0 \frac{|u_0|^2}{2} + A \frac{\rho_0^\gamma}{\gamma - 1} \right)(x)dx$$

Does capillarity provide additional regularity?

Energy equality in the barotropic case:

$$\int_\Omega u \cdot \operatorname{div} K = \int_\Omega u \cdot \sigma\rho\nabla\Delta\rho = -\sigma \int_\Omega \operatorname{div}\ (\rho u)\Delta\rho$$

$$= \sigma \int_\Omega \partial_t\rho\Delta\rho = -\frac{d}{dt}\int_\Omega \frac{\sigma}{2}|\nabla\rho|^2,$$

leads to propagation of H^1 regularity of the density.

$$\frac{d}{dt}\int_\Omega \left(\frac{\sigma}{2}|\nabla\rho|^2 + \frac{A\rho^\gamma}{\gamma-1} + \rho\frac{|u|^2}{2}\right) + \int_\Omega 2\mu(\rho)D(u) : D(u)$$

$$+ \int_\Omega \lambda(\rho)|\operatorname{div} u|^2 = \int_\Omega \rho f \cdot u$$

Does capillarity provide additional regularity?

The case of constant λ and μ: local existence and uniqueness of smooth solutions

- With "large" initial data (away from equilibrium states $\rho = constant$ and $u = 0$).
- Still, as long as the density stays away from zero (positive lower bound on ρ).
- Regularizing effect of Korteweg stress can be identified on the linearized system:

$$\rho_0 \in H^s(R^d) \implies \rho \in L^\infty((0,T); H^s(R^d)) \cap L^2((0,T); H^{s+1}(R^d)).$$

Related papers

- Hattori and Li (1996): $(\rho_0, m_0) \in H^s(R^d) \times H^{s-1}(R^d)^d$ with $s > d/2 + 4$.
- Danchin, D. (1999): in scaling invariant spaces $B^{d/2}_{2,1}(R^d)$.

However, this regularizing effect does not seem to be preserved for the non linear system...

Global existence of weak solutions

Surprising cancellation of non linear terms occurs for degenerate viscosities $\mu(\rho) = \nu\rho$ and $\lambda(\rho) = 0$.

$$\partial_t\rho + \mathrm{div}(\rho u) = 0, \quad \text{and} \quad \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p = \mathrm{div}(2\nu\rho D(u)) + \sigma\rho\nabla\Delta\rho,$$

Which formally leads to additional regularity globally in time provided $\nabla\sqrt{\rho_0}$ and $\nabla\rho_0 \in L^2(\Omega)$

$$\rho \in L^2(0,T;H^2(\Omega)), \qquad \nabla\rho \quad \text{and} \quad \nabla\sqrt{\rho} \in L^\infty(0,T;L^2(\Omega)^d)$$

$$\sqrt{\rho}u \in L^\infty(0,T;L^2(\Omega)^d) \quad \text{and} \quad \sqrt{\rho}\nabla u \in L^2(0,T;L^2(\Omega))^{d\times d}.$$

Difficulty:
Compactness of $\rho_n u_n \otimes u_n$ because u_n is no longer bounded in $L^2(0,T;H^1)$.

Global existence of weak solutions

Definition: Weak solutions satisfy the energy inequality and multiplication of the momentum equation by ρ yields:

For all $\varphi \in C^\infty((0,T)\times\Omega)^d$ such that $\varphi(T,\cdot)=0$, one has

$$0 = \int_\Omega \rho_0 u_0 \cdot \rho_0 \varphi(0,\dot{}) + \int_0^T \int_\Omega \Big(\rho^2 u \cdot \partial_t \varphi + \rho u \otimes \rho u : D(\varphi) - \rho^2 (u\cdot\varphi)\,\mathrm{div}\,u$$

$$-\nu\rho D(u) : D(\varphi) - \nu \rho D(u) : \varphi \otimes \nabla\rho + \Xi(\rho)\,\mathrm{div}\,\varphi - \sigma\rho^2 \Delta\rho\,\mathrm{div}\,\varphi - 2\sigma\rho(\varphi\cdot\nabla\rho)\Delta\rho\Big)$$

Theorem (D. Bresch, C.K. Lin, D. in 2001)

There exists a global weak solution (ρ, u).

Why degenerate viscosities may help?

- Behavior when density ρ vanishes.
- Dimensional analysis.
- Density fluctuations modeling.

Behavior when density ρ vanishes

When the density ρ tends to zero

- From a physical viewpoint, rarefied gas models would be more relevant.
- Still Navier–Stokes numerical codes need to be robust near regions where $\rho \to 0$.
- The compressible Navier–Stokes equations become in vacuum

$$-\operatorname{div}(2\overline{\mu}D(u)) - \nabla(\overline{\lambda}\operatorname{div} u) = 0$$

where $\overline{\mu} = \mu(\rho = 0, \theta)$ and $\overline{\lambda} = \lambda(\rho = 0, \theta)$.

Remark. D. Hoff, D. Serre. SIAM J. Applied Math. 1991. Failure of continuous dependence on initial data for the Navier-Stokes equations of compressible flow.

