Fluctuation-Dissipation Theorem with Application to Climate Change Studies

Xiaoming Wang
Florida State University

NTU International Science Conference on Climate Change: multidecadal and beyond, Sept. 17-20, 2012

Abstract climate models

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X},t) + \sigma(\mathbf{X},t)\dot{W}, \ \mathbf{X} \in \mathbb{R}^N, \ \mathbf{F} = (F_1, \cdots, F_N),$$

$\mathbf{X} \in \mathbb{R}^N, \dot{W} \in \mathbb{R}^M, \sigma \in M^{N\times M}$ $(N \gg 1)$

- Lorenz 63 and 96 atmospheric models with noise
- Barotropic model (2D NS system and related systems) spatially discretized with noise
- Primitive equation systems (Atmospheric GCMs) spatially discretized with noise

* source of noise: stochastic parametrizations (back-scattering from unresolved processes), rounding errors, ...

Climate and Climate Change

- **Climate** is the long time statistics of the system (distribution of the "weather"): invariant measure
- *Bogliubov-Krylov Theorem*: There is (at least one) invariant measure (equilibrium PDF) for most systems. In typical cases there is "physical" (or Sinai-Ruelle-Bowen or SRB) measure on the attractor.

Climate and Climate Change

- **Climate** is the long time statistics of the system (distribution of the "weather"): invariant measure
- *Bogliubov-Krylov Theorem*: There is (at least one) invariant measure (equilibrium PDF) for most systems. In typical cases there is "physical" (or Sinai-Ruelle-Bowen or SRB) measure on the attractor.
- **Climate change** is their response to change in parameters of dynamics

Climate and Climate Change

- **Climate** is the long time statistics of the system (distribution of the "weather"): invariant measure
- *Bogliubov-Krylov Theorem*: There is (at least one) invariant measure (equilibrium PDF) for most systems. In typical cases there is "physical" (or Sinai-Ruelle-Bowen or SRB) measure on the attractor.
- **Climate change** is their response to change in parameters of dynamics
- potential challenge using direct approach
  - development of fast and accurate numerical scheme for climate
  - large system + small time step + long time integration (some notable recent progress)
  - governing equations may not be known although observations may be available

Long time statistics

- $\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X})$, with solution $\mathbf{X}(t)$, climate (invariant measure) μ
- functional $A(\mathbf{X})$

$$\langle A(\mathbf{X}) \rangle = LIM_{t\to\infty} \frac{1}{t} \int_0^t A(\mathbf{X}(s)) ds$$

  statistical equilibrium

- *Birkhoff's Theorem:* Time-averaging and spatial-averaging are equivalent for ergodic system (ergodic invariant measure)

$$LIM_{t\to\infty} \frac{1}{t} \int_0^t A(\mathbf{X}(s)) ds = \langle A \rangle = \int_H A(\mathbf{X}) d\mu(\mathbf{X}), a.s.$$

  (essentially independent of the initial data)

Response operator

- Perturbed climate model $\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}) + \delta\mathbf{f}$, solution $\mathbf{X}^1$, new climate μ^1
- statistics of the perturbed system

$$\langle A(\mathbf{X}^1)\rangle := LIM_{t\to\infty}\frac{1}{t}\int_0^t A(\mathbf{X}^1(s))ds = \int_H A(\mathbf{X})d\mu^1(\mathbf{X})$$

- Changes in long time statistics and the response operator

$$\delta\langle A\rangle = \langle A(\mathbf{X}^1)\rangle - \langle A(\mathbf{X})\rangle := M(\delta f)$$

- For small δf, the response operator M can be expected to be linear

Direct approach to compute the (linear) response operator M

- Precise formula, distributed at statistical equilibrium

$$\delta < A > (\delta f) =< A(\mathbf{X}^1(\delta f)) > - < A(\mathbf{X}) >$$

$$< A(\mathbf{X}^1) > (\delta f) =< A(\mathbf{X}) > + \left[\frac{\partial(\int_H A d\mu^1)(\delta f)}{\partial \delta f}\right]_{\delta f = 0} \delta f$$

$$\delta < A >= M\delta f, \quad M = \left[\frac{\partial(\int A d\mu^1)(\delta f)}{\partial \delta f}\right]_{\delta f = 0}$$

Direct approach to compute the (linear) response operator M

- Precise formula, distributed at statistical equilibrium

$$\delta < A > (\delta f) =< A(\mathbf{X}^1(\delta f)) > - < A(\mathbf{X}) >$$

$$< A(\mathbf{X}^1) > (\delta f) =< A(\mathbf{X}) > + \left[\frac{\partial (\int_H A d\mu^1)(\delta f)}{\partial \delta f} \right]_{\delta f = 0} \delta f$$

$$\delta < A >= M \delta f, \quad M = \left[\frac{\partial (\int A d\mu^1)(\delta f)}{\partial \delta f} \right]_{\delta f = 0}$$

- Not particularly useful

Fluctuation-Dissipation Theorem

- dissipation and fluctuation of a given system are related
- System response to an external perturbation may be expressed in terms of the fluctuation properties of the system in thermal equilibrium
- applies to large system of identical particles (Green-Kubo) among others
- application to climate system proposed by Leith (1975)

FDT theory applied to climate systems

- $$M(t) = \int_0^t \langle R(\tau)\rangle d\tau \quad (\langle A(\mathbf{X}^1(t))\rangle - \langle A(\mathbf{X})\rangle = M(t)\delta f)$$

$$M = M(\infty) = \int_0^\infty \langle R(\tau)\rangle d\tau$$

- For Gaussian equilibrium: $p_e(\mathbf{X}) = c\ \exp(-(C^{-1}(0)\mathbf{X}, \mathbf{X}))$

$$M(t) = \int_0^t \langle A(\mathbf{X}(s+\tau))\mathbf{X}(s)^T\rangle C^{-1}(0)d\tau$$