$\Longrightarrow$ μ and λ need to vanish…

Dimensional analysis

Dimensional analysis may help find relevant velocities linked to density / viscosity variations:

Define the gradient length associated to viscosity variations

$$L_\mu = \frac{\mu}{\|\nabla \mu\|}$$

A Reynolds number can be defined by

$$Re = \frac{\rho V L_\mu}{\mu},$$

where V denotes a velocity scale associated to viscosity variations,

$$V \sim \frac{\mu}{\rho L_\mu} = \frac{\|\nabla \mu(\rho)\|}{\rho}$$

hence velocity like $\nabla \mu / \rho$ seem to play a role…

Density fluctuations modeling: Favre averaging

In statistical derivation of turbulence models, Favre average is adapted to heterogeneous flows (compressibility / large density variations):

$$\widetilde{a} = \frac{\overline{\rho a}}{\overline{\rho}}$$

Favre fluctuations

$$a = \tilde{a} + a''$$

Example: take ensemble average of mass conservation equation

$$\overline{\partial_t \rho + \text{ div } (\rho u)} = 0$$

$$\partial_t \overline{\rho} + \text{ div } (\overline{\rho u}) = 0$$

and since $\overline{\rho u} = \overline{\rho}\widetilde{u}$, one has

$$\partial_t \overline{\rho} + \text{ div } (\overline{\rho}\widetilde{u}) = 0.$$

Boussinesq–Reynolds closure

Averaging fluid mechanics models leads to correlations of fluctuations of physical quantities a with velocity fluctuations u''

$$\overline{\rho u'' a''}$$

Boussinesq-Reynolds closure then leads to

$$\overline{\rho u'' a''} = -\overline{\rho} \frac{\nu_t}{\sigma_a} \nabla \widetilde{a}$$

where ν_t is the turbulent viscosity, σ_a is a constant (Prandtl–Schmidt number).

Boussinesq–Reynolds closure

Example: difference between Favre and Reynolds velocity

$$\begin{aligned} \overline{u} &= \overline{\rho u v} \qquad \text{where} \quad v = 1/\rho \\ &= \widetilde{u} + \overline{\rho u'' v''} = \overline{u} - \overline{\rho}\frac{\nu_t}{\sigma_t}\nabla\widetilde{v} \end{aligned}$$

since $\widetilde{v} = \overline{\rho v}/\overline{\rho} = 1/\overline{\rho}$, one ends up with

$$\begin{aligned} \overline{u} &= \widetilde{u} - \overline{\rho}\frac{\nu_t}{\sigma_t}\nabla\frac{1}{\overline{\rho}} \\ &= \widetilde{u} + \frac{\nu_t}{\sigma_t}\frac{\nabla\overline{\rho}}{\overline{\rho}}. \end{aligned}$$

Compressible barotropic models

Navier-Stokes equations for viscous compressible and barotropic fluids with $\mu(\rho)$ and $\lambda(\rho)$

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) = \mathrm{div}(2\mu(\rho) D(u)) + \nabla(\lambda(\rho)\mathrm{div} u) - \nabla p(\rho) + \rho f, \\ \rho_{|t=0} = \rho_0, \\ \rho u_{|t=0} = m_0. \end{cases}$$

Compressible barotropic models - Energy estimates

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|u|^2 + \int_\Omega \left(2\mu(\rho)|D(u)|^2 + \lambda(\rho)|\mathrm{div} u|^2\right) + \int_\Omega \nabla p\cdot u = \int \rho f\cdot u.$$

$\int_\Omega \nabla p \cdot u$ is computed by using the mass conservation equation. One has

$$\begin{aligned}\int_\Omega \nabla p\cdot u &= \int_\Omega p'(\rho)\nabla\rho\cdot u = \int_\Omega \frac{p'(\rho)}{\rho}\nabla\rho\cdot\rho u = \int_\Omega \nabla\Pi'(\rho)\cdot\rho u\\ &= -\int_\Omega \Pi'(\rho)\mathrm{div}(\rho u) = \int_\Omega \Pi'(\rho)\partial_t\rho = \frac{d}{dt}\int_\Omega \Pi(\rho)\end{aligned}$$

$$\text{where}\quad \Pi(\rho) = \rho\int_0^\rho \frac{p(s)}{s^2}\,ds.$$

so that $\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|u|^2 + \frac{d}{dt}\int_\Omega \Pi(\rho) + \int_\Omega \left(2\mu(\rho)|D(u)|^2 + \lambda(\rho)|\mathrm{div} u|^2\right) = \int \rho f\cdot u.$

$\Longrightarrow \sqrt{\rho}u \in L^\infty(0,T;(L^2(\Omega))^d), \quad \sqrt{\mu}D(u)\in L^2((0,T)\times\Omega)^d,$
$\Pi(\rho)\in L^\infty(0,T;L^1(\Omega)).$

Compactness of weak solutions

Compactness (Stability)

- Compactness on ρu using Aubin-Lions-Simon lemma if non degenerate viscosity μ...
- ... but compactness problem on ρ to pass to the limit in the pressure term ρ^γ:

This is the main difficulty in P.–L. Lions and E. Feireisl work. More information on the density ρ?.
If degeneracy in μ with respect to ρ: More information on u as well?.

Two cases:

- Constant μ and $\lambda \implies$ non degenerate.
- Case when μ and/or λ depend on ρ.

Compactness of weak solutions

Case of constant viscosities. P.–L. LIONS, E. FEIREISL
More information on ρ ?

$$\partial_t \mathrm{div}(\rho u) + \mathrm{div}\big(\mathrm{div}(\rho u \otimes u)\big) - \Delta((2\mu+\lambda)\mathrm{div}u - a\rho^\gamma) = \mathrm{div}(\rho f),$$

$\Longrightarrow$ some compactnes on the effective viscous flux $F = (2\mu+\lambda)\mathrm{div}u - a\rho^\gamma$ eventually allows to control density oscillations.

Remark. μ and λ need to be constant (at least μ needs to be constant).

Case of density dependent viscosities. A. VAIGANT, A. KAZHIKHOV, D. BRESCH, D., A. MELLET, A. VASSEUR
More information on ρ?

- Yes if $\mu = \text{cste}$, $\lambda = \rho^\beta$, $\beta \geq 3$ (A. Vaigant, A. Kazhikhov).
- Yes if $\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))$ (D. Bresch, D., A. Mellet, A. Vasseur).

Mathematical entropy in the degenerate case

Additional mathematical entropy if $\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))$.
(ref. D. Bresch, D., C.R. Acad. Sciences Section Mécanique, (2004)).

$$\partial_t \rho + \text{div}(\rho u) = 0$$

Multiplying by $\mu'(\rho)$, one gets

$$\partial_t \mu(\rho) + \text{div}(\mu(\rho) u) + (\mu'(\rho)\rho - \mu(\rho))\text{divu} = 0$$

Taking the gradient of the equations

$$\partial_t \nabla \mu(\rho) + \text{div}(u \otimes \nabla \mu(\rho)) + \text{div}(\mu(\rho)^t \nabla u) + \nabla\big((\mu'(\rho)\rho - \mu(\rho))\text{div}u\big) = 0$$

Which writes

$$\partial_t(\rho \nabla \varphi(\rho)) + \text{div}(\rho u \otimes \nabla \varphi(\rho)) + \text{div}(\mu(\rho) D(u)) - \text{div}(\mu(\rho) A(u)) + \frac{1}{2} \nabla\big(\lambda(\rho)\text{div}u\big) = 0$$

where $\varphi'(s) = \mu'(s)/s$ and $A(u) = (\nabla u - {}^t\nabla u)/2$.

Mathematical entropy in the degenerate case

Summing this equation multiplied by 2 with the momentum conservation equation, one gets

$$\partial_t(\rho(u+v)) + \mathrm{div}(\rho u \otimes (u+v)) - \mathrm{div}(2\mu(\rho)A(u)) + \nabla p = \rho f,$$

where

$$v = 2\nabla\mu(\rho)/\rho \quad \text{and} \quad A(u) = (\nabla u - {}^t\nabla u)/2.$$

Multiplying this equation by $u+v$ and the mass conservation equation by $|u+v|^2/2$ one gets the additional entropy estimate

$$\frac{d}{dt}\int_\Omega \left(\rho\frac{|u+v|^2}{2} + \pi(\rho)\right) + \int_\Omega 2\mu(\rho)|A(u)|^2 + \int_\Omega \frac{p'(\rho)}{\rho}|\nabla\rho|^2$$
$$= \int_\Omega \rho f \cdot u + \int_\Omega 2\rho f \cdot \nabla\varphi.$$

A priori bounds

Assuming $f \equiv 0$ and $p'(\rho) \geq 0$ then such BD entropy leads to the following bounds

$$\sqrt{\rho}(u+v) \in L^\infty(0,T;(L^2(\Omega))^d), \tag{1}$$

$$\sqrt{\mu(\rho)}A(u) \in L^2(0,T;L^2(\Omega))^{d\times d},$$

$$\sqrt{\frac{p'(\rho)}{\rho}}\nabla\rho \in L^2(0,T;(L^2(\Omega))^d), \tag{2}$$

$$\pi(\rho) \in L^\infty(0,T;L^1(\Omega))$$

Using the the energy equality, one gets

$$\sqrt{\rho}v \in L^\infty(0,T;(L^2(\Omega))^d), \quad \sqrt{\frac{p'(\rho)}{\rho}}\nabla\rho \in L^2(0,T;(L^2(\Omega))^d).$$

$\Longrightarrow$ More information on ρ!

Difficulties

Degenerate model since only $\sqrt{\mu}\nabla u$ is bounded in $L^2((0,T)\times\Omega)$
$\Longrightarrow$ More esimates on u required, such as:

$$\sup_{t\geq 0}\int_\Omega \rho|u|^{2+\epsilon} < +\infty$$

or

$$\int_0^T\int_\Omega |\nabla u|^q < +\infty \quad \text{for} \quad 1 < q < 2.$$

Example: quadratic drag force $-\alpha\rho|u|u$ in the momentum conservation equation.