Einstein's relation

- Brownian motion with friction

$$m\frac{du}{dt} = -m\gamma u + \sigma\frac{dW}{dt}, \frac{dx}{dt} = u$$

- equilibrium distribution

$$p(u) = C\exp[-\frac{mu^2}{2kT}], kT = \frac{\sigma^2}{2m\gamma}$$

- diffusion coefficient

$$\begin{aligned} D &= \lim_{t\to\infty}\frac{1}{2t}\langle|x(t)-x(0)|^2\rangle \\ &= \lim_{t\to\infty}\frac{1}{t}\int_0^t dt_1\int_0^{t-t_1} ds\langle u(t_1)u(t_1+s)\rangle \\ &= \int_0^\infty \langle u(t_0)u(t_0+t)\rangle dt = \frac{\sigma^2}{2m^2\gamma^2} \end{aligned}$$

- Einstein's relation (dissipation-fluctuation relation)

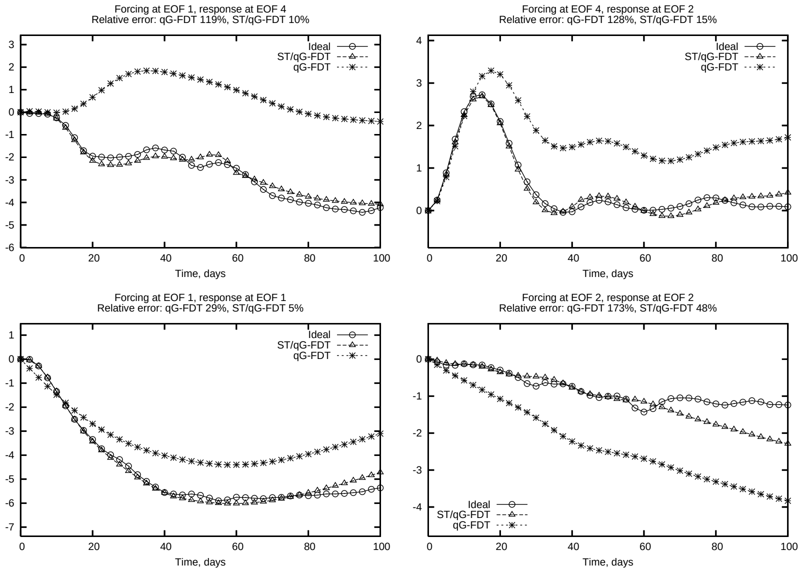
$$\mu = \frac{1}{m\gamma} = \frac{D}{kT} = \frac{1}{kT}\int_0^\infty \langle u(t_0)u(t_0+t)\rangle dt$$

Linear response theory

- $$m\frac{du}{dt} = -m\gamma u + \sigma\frac{dW}{dt} + K(t),\, \frac{dx}{dt} = u,\, K(t) = K_0\cos\omega t$$
- long time linear response
$$\begin{aligned} \delta\langle u(t)\rangle &= \mathcal{R}\mu(\omega)K_0\exp(i\omega t) \\ \mu(\omega) &= \frac{1}{m}\frac{1}{i\omega+\gamma},\ (mobility, admittance) \end{aligned}$$
- $$\begin{aligned} \mu(\omega) &= \frac{1}{m\langle u^2\rangle}\int_0^\infty \langle u(t_0)u(t_0+t)\rangle e^{-i\omega t}dt \\ &= \frac{1}{kT}\int_0^\infty \langle u(t_0)u(t_0+t)\rangle e^{-i\omega t}dt \end{aligned}$$
$$\int_0^\infty \langle u(t_0)u(t_0+t)\rangle e^{-i\omega t}dt = \frac{\langle u^2\rangle}{i\omega+\gamma}$$

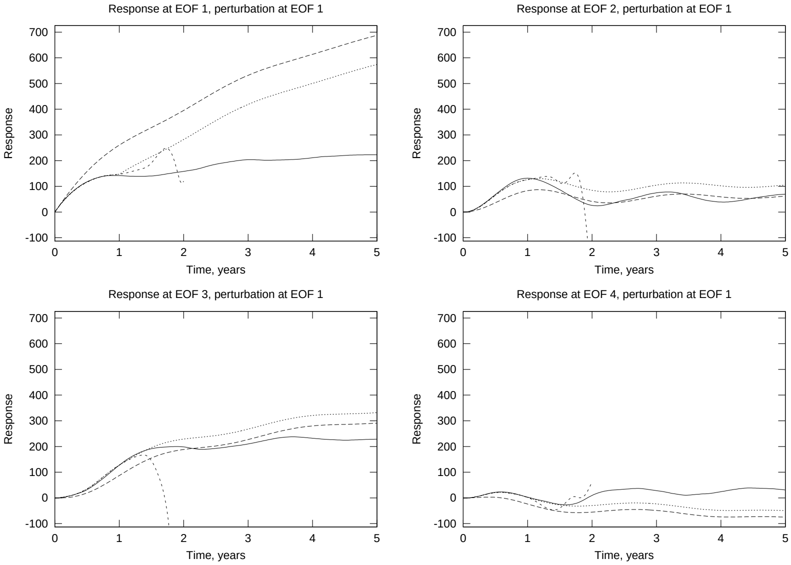
Abramov & Majda, J. Atmos. Sci., 2009

- *"A new algorithm for low frequency climate response"*
- T21 barotropic climate model on a sphere with realistic Earth topography, 500 mbar regime (Selten 1995)
- Able to predict response at four leading EOFs for both mean state and variance with blended ST/qG response