A priori bounds: a new estimate due to Mellet-Vasseur, 2007

Mutliplying the momentum conservation equation by $(1+\ln(1+|u|^2))u$ if $d\lambda + 2\mu \geq \nu\mu$, leads to

$$\frac{d}{dt}\int \rho \frac{1+|u|^2}{2}\ln(1+|u|^2) + \frac{\nu}{2}\int \mu(\rho)(1+\ln(1+|u|^2))|D(u)|^2$$
$$\leq c\int \mu(\rho)|\nabla u|^2 + c\left(\int \rho\left(2+\ln(1+|u|^2)\right)^{2/\delta}\right)\left(\int \frac{\rho^{2\gamma-\delta/2}}{\mu(\rho)}\right)^{(2-\delta)/2} \quad (3)$$

this estimate together with the energy estimates are sufficient for getting compactness on $\sqrt{\rho}u$ in $L^2(0,T;L^2(\Omega)^{d\times d})$.

Theorem (A. Mellet, A. Vasseur, 2007)

Stability of sequences of weak solutions to barotropic NS with $p(\rho) = a\rho^\gamma$ and $\gamma > 1$.

Construction of approximate solutions

Viscosities μ and λ satisfying $\lambda(\rho) = 2(\rho\mu'(\rho) - \mu(\rho))$

Open problem: regularize the system while preserving estimates

- Energy.
- Additional entropy.
- Mellet-Vasseur estimate.

A way to proceed: add a drag force term or a cold pressure component close to vacuum $\Longrightarrow$ one solves a modified (CNS).

Construction of approximate solutions

$$\begin{aligned}
&\partial_t\rho + \mathrm{div}(\rho u) = 0,\\
&\partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) - 2\mathrm{div}(\mu(\rho)D(u)) - \nabla(\lambda(\rho)\mathrm{div}u) + \nabla p(\rho)\\
&\qquad\qquad + r_1\rho|u|u\\
&\qquad\qquad - \varepsilon\rho\cdot\nabla(\mu'(\rho)\Delta^s\mu(\rho)) + \eta\Delta^2 u = 0.
\end{aligned}\tag{4}$$

If r_1 is fixed then one lets η tend to 0 then ε and finally r_2.

Convergence when $\varepsilon\to 0$ in the case $\mu(\rho)\sim\rho$ and $\lambda = 0$.

Proven in the Shallow water case $\mu(\rho)=\rho$: D. Bresch , B. D. On the construction of approximate solutions for 2D viscous shallow water model with extension to compressible N.S. models. *J. Maths Pures et Appliquées, 86, 362–368, (2006).*

The full Navier–Stokes equations

The full Navier–Stokes equations for compressible viscous and heat conducting fluids writes as

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p = \mathrm{div} S + \rho f, \\ \partial_t\left(\rho E\right) + \mathrm{div}\left(\rho u H\right) = \mathrm{div}(S \cdot u) + \mathrm{div}(\kappa \nabla \theta) + \rho f \cdot u, \\ E = e + \dfrac{|u|^2}{2}, \quad H = h + \dfrac{|u|^2}{2}, \quad h = e + \dfrac{p}{\rho}, \end{cases}$$

where

- $u \in \mathbb{R}^3$ fluid velocity, ρ density, p pressure, θ temperature.
- $\kappa(\rho, \theta)$ thermal conductivity coefficient, $\Sigma = S - pI$ stress tensor.
- e specific internal energy, h specific enthalpy.
- E specific total energy, H associated specific total enthalpy.
- f external bulk forces.

Difficulties for the full Navier Stokes equations

Difficulties of the complete model

All the previous difficulties except for the pressure $\Longrightarrow$ two possible ways:

- μ and λ constants.
- μ and λ depending on ρ satisfying $\lambda(\rho) = 2(\rho\mu'(\rho) - \mu(\rho))$.

Other difficulties linked to the coupling with the heat equation?

- A priori estimates not sufficient to define weak solutions for perfect gases…
- Temperature compactness.

Difficulties for the full Navier Stokes equations

A priori estimates not sufficient to define weak solutions.

$$\rho,\ \rho|u|^2,\ \rho\theta \text{ and } \rho\log\rho \in L^\infty(0,T;L^1(\Omega)),$$
$$k(\theta)|\nabla\log\theta|^2,\ |D(u)|^2/\theta \in L^1(0,T;L^1(\Omega)).$$

but

$$u\left(\rho\frac{|u|^2}{2}+\rho e+p\right)=u\left(\rho\frac{|u|^2}{2}+(R+C_0)\rho\theta\right) \text{ is not necessarily integrable.}$$

$\Longrightarrow$ $p = p_c(\rho) + p(\rho,\theta)$ with p_c cold pressure to get more information on u.
This kind of pressure is used in E. Feireisl work and D. Bresch, B. D. as well.

- E. Feireisl : $p(\rho,\theta) = p_c(\rho) + \theta p_\theta(\rho)$ with $p_\theta(\rho) \ll p_c$.
- Didier Bresch, B. D.: $p(\rho,\theta) = r\rho\theta + p_c(\rho)$ with $p_c \equiv 0$ away from 0.

Compactness in temperature.

How to prove that

- $\rho|u|^2u \in L^p((0,T)\times\Omega), p>1$ and $\kappa\nabla\theta \in L^p((0,T)\times\Omega), p>1$???

A priori bounds

Important inequalities:

- Estimates on total energy (conservative form on E).
- Entropy estimates and variants.
- Additional entropy if λ and μ linked to eachother and depending on ρ.

A priori bounds

Physical Energy

$$\frac{d}{dt}\int_\Omega \rho(e+\frac{|u|^2}{2}) = \int \rho f_{\text{ext}}\cdot u.$$

$\Longrightarrow$

$$\int_\Omega \rho(e+\frac{|u|^2}{2})(t,x) \le \int_\Omega (G_0+\frac{|m_0|^2}{2\rho_0})\,dx + \int_0^t\int_\Omega \rho f_{\text{ext}}\cdot u.$$

In this estimate, no viscosity!!!

A priori bounds

Estimates on the temperature

Two formulations:

- Temperature evolution equations,
- Entropy equations.

Temperature equations:

$$C_v(\partial_t(\rho\theta)+\mathrm{div}(\rho\theta u)+\Gamma\rho\theta\mathrm{div}u) = \rho f_{\mathrm{int}}\cdot u+2\mu S(u) : S(u)+(\lambda+\frac{2\mu}{d})|\mathrm{div}u|^2+\mathrm{div}(\kappa\nabla\theta)$$

where $S(u) = D(u) - \frac{1}{d}\mathrm{div}u\mathrm{Id}$ with $\Gamma = \rho^{-1}\partial p/\partial e|_\rho$. This inequality leads to temperature positivity.

Entropy equation:

$$\theta(\partial_t(\rho s) + \mathrm{div}(\rho u s)) = \rho f_{\mathrm{int}} \cdot u + 2\mu D(u) : D(u) + \lambda|\mathrm{div}u|^2 + \mathrm{div}(\kappa\nabla\theta).$$

Dividing by θ and integrating, this identity leads to

$$\int_\Omega \left(\frac{1}{\theta}\rho f_{\mathrm{int}} \cdot u + 2\frac{\mu}{\theta}D(u) : D(u) + \frac{\lambda}{\theta}|\mathrm{div}(u)|^2\right) + \int_\Omega \frac{\kappa}{\theta^2}|\nabla\theta|^2 = \frac{d}{dt}\int_\Omega \rho s.$$

Then, one shows that entropy can be controled, and therefore the left hand side.

A priori bounds: estimates on the velocity

One the one hand, multiply the momentum conservation equation by u

$$\frac{1}{2}\frac{d}{dt}\int \rho|u|^2 + 2\int_\Omega \mu|D(u)|^2 + \int_\Omega \lambda|\mathrm{div} u|^2 + \int_\Omega \nabla p \cdot u = \int_\Omega f \cdot u.$$

On the other hand,

$$\partial_t(\rho(u + 2\nabla\varphi)) + \mathrm{div}(\rho u \otimes (u + 2\nabla\varphi)) - 2\mathrm{div}(\mu(\rho)A(u)) + \nabla p = \rho f.$$

Then, multiplying by $u + 2\nabla\varphi$

$$\frac{d}{dt}\int_\Omega \rho|u + 2\nabla\varphi|^2 + 2\int_\Omega \mu(\rho)|A(u)|^2 + \int_\Omega \nabla p \cdot (u + \nabla\varphi) = \int_\Omega \rho f \cdot (u + \nabla\varphi).$$

$\Longrightarrow$ The problem is to control the pressure term, which is done using estimates on the temperature.

Theorem (D. Bresch, D., 2007)

Under assumptions on cold pressure, viscosity law exponents . . . stability property holds for sequences of weak solutions.

Complex free surface flows

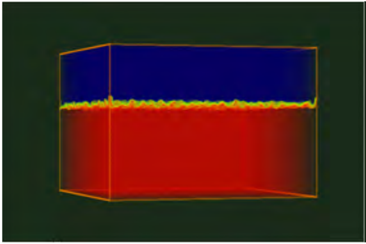
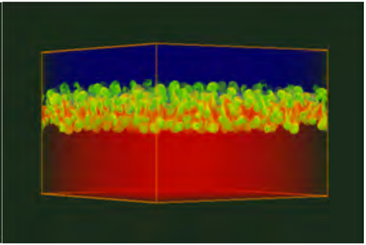
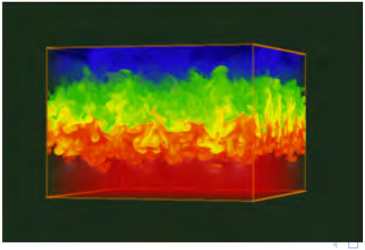
Interfaces subject to Variable accelerations $g(t)$, shocks, shear
Mixing induced by instabilities