Forcing at EOF 1, response at EOF 4
Relative error: qG-FDT 119%, ST/qG-FDT 10%
Forcing at EOF 4, response at EOF 2
Relative error: qG-FDT 128%, ST/qG-FDT 15%
Forcing at EOF 1, response at EOF 1
Relative error: qG-FDT 29%, ST/qG-FDT 5%
Forcing at EOF 2, response at EOF 2
Relative error: qG-FDT 173%, ST/qG-FDT 48%
Ideal
ST/qG-FDT
qG-FDT
Time, days

Abramov & Majda, JPO 2011

- *"Low Frequency Climate Response of Quasigeostrophic Wind-Driven Ocean Circulation"*
- 1.5-layer quasigeostrophic model with wind stress (McCalpin & Haidvogel, JPO 1996)
- Flow with boundary layer separation like Gulf Stream or Kuroshio
- Four leading EOFs have both jet and meandering patterns


Response at EOF 1, perturbation at EOF 1
Response at EOF 2, perturbation at EOF 1
Response at EOF 3, perturbation at EOF 1
Response at EOF 4, perturbation at EOF 1
Response
Time, years
700
600
500
400
300
200
100
0
-100
0
1
2
3
4
5

Gritsun & Dymnikov & Branstator, 2002, Gritsun & Branstator & Majda 2008

NCAR atmospheric GCM CCM0 (R15, 9 levels, state of the art 1980)

- Data: 4 Million days, perpetual January.
- Response operators constructed for $A = <\psi>, <\psi^2>, <(\psi')^2>, <precip>, <\nabla \cdot \vec{u}>, <u'v>$ $(...)'$ band passed (1-14days) component
- $$M = \int_0^t < A(u(t+\tau))u(t)^T > C^{-1}(0)d\tau$$

Typical dimension reduction for AGCM (CCM0)
9 pressure levels, R15 resolution, independent variables are psi, div, T, Ps, q.

1) Use only T and Psi from all pressure levels.
2) Calculate EOF for each data field. Project T and Psi onto 300 (T) and 100 (Psi) leading EOFs. Operator dimension goes to 3600 (from 20000).
3) Calculate 3D EOFS of the 3600- component vector. Project data onto 2000 leading 3D EOFs.
4) Calculate covariances in the space of 2000 leading 3D EOFs.

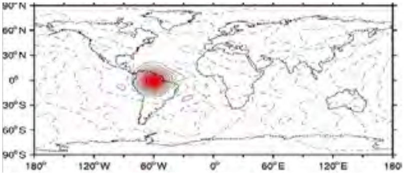


Figure: δf

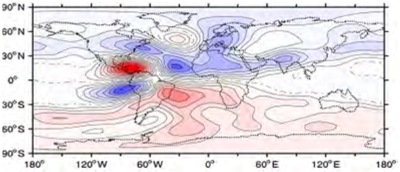


Figure: $\delta < \psi >$

$$GCM + \delta f = \delta < \psi >$$

Can FDT predict this response?
$M\delta f$

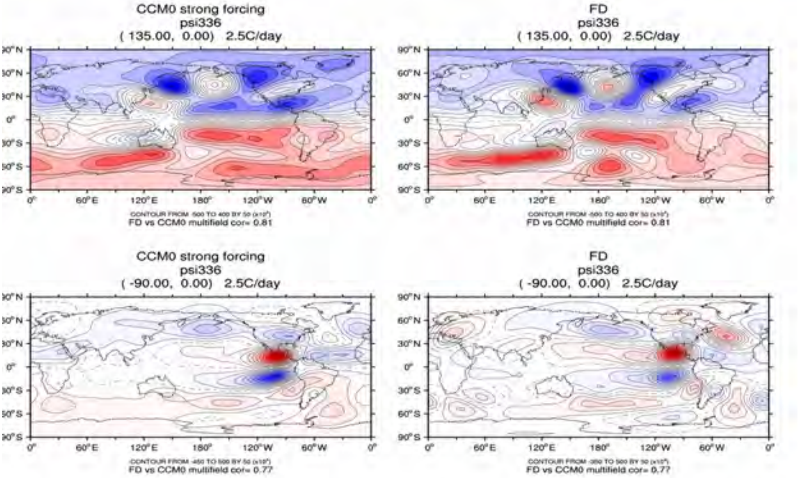


Figure: CCM0 (left) and FDT operator $M\delta f$ (right) responses to 2 equator heating anomalies (Psi336).

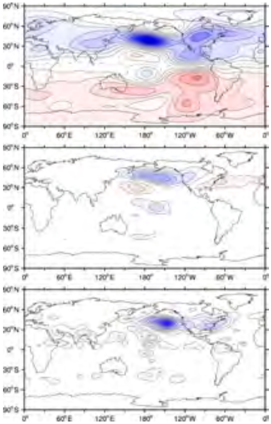


Figure: AGCM

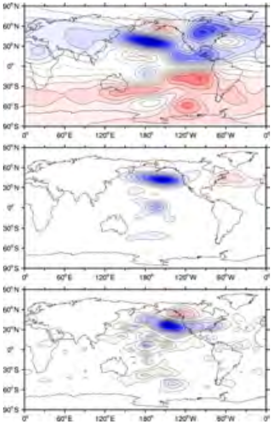


Figure: FDT

Response of the streamfunction, high frequency variance and meridional momentum flux onto the heating at (165W,0N), sigma=336, $\bar{\psi}_{100}$(top), $Var\psi^{bp}_{100}$(middle), $u^{bp}_{300}v^{bp}_{300}$(bottom).