1. Rayleigh-Taylor
2. Richtmyer-Meshkov
3. Kelvin-Helmholtz

…in the fully turbulent regime

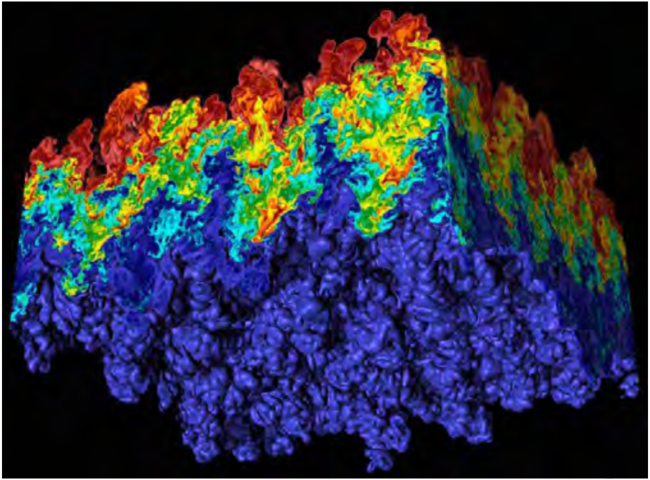
Rayleigh Taylor Instabilities (1/3)

Rayleigh-Taylor (acceleration induced) instabilities

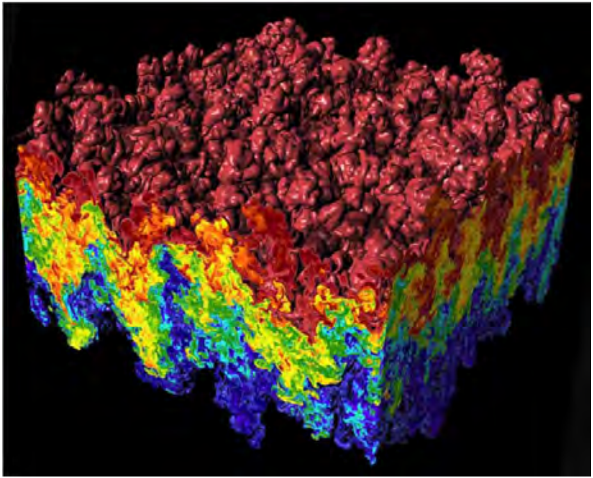
Rayleigh Taylor Instabilities (2/3)

B. Cabot, A. Cook, P. Miller (2006).
3 to 1 density ratio *Atwood* $= 0.5$, 3072^3 simulation



Rayleigh Taylor Instabilities (3/3)

B. Cabot, A. Cook, P. Miller (2006).
grid spacing $\Delta \sim \eta$, Kolmogorov scale



Four equations model

The four equations - one pressure two phase model writes as:

$$
\begin{aligned}
\alpha_+ + \alpha_- &= 1, \\
\partial_t(\alpha^+\rho^+) + \operatorname{div}(\alpha^+\rho^+u^+) &= 0, \\
\partial_t(\alpha^-\rho^-) + \operatorname{div}(\alpha^-\rho^-u^-) &= 0, \\
\partial_t(\alpha^+\rho^+u^+) + \operatorname{div}(\alpha^+\rho^+u^+\otimes u^+) + \alpha^+\nabla P &= -D + \alpha^+\rho^+g + \operatorname{div} S^+, \\
\partial_t(\alpha^-\rho^-u^-) + \operatorname{div}(\alpha^-\rho^-u^-\otimes u^-) + \alpha^-\nabla P &= +D + \alpha^-\rho^-g + \operatorname{div} S^-, \\
P = P_-(\rho_-) = P_+(\rho_+),
\end{aligned}
$$

with

$$D = \text{momentum exchange} \sim (u^+ - u^-)|u^+ - u^-|^\beta \text{or interfacial pressure } \pi\nabla\alpha^+,$$

$$0 \le \alpha_\pm \le 1, \qquad S^\pm \text{ denotes the stress tensor of phase } \pm .$$

Difficulties

- Non hyperbolicity for $|u^+ - u^-| < C_0$ and ill posedness
- Non linear and non conservative equations:

  Existence of weak solutions is open
- Regularizations
  - Hyperbolization
  - Viscosity at high frequency
  - Capillarity

The model

Introducing viscosity and capillarity effects on bifluid system and write:

$$\begin{aligned}
&\alpha^+ + \alpha^- = 1\,,\\
&\partial_t(\alpha^\pm\rho^\pm) + \mathrm{div}(\alpha^\pm\rho^\pm u^\pm) = 0\,,\\
&\partial_t(\alpha^\pm\rho^\pm u^\pm) + \mathrm{div}(\alpha^\pm\rho^\pm u^\pm \otimes u^\pm)\\
&\qquad\qquad +\alpha^\pm\nabla p = \mathrm{div}(\alpha^\pm\tau^\pm) + \sigma^\pm\alpha^\pm\rho^\pm\nabla\Delta\left(\alpha^\pm\rho^\pm\right)\,.
\end{aligned}$$

with

$$\tau^\pm = 2\mu^\pm D(u^\pm) + \lambda^\pm\,\mathrm{div}\,u^\pm \mathrm{Id}$$

$$p = p_\pm(\rho^\pm) = A^\pm(\rho^\pm)^{\gamma^\pm} \quad \text{where } \gamma^\pm \text{ are given constants greater than } 1$$

Assume $\mu_\pm(\rho^\pm) = \nu^\pm\rho^\pm, \qquad \lambda_\pm(\rho^\pm) = 0.$
Then, using the mathematical structure associated with the additional entropy

Theorem (D. Bresch, J.M. Ghidaglia, E. Grenier, D., 2010)

Stability of weak solutions.

In progress: the incompressible limit in the two phase framework

When one of the two phases becomes incompressible $\rho^+ \to$ constant and phase $-$ remain compressible

- Regular limit
- No acoustic waves

Double incompressible limit: sound speeds C^+ and C^- are of same order of magnitude $\to \infty$.

- No relevant sound speed for two phase mixtures.
- Asymptotic behavior of the spectrum of linearized operator.

The double incompressible limit

The scaled two phase model for the double incompressible limit can be written as

$$
\begin{aligned}
\alpha_+ + \alpha_- &= 1,\\
\partial_t(\alpha^+\rho^+) + \operatorname{div}(\alpha^+\rho^+u^+) &= 0,\\
\partial_t(\alpha^-\rho^-) + \operatorname{div}(\alpha^-\rho^-u^-) &= 0,\\
\partial_t(\alpha^+\rho^+u^+) + \operatorname{div}(\alpha^+\rho^+u^+\otimes u^+) + \frac{\alpha^+}{\varepsilon^2}\nabla P &= -D + \alpha^+\rho^+ g + \operatorname{div} S^+,\\
\partial_t(\alpha^-\rho^-u^-) + \operatorname{div}(\alpha^-\rho^-u^-\otimes u^-) + \frac{\alpha^-}{\varepsilon^2}\nabla P &= +D + \alpha^-\rho^- g + \operatorname{div} S^-,\\
P = P_-(\rho_-) = P_+(\rho_+),
\end{aligned}
$$

Perspectives

…in progress

- Construction of approximate solutions in the case of general $\mu(\rho)$ and $\lambda(\rho)$ satisfying $\lambda(\rho) = 2(\rho\mu'(\rho) - \mu(\rho))$ (with D. Bresch, 2013).

- Incompressible limit of two phase flow models with degenerate viscosities (with D. Bresch, J.M. Ghidaglia and E. Grenier, 2013).

Perspectives

非常感謝

Reformulation of the two phase system

The two phase system can be rewritten in terms of (α^+, P, v_r, U) variables

$$\partial_t \alpha^+ + V_\rho \cdot \nabla \alpha^+ + B_\rho \operatorname{div} v_r + \frac{B_\rho}{P}\left(\frac{\rho^+}{\gamma^+} - \frac{\rho^-}{\gamma^-}\right)\left(\hat{V}_\rho \cdot \nabla P + A_\gamma P \operatorname{div} U\right) = 0 \quad (5)$$

$$\partial_t P + \hat{V}_\gamma \cdot \nabla P + P A_\gamma \operatorname{div} U = 0 \quad (6)$$

where

$$V_\rho = \frac{\frac{\rho^+ u^+}{\alpha^+} + \frac{\rho^- u^-}{\alpha^-}}{\frac{\rho^+}{\alpha^+} + \frac{\rho^-}{\alpha^-}}, \hat{V}_\rho = \frac{\frac{\alpha^+ u^+}{\rho^+} + \frac{\alpha^- u^-}{\rho^-}}{\frac{\alpha^+}{\rho^+} + \frac{\alpha^-}{\rho^-}}, A_\rho = \frac{1}{\frac{\alpha^+}{\rho^+} + \frac{\alpha^-}{\rho^-}}, B_\rho = \frac{1}{\frac{\rho^+}{\alpha^+} + \frac{\rho^-}{\alpha^-}},$$

$$V_\gamma = \frac{\frac{\gamma^+ u^+}{\alpha^+} + \frac{\gamma^- u^-}{\alpha^-}}{\frac{\gamma^+}{\alpha^+} + \frac{\gamma^-}{\alpha^-}}, \hat{V}_\gamma = \frac{\frac{\alpha^+ u^+}{\gamma^+} + \frac{\alpha^- u^-}{\gamma^-}}{\frac{\alpha^+}{\gamma^+} + \frac{\alpha^-}{\gamma^-}}, A_\gamma = \frac{1}{\frac{\alpha^+}{\gamma^+} + \frac{\alpha^-}{\gamma^-}}, B_\gamma = \frac{1}{\frac{\gamma^+}{\alpha^+} + \frac{\gamma^-}{\alpha^-}},$$

$$U = \alpha^+ u^+ + \alpha^- u^-, \quad v_r = \rho^+ u^+ - \rho^- u^-, \quad u_r = u^+ - u^-, \quad \text{and} \quad K = \alpha^+ \rho^- + \alpha^- \rho^+.$$