Set-up of the problem with seasonal impact

- Generic finite dimensional (random) dynamical system (climate model)

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}, t) + \sigma(\mathbf{X}, t)\dot{W}, \ \mathbf{X} \in \mathbb{R}^N, \ \mathbf{F} = (F_1, \cdots, F_N),$$

$\mathbf{X} \in \mathbb{R}^N, \dot{W} \in \mathbb{R}^M, \sigma \in M^{N\times M} \ (N \gg 1)$

- **Fokker-Planck equation**

$$\frac{\partial \bar{p}}{\partial t} = -\nabla \cdot (\bar{p}\mathbf{F}) + \frac{1}{2}\nabla \cdot \nabla \cdot (Q\bar{p})(:\overset{def}{=} L_{FP}\bar{p}),$$

$$\bar{p}(\mathbf{X}, t)\Big|_{t=0} = \bar{p}_0(\mathbf{X}).$$

$Q = \sigma\sigma^T \geq 0, \ Q \in M^{N\times N}.$

- Fokker-Planck equation reduces to **Liouville equation** if $\sigma \equiv 0$.
- Example (seasonal cycle of solar heating): $\mathbf{F}(\mathbf{X}, t) = \mathbf{F}(\mathbf{X}) + \mathbf{f}(t), \sigma(\mathbf{X}, t) = \sigma(\mathbf{X})$

Issues

- Is the climate (distribution of the weather) unique?
- How does the climate change under perturbation?
- Is there any easy way to compute/estimate the change of statistical quantities (mean temperature) due to perturbation to the system the initial data, forcing, or noise $(p_0, \mathbf{F}, \sigma)$ via past history of the system?
- Stationary forcing case is known (unique ergordicity, Green-Kubo formula, ...)

- Uniqueness of invariant measure: Vishik, Fursikov, Flandoli, Maslov, Kuksin, Shirikyan, Da Prato, Zabzcyk, Debussche, Mattingly, E, Sinai, Msmoudi, Young, Hairer, Liu, Ekmann, ...
- Fluctuation-dissipation theory applied to climate: Leith, Bell, Dymnikov, Gritsun, Branstator, Franzke, Majda, Abramov, ...

Skew product

- Assume $\mathbf{F}, \sigma$ periodic in t with period T_0

$$\begin{aligned} \frac{d\mathbf{X}}{dt} &= \mathbf{F}(\mathbf{X}, s) + \sigma(\mathbf{X})\dot{W}, \\ \frac{ds}{dt} &= 1, s \in \mathbb{S}^1 = \mathbb{R}^1/\text{mod}\, T_0 \end{aligned}$$

- Alternative form

$$\frac{d\hat{\mathbf{X}}}{dt} = \hat{\mathbf{F}}(\hat{\mathbf{X}}) + \hat{\sigma}(\hat{\mathbf{X}})\dot{W}, \;\; \hat{\mathbf{X}} \in \mathbb{R}^N \times \mathbb{S}^1,$$

$$\hat{\mathbf{X}} = \begin{pmatrix} \mathbf{X} \\ s \end{pmatrix} \quad \hat{\mathbf{F}}(\hat{\mathbf{X}}) = \begin{pmatrix} \mathbf{F}(\mathbf{X}, s) \\ 1 \end{pmatrix}, \quad \hat{\sigma}(\hat{\mathbf{X}}) = \begin{pmatrix} \sigma(\mathbf{X}, s) \\ 0 \end{pmatrix}.$$

- Skew-product FPE

$$\begin{aligned} \frac{\partial \hat{p}}{\partial t} &= -\nabla \cdot (\hat{p}\mathbf{F}(\mathbf{X}, s)) - \frac{\partial \hat{p}}{\partial s} + \frac{1}{2}\nabla \cdot \nabla \cdot (Q\hat{p}) (:\stackrel{def}{=} \hat{L}_{FP}\hat{p}), \\ \hat{p}(\hat{\mathbf{X}}, t)\Big|_{t=0} &= \bar{p}_0(\mathbf{X}) \times \delta_0(s) \end{aligned}$$

PDF relationship

- $\bar{p}(\mathbf{X}, t)$: solution of the time-dependent FPE $\Rightarrow$

$$\hat{p}(\hat{\mathbf{X}}, t) = \bar{p}(\mathbf{X}, t) \times \delta_0(s - t)$$

  is a solution to the skew-product FPE
- $\hat{p}(\hat{\mathbf{x}}, t)$: smooth solution of the skew-product FPE $\Rightarrow$

$$p(\mathbf{x}, t) \stackrel{def}{=} T_0 \hat{p}(\begin{pmatrix} \mathbf{x} \\ t \end{pmatrix}, t)$$

  is a solution to the time-dependent FPE
- $\hat{p}^{eq}(\hat{\mathbf{x}})$: equilibrium solution of the skew-product FPE $\Rightarrow$

$$p_{per}(\mathbf{x}, t) \stackrel{def}{=} T_0 \hat{p}^{eq}(\begin{pmatrix} \mathbf{x} \\ t \end{pmatrix})$$

  is a time periodic solution to the time-dependent FPE with period T_0.

- Comparison

	original formulation	skew-product
	non-stationary	stationary
	full rank noise	degenerate noise
	smooth pdf	singular pdf

- Strategy for linear response/FDT theory for time periodic system: use skew-product system but need to deal with singular pdf and time shift (phase)

Time periodic climate

- Fact(Majda&W.2010): Dissipative system possesses at least one time periodic statistical solution p_{per} which is associated with a statistical equilibrium $\hat{p}^{eq}$ of the skew-product system
- Fact(Majda&W.2010): Dissipative system + generic noise ($\text{rank}(Q) = N$), then p_{per} captures all asymptotic statistical properties of the original system in the sense that for any statistical solution p

$$\lim_{t\to\infty} \mathcal{P}(p(t), p_{per}(t)) = \lim_{t\to\infty} \mathcal{P}(p_{per}(t), p(t)) = 0.$$

$$\mathcal{P}(p_1, p_2) = \int p_1(\mathbf{X}) \ln \frac{p_1(\mathbf{X})}{p_2(\mathbf{X})} \, d\mathbf{X}$$

- Fact(Majda&W.2010):

$$\int \Phi(\mathbf{x}) p_{per}(\mathbf{x}, s_0) \, d\mathbf{x} = \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \Phi(\mathbf{X}(s_0 + kT_0)), a.s.$$

- Prototype dissipative system

$d\mathbf{X}$

Perturbed system

- Perturbed system

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}, t) + \mathbf{a}(\mathbf{X}) \bullet \delta\tilde{\mathbf{F}}(t) + (\sigma(\mathbf{X}) + \delta\tilde{\sigma}(\mathbf{X}))\dot{W},$$

$\mathbf{a} \bullet \mathbf{w}$: Hadamard (or Schur, or entrywise) product

$$(\mathbf{a} \bullet \mathbf{w})_j = a_j w_j.$$

- **perturbed FPE**

$$\begin{aligned}\frac{\partial \hat{p}^\delta}{\partial t} &= -\nabla \cdot (\hat{p}^\delta \mathbf{F}) + \frac{1}{2}\nabla \cdot \nabla \cdot (Q\hat{p}^\delta) - \frac{\partial \hat{p}^\delta}{\partial s} - \delta\nabla \bullet (\mathbf{a}(\mathbf{X})\hat{p}^\delta) \\ &\quad + \frac{\delta^2}{2}\nabla \cdot \nabla \cdot (\tilde{Q}\hat{p}^\delta) + \frac{\delta}{2}\nabla \cdot \nabla \cdot ((\sigma\tilde{\sigma}^T + \tilde{\sigma}\sigma^T)\hat{p}^\delta), \\ \hat{p}^\delta(\mathbf{X}, 0) &= \hat{p}_0^\delta = \bar{p}_0(\mathbf{X}) \times \delta_0(s) + \delta p_0'(\mathbf{X}) \times \delta_0(s),\end{aligned}$$

$\tilde{Q} = \tilde{\sigma}^T\tilde{\sigma}$, $\delta p_0'$: initial errors in mean, variance, etc.

linear response calculation

- Assume

$$\hat{p}^{\delta} = \bar{\hat{p}} + \delta \hat{p}' + \mathcal{O}(\delta^2).$$

- **Approximate Linear Response Dynamics** (sensitivity)

$$\begin{aligned} \frac{\partial \hat{p}'}{\partial t} &= -\nabla \cdot (\hat{p}' \mathbf{F}) - \frac{\partial \hat{p}'}{\partial s} + \frac{1}{2} \nabla \cdot \nabla \cdot (Q \hat{p}') \\ &\quad -\nabla \bullet (\mathbf{a}(\mathbf{X}) \bar{\hat{p}}) \cdot \tilde{\mathbf{F}}(t) + \frac{1}{2} \nabla \cdot \nabla \cdot ((\sigma \tilde{\sigma}^T + \tilde{\sigma} \sigma^T) \bar{\hat{p}}) \\ &\overset{def}{=} \hat{L}_{FP} \hat{p}' + \mathbf{L_a} \bar{\hat{p}} \cdot \tilde{\mathbf{F}} + L_{\sigma} \bar{\hat{p}}, \\ \hat{p}' \Big|_{t=0} &= p_0'(\mathbf{X}) \times \delta_0(s). \end{aligned}$$

$$\mathbf{L_a} p = -\nabla \bullet (\mathbf{a} p), \quad L_{\sigma} p = \frac{1}{2} \nabla \cdot \nabla \cdot ((\sigma \tilde{\sigma}^T + \tilde{\sigma} \sigma^T) p).$$

- **perturbative pdf**

$$\hat{p}'(t) = e^{t \hat{L}_{FP}} \hat{p}_0' + \int_0^t [e^{(t-\tau)\hat{L}_{FP}} \mathbf{L_a} \bar{\hat{p}}(\tau)] \cdot \tilde{\mathbf{F}}(\tau)\, d\tau + \int_0^t e^{(t-\tau)\hat{L}_{FP}} L_{\sigma} \bar{\hat{p}}(\tau)$$

Perturbation in statistics

- Statistics $A(\hat{\mathbf{X}})$

$$
\begin{aligned}
E^\delta(A)(t) &= \int A(\hat{\mathbf{X}},t)\hat{p}^\delta(\hat{\mathbf{X}},t)\,d\hat{\mathbf{X}} = E^0 + \delta E' + \mathcal{O}(\delta^2) \\
\delta E'(t) &= \delta \int A(\hat{\mathbf{X}},t)\hat{p}'(\hat{\mathbf{X}},t)\,d\hat{\mathbf{X}}
\end{aligned}
$$

- Example:

$$
\begin{aligned}
\bar{p} &= p^G(\mathbf{x}) = \mathcal{Z}^{-1}\exp(-\beta E(\mathbf{x})), \\
A(\mathbf{x}) &= |\mathbf{x}|^2 \\
\delta E'(A) &= \text{"change in temperature"}
\end{aligned}
$$