Reformulation of the two phase system

$$\partial_t v_r + V_\rho \cdot \nabla v_r + A_\gamma \left(\frac{\rho^+ u^+}{\gamma^+} - \frac{\rho^- u^-}{\gamma^-} \right) \operatorname{div} U + A_\rho u_r \cdot \nabla U - A_\rho u_r \left(u_r \cdot \nabla \alpha^+ \right)$$
$$+ \left(\frac{\rho^+ u^+}{\gamma^+} - \frac{\rho^- u^-}{\gamma^-} \right) (\hat{V}_\gamma - V_\rho) \cdot \frac{\nabla P}{P} - \pi \left(\frac{1}{\alpha^+} + \frac{1}{\alpha^-} \right) \nabla \alpha^+ = 0. \qquad (7)$$

and

$$\partial_t U + \hat{V}_\rho \cdot \nabla U + \frac{1}{A_\rho \varepsilon^2} \nabla P + B_\rho u_r \operatorname{div} v_r + u_r B_\rho \left(\frac{\rho^+}{\gamma^+} - \frac{\rho^-}{\gamma^-} \right) A_\gamma \operatorname{div} U$$
$$+ B_\rho u_r \cdot \nabla v_r + u_r (V_\rho - \hat{V}_\rho) \cdot \nabla \alpha^+ - B_\rho^2 u_r \left(u_r \cdot \frac{\nabla P}{P} \right) \left(\frac{{\rho^+}^2}{\alpha^+ \gamma^+} + \frac{{\rho^-}^2}{\alpha^- \gamma^-} \right)$$
$$+ \pi \left(\frac{1}{\rho^-} - \frac{1}{\rho^+} \right) \nabla \alpha^+ + U \left(\frac{\alpha^+ u^+ \rho^-}{\gamma^-} + \frac{\alpha^- u^- \rho^+}{\gamma^+} \right) \cdot \frac{\nabla P}{PK}$$
$$- \left(\frac{\alpha^+ u^+ \rho^-}{\gamma^-} + \frac{\alpha^- u^- \rho^+}{\gamma^+} \right) U \cdot \frac{\nabla P}{PK} = 0. \qquad (8)$$

First step: uniform local existence

Methodology ”à la Métivier Schochet” to prove local existence over an interval $[0,T]$ independent of ε

$$\begin{cases} a(\partial_t q + v\cdot\nabla q) + \dfrac{1}{\varepsilon}\mathrm{div}_{\mathbf{x}} u = 0, \\ b(\partial_t m + v\cdot\nabla m) + \dfrac{1}{\varepsilon}\nabla\psi = 0 \end{cases} \tag{9}$$

with $a(t,x)$ and $b(t,x)$ known and positive, and

$$q = \frac{1}{\varepsilon}Q(t,x,\varepsilon\psi), \quad m = \mu(t,x)u, \quad v = V(t,x,u,q), \tag{10}$$

where Q, μ and V are smooth functions of their arguments with $Q(t,x,0)=0$, $\partial_\theta Q > 0$ and $\mu > 0$. Note that q is not singular in ε and satisfies

$$q = Q_1(t,x,\varepsilon\psi)\psi, \qquad \text{with} \quad Q_1 > 0. \tag{11}$$

a and b depend on functions satisfying a non singular equation (like α^+ and v_r).

Theorem $\exists T > 0, \varepsilon_0 > 0$ such that

$$\sup_{t\in[0,T],\varepsilon<\varepsilon_0} \|(u^\varepsilon(t),\psi^\varepsilon(t))\|_{H^s} \le C_T\|(u_0^\varepsilon,\psi_0^\varepsilon)\|_{H^s}$$

Second step: analysis of the spectrum

Asymptotic behavior of solutions depend on averaging phenomena in variable t/ε, close to what happens in systems like

$$\begin{aligned} \varepsilon\dot{\varphi} &= \omega(I) + \varepsilon g(I,\varphi), & \varphi(0) = \varphi^* \\ \dot{I} &= f(I,\varphi), & I(0) = I^* \end{aligned} \tag{12}$$

Under generic non resonance assumptions, for a.e. initial data (φ^*, I^*), I^ε converges to I such that

$$\dot{I} = \frac{1}{|\mathbb{T}^d|}\int_{\mathbb{T}^d} f(I,\varphi)d\varphi, \qquad I(0) = I^*.$$

Reduction of (5)(6)(7)(8) to (12) requires constant multiplicity of the spectrum of the linearized operator.

Second step: analysis of the spectrum

Detailed description of the spectrum of system (5)(6)(7)(8) is needed to complete the analysis.
Eigenvalues are the roots of the polynomial in X

$$P(X) = (X-\lambda)^2(X+\lambda)^2 - K_1(X-\lambda)^2 - K_2(X+\lambda)^2 + K_3$$

with

$$K_1 = \alpha^-\rho^+ - \alpha^+\pi/C^2, \qquad K_2 = \alpha^+\rho^- - \alpha^-\pi/C^2,$$
$$K_3 = -\pi/\gamma^2, \qquad \lambda = u_r/2\gamma \qquad \gamma^2 = C^2/(\alpha^-\rho^+ + \alpha^+\rho^-),$$

- Resonances ? Eigenvalue crossings ?
- Wave dispersion / damping depending on domain geometry ?