Perturbation in stat:II

- $$\begin{aligned} E'(A)(t) &\stackrel{def}{=} \int \hat{p}_0'(\mathbf{x}) e^{t\hat{L}_{FP}^T} A(\hat{\mathbf{x}})\, d\hat{\mathbf{x}} \\ &+ \int_0^t \vec{R}_{\mathbf{a},A}(t,\tau) \cdot \tilde{\mathbf{F}}(\tau)\, d\tau + \int_0^t R_{\sigma,A}(t,\tau)\, d\tau. \\ \vec{R}_{\mathbf{a},A}(t,\tau) &= \int \mathbf{L}_{\mathbf{a}}^T [e^{(t-\tau)\hat{L}_{FP}^T} A(\hat{\mathbf{x}})] \bar{\hat{p}}(\hat{\mathbf{x}},\tau)\, d\hat{\mathbf{x}} \\ R_{\sigma,A}(t,\tau) &= \int [L_\sigma^T e^{(t-\tau)\hat{L}_{FP}^T} A(\hat{\mathbf{x}})] \bar{\hat{p}}(\hat{\mathbf{x}},\tau)\, d\hat{\mathbf{x}}. \end{aligned}$$
- Fast decay of the correlation functions needed for FDT approach be applicable

Correlation representation of linear response

- Fact(Majda&W.2010): assume smooth positive $\bar{\hat{p}}$

$$\vec{R}_{\mathbf{a},A}(t,\tau) = < A(\hat{\mathbf{X}}(t))\hat{\mathbf{B}}_{\mathbf{a}}(\hat{\mathbf{X}}(\tau)) >,$$
$$R_{\sigma,A}(t,\tau) = < A(\hat{\mathbf{X}}(t))\hat{B}_{\sigma}(\hat{\mathbf{X}}(\tau)) >,$$

$$\hat{\mathbf{B}}_{\mathbf{a}}(\hat{\mathbf{X}},\tau) = \frac{\mathbf{L}_{\mathbf{a}}\bar{\hat{p}}(\hat{\mathbf{X}},\tau)}{\bar{\hat{p}}(\hat{\mathbf{X}},\tau)}, \quad \hat{B}_{\sigma}(\hat{\mathbf{X}},\tau) = \frac{L_{\sigma}\bar{\hat{p}}(\hat{\mathbf{X}},\tau)}{\bar{\hat{p}}(\hat{\mathbf{X}},\tau)},$$

- Fact(Majda&W.2010): assume smooth positive $\bar{p}$ $(\hat{p}(\hat{\mathbf{X}},t) = \bar{p}(\mathbf{X},t) \times \delta_0(s-t))$

$$\vec{R}^{T}_{\mathbf{a},A}(t,\tau) = < A(\hat{\mathbf{X}}(t))\mathbf{B}_{\mathbf{a}}(\hat{\mathbf{X}}(\tau)) >$$
$$R_{\sigma,A}(t,\tau) = < A(\hat{\mathbf{X}}(t))B_{\sigma}(\hat{\mathbf{X}}(\tau)) >,$$

$$\mathbf{B}_{\mathbf{a}}(\hat{\mathbf{X}},\tau) = \frac{L_{\mathbf{a}}\bar{p}(\mathbf{X},\tau)}{\bar{p}(\mathbf{X},\tau)}, \quad B_{\sigma}(\hat{\mathbf{X}},\tau) = \frac{L_{\sigma}\bar{p}(\mathbf{X},\tau)}{\bar{p}(\mathbf{X},\tau)}.$$

Special cases

- Perturbation away from equilibrium $L_{FP}\bar{p} \equiv 0$,

$$R_{A,B}(t,\tau) = < A(\mathbf{X}(t))B(\mathbf{X}(\tau)) >= R_{A,B}(t-\tau,0)$$
$$= \lim_{T\to\infty} \frac{1}{T}\int_{T_0}^{T+T_0} A(\mathbf{X}(s+t-\tau))B(\mathbf{X}(s))\,ds$$

- $\bar{p}$ is Gaussian and external forcing perturbation only $\mathbf{a}(\mathbf{x}) \equiv \mathbf{a}, \tilde{\mathbf{F}}(t) \equiv \tilde{\mathbf{F}}$, change in mean $(A(\mathbf{x}) = \mathbf{x})$

$$\mathbf{B_a}\bar{p} = \text{linear function in } \mathbf{x}$$

$$R_{A,\mathbf{B}} :\approx \int \mathbf{x}(t) \otimes \mathbf{x}(\tau)\bar{p}(\mathbf{x},\tau)\,d\mathbf{x}$$

- Combination of the two

$$R_{A,\mathbf{B}}(t,0) :\approx \text{auto-correlation}$$

- Fluctuation-Dissipation interpretation: for stationary solution to Langevin equation $\frac{dv}{dt} + \gamma v = \sigma \frac{dW}{dt}$

$$< v(t_2)v(t_1) > = \frac{\sigma}{2\gamma} e^{-\gamma|t_1-t_2|}$$

Zero Noise Van Kampen Adjoint Form, initial data

- SDE $\Rightarrow$ ODE, adjoint FPE $\Rightarrow$ linear transport equation
- $$e^{t\hat{L}_{FP}^{T}} A(\hat{\mathbf{x}}, t) = A(\hat{\mathbf{X}}(\hat{\mathbf{x}}, t), t)$$
- $$\int \hat{p}_0'(\hat{\mathbf{x}}) e^{t\hat{L}_{FP}^{T}} A(\hat{\mathbf{x}})\, d\hat{\mathbf{x}} = \int p_0'(\mathbf{x}) A(\hat{\mathbf{X}}(\begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix}, t))\, d\mathbf{x}.$$

Zero Noise Tangent map approach

$$e^{(t-\tau)\hat{L}_{FP}^T} A(\hat{\mathbf{x}}) = A(\hat{\mathbf{X}}(\hat{\mathbf{x}}, t-\tau))$$

- Linear response operator

$$\vec{R}^T_{\mathbf{a},A}(t,\tau) = \vec{R}^T(t,\tau)$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{S}^1} \nabla_{\mathbf{x}} A(\hat{\mathbf{X}}(\begin{pmatrix} \mathbf{x} \\ s \end{pmatrix}, t-\tau)) \bullet \mathbf{a}(\mathbf{x})\, \bar{\hat{p}}(\begin{pmatrix} \mathbf{x} \\ s \end{pmatrix}, \tau)\, ds\, d\mathbf{x}.$$

- Tangent map

$$\nabla_{\mathbf{x}} \mathbf{X}(\hat{\mathbf{x}}, t'+t)$$
$$= \exp(\int_{t'}^{t'+t} \nabla_{\mathbf{X}} \mathbf{F}(\mathbf{X},\tau)\Big|_{\mathbf{X}=\mathbf{X}(\hat{\mathbf{x}},\tau)} d\tau) \nabla_{\mathbf{x}} \mathbf{X}(\hat{\mathbf{x}}, t') (\stackrel{def}{=} T^t_{(\hat{\mathbf{x}},t')} \nabla_{\mathbf{x}} \mathbf{X}(\hat{\mathbf{x}}$$

- Case $\bar{\hat{p}} = \bar{p} \times \delta_0(s-t), A(\hat{\mathbf{X}}) = A(\mathbf{X})$

$$\vec{R}^T_{\mathbf{a},A}(t,\tau) = \int \nabla_{\mathbf{x}} A(\mathbf{X}_\tau(\mathbf{x}, t-\tau)) \bullet \mathbf{a}(\mathbf{x}) \bar{p}(\mathbf{x},\tau)\, d\mathbf{x}$$

$\mathbf{X}_\tau(\mathbf{x}, t-\tau)$: $\mathbf{X}$ component of $\hat{\mathbf{X}}(\begin{pmatrix} \mathbf{x} \\ \tau \end{pmatrix}, t-\tau)$

- Case ensemble prediction

$$\bar{p}(\mathbf{X}, t) = \sum_{j=1}^{R} p_j \delta(\mathbf{X} - \mathbf{X}_j(t)), \sum_{j=1}^{R} p_j = 1$$

$$\vec{R}^T_{\mathbf{a},A}(t,\tau) = \sum_{j=1}^{R} p_j \nabla_{\mathbf{x}} A(\mathbf{X}_\tau(\mathbf{x}, t-\tau)) \bullet \mathbf{a}(\mathbf{x}) \Big|_{\mathbf{x}=\mathbf{x}_j(\tau)}$$

- Case $\bar{\hat{p}} = \hat{p}^{eq}$

$$\begin{aligned} \vec{R}^T_{\mathbf{a},A}(t) &= \int_{\mathbb{R}^N} \left(\int_{\mathbb{S}^1} \nabla_{\mathbf{x}} A(\mathbf{X}(\begin{pmatrix} \mathbf{x} \\ s \end{pmatrix}, t)) \bullet \mathbf{a}(\mathbf{x})\, p_{per}(\mathbf{x}, s)\, ds \right) d\mathbf{x} \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{T^*}^{T+T^*} \nabla_{\mathbf{x}} A(\mathbf{X}(\hat{\mathbf{x}}, t+\tau)) \bullet \mathbf{a}(\mathbf{X}(\hat{\mathbf{x}}, \tau))\, d\tau \end{aligned}$$

Quasi-Gaussian Approximation

- Approximate $\bar{p}$ via a Gaussian $p^G(\mathbf{X}, t)$ with the same mean $(\bar{\mathbf{X}}(t))$ and second moments (covariance matrix $\mathcal{C}$)
- Approximate the linear response operators

$$\begin{aligned}(\vec{R}^G_{\mathbf{a},A})^T(t,\tau) &= < A(\hat{\mathbf{X}}(t))\mathbf{B}^G_{\mathbf{a}}(\mathbf{X}(\tau)) >,\\ R^G_{\sigma,A}(t,\tau) &= < A(\hat{\mathbf{X}}(t))B^G_{\sigma}(\mathbf{X}(\tau)) >,\end{aligned}$$

$$\mathbf{B}^G_{\mathbf{a}}(\mathbf{X},\tau) = \frac{\mathbf{L}_{\mathbf{a}} p^G(\mathbf{X},\tau)}{p^G(\mathbf{X},\tau)}, \quad B^G_{\sigma}(\mathbf{X},\tau) = \frac{L_{\sigma} p^G(\mathbf{X},\tau)}{p^G(\mathbf{X},\tau)}.$$

- Ensemble approximation

$$\begin{aligned}\vec{R}^G_{\mathbf{a},A}(t,\tau) &= \sum_{j=1}^{R} p_j A(\mathbf{X}_j(t))\mathbf{B}^G_{\mathbf{a}}(\mathbf{X}_j(\tau)),\\ R^G_{\sigma,A}(t,\tau) &= \sum_{j=1}^{R} p_j A(\mathbf{X}_j(t))B^G_{\sigma}(\mathbf{X}_j(\tau)).\end{aligned}$$

- Short time accuracy for linear functional

- special functionals

$$A\begin{pmatrix} \mathbf{x} \\ s \end{pmatrix} = \tilde{A}(\mathbf{x})\psi(s)$$

- Linear response operator (around statistical equilibrium)

$$\begin{aligned}
\vec{R}^{T}_{\mathbf{a},A}(t) &= < A(\hat{\mathbf{X}}(t))\hat{B}_{\mathbf{a}}(\hat{\mathbf{X}}(0)) > \\
&= \mathbb{E}\int_{\mathbb{R}^N}\int_{\mathbb{S}^1} A(\hat{\mathbf{X}}(t))\hat{B}_{\mathbf{a}}(\hat{\mathbf{X}}(0))\hat{p}^{eq}(\hat{\mathbf{x}})\, ds\, d\mathbf{x} \\
&= \mathbb{E}\frac{1}{T_0}\int_{\mathbb{R}^N}\int_0^{T_0} \tilde{A}(\mathbf{X}(t,s,\mathbf{x}))\psi(s+t)\hat{B}_{\mathbf{a}}(\mathbf{x},s)p_{per}(\mathbf{x},s)\, ds
\end{aligned}$$

Computational Algorithm 1

- $$s_j = \frac{(2j-1)T_0}{2L}, 1 \le j \le L$$
- $$\begin{aligned} & \vec{R}^{T}_{\mathbf{a},A}(t) \\ \approx \quad & \frac{1}{L}\sum_{j=1}^{L}\int_{\mathbb{R}^N} \mathbb{E}\tilde{A}(\mathbf{X}(t,s_j,\mathbf{x})\psi(s_j+t)\hat{B}_{\mathbf{a}}(\mathbf{x},s_j)\,d\mathbf{x} \\ \approx \quad & \frac{1}{LK}\sum_{j=1}^{L}\sum_{k=1}^{K}\mathbb{E}\tilde{A}(\mathbf{X}(t,s_j,\mathbf{x}(s_j+kT_0))\psi(s_j+t+kT_0)\hat{B}_{\mathbf{a}}(\mathbf{x}(s_j+ \end{aligned}$$
-
  - Direct FDT algorithm: $N \le 4$, $\hat{B}^{eq}_{\mathbf{a}}(\mathbf{x}, s_j) = \frac{\mathbf{L_a} p_{per}(\mathbf{x},s_j)}{p_{per}(\mathbf{x},s_j)}$
  - Quasi-Gaussian algorithm: $N \gg 1$, $\hat{B}^{G,eq}_{\mathbf{a}}(\mathbf{x}, s_j) = \frac{\mathbf{L_a} p^G_{per}(\mathbf{x},s_j)}{p^G_{per}(\mathbf{x},s_j)}$

Computational Algorithm 2: zero noise

$$
\begin{aligned}
&\vec{R}^T_{\mathbf{a},A}(t) \\
= \;& \int_{\mathbb{R}^N}\int_{\mathbb{S}^1} \nabla_{\mathbf{x}} A(\hat{\mathbf{X}}(\begin{pmatrix} \mathbf{x} \\ s \end{pmatrix}, t)) \bullet \mathbf{a}(\mathbf{x}) \hat{p}^{eq}(\mathbf{x}, s)\, ds\, d\mathbf{x} \\
\approx \;& \frac{1}{L}\sum_{j=1}^{L}\int_{\mathbb{R}^N} \nabla_{\mathbf{x}} A(\hat{\mathbf{X}}(\begin{pmatrix} \mathbf{x} \\ s_j \end{pmatrix}, t)) \bullet \mathbf{a}(\mathbf{x}) p_{per}(\mathbf{x}, s_j)\, d\mathbf{x} \\
\approx \;& \frac{1}{LK}\sum_{j=1}^{L}\sum_{k=0}^{K} \nabla_{\mathbf{x}} A(\hat{\mathbf{X}}(\begin{pmatrix} \mathbf{x}(s_j + kT_0) \\ s_j \end{pmatrix}, t)) \bullet \mathbf{a}(\mathbf{x}(s_j + kT_0))
\end{aligned}
$$

3 mode triad model Gershgorin & Majda 2010

- Exactly solvable 3 mode model

$$
\begin{aligned}
\frac{du_1}{dt} &= -\gamma_1 u_1 + f_1(t) + \sigma_1 \dot{W}_1, \\
\frac{du_2}{dt} &= (-\gamma_2 + i(\omega_0 + a_0 u_1))u_2 + f_2(t) + \sigma_2 \dot{W}_2,
\end{aligned}
$$

- qG-FDT possesses high skill for the mean response to the changes in forcing even in highly non-Gaussian regime.
- qG-FDT performance not so good for the variance response to the perturbations of dissipation in the strongly non-Gaussian regime.

Information content

- Fact(Majda&W.2010): For time independent perturbation

$$
\begin{aligned}
&\mathcal{P}(\hat{p}^{\delta}(T), \bar{\hat{p}}(T)) \\
= \quad & -\frac{\delta^2}{2}\int_0^T \int \bar{\hat{p}} \nabla \frac{\hat{p}'}{\bar{\hat{p}}} \cdot Q \nabla \frac{\hat{p}'}{\bar{\hat{p}}} + \delta^2 \int_0^T \int \frac{\hat{p}'}{\bar{\hat{p}}} \mathbf{L_a} \bar{\hat{p}} \cdot \tilde{\mathbf{F}} \\
& + \delta^2 \int_0^T \int \frac{\hat{p}'}{\bar{\hat{p}}} L_{\sigma} \bar{\hat{p}} + \frac{\delta^2}{2} \int \frac{\hat{p}_0'(\hat{\mathbf{x}})^2}{\bar{\hat{p}}_0(\hat{\mathbf{x}})} + \mathcal{O}(\delta^3).
\end{aligned}
$$

- Most sensitive direction without perturbation in initial distribution can be derived similar to the stationary situation in the case of zero noise or the case of no perturbation in noise.

Conclusion

- Linear response FDT theory can be extended to the case with time periodic forcing
-
    - Accurate and efficient algorithm?
    - Application to (high dimension) climate models ?
    - Generalisation to infinite dimensional model?
    - Impact of model errors?

Thank You!