Response, Stochasticity and Memory Effects in Multi-level Dynamical Systems

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Goals

- Recent results of the perturbation theory for non-equilibrium statistical mechanics
  - Deterministic & Stochastic Perturbations
- General properties
- Applications on system of geophysical fluid dynamical interest
  - Lorenz 96 – various observables
- What is a parametrization?
  - Mori Zwazig and Ruelle approaches
- Try to convince you this is a useful framework

Major theoretical challenges for complex, non-equilibrium systems

- Mathematics: Stability prop of time mean state say nothing on the prop of the system
  - Cannot define a simple theory of the time-mean properties relying only on the time-mean fields.
- Physics: “no” fluctuation-dissipation theorem for a chaotic dissipative system
  - non-equivalence of external/ internal fluctuations ➔ Climate Change is hard to parameterise
- Numerics: Complex systems feature multiscale properties, they are stiff numerical problems, hard to simulate “as they are”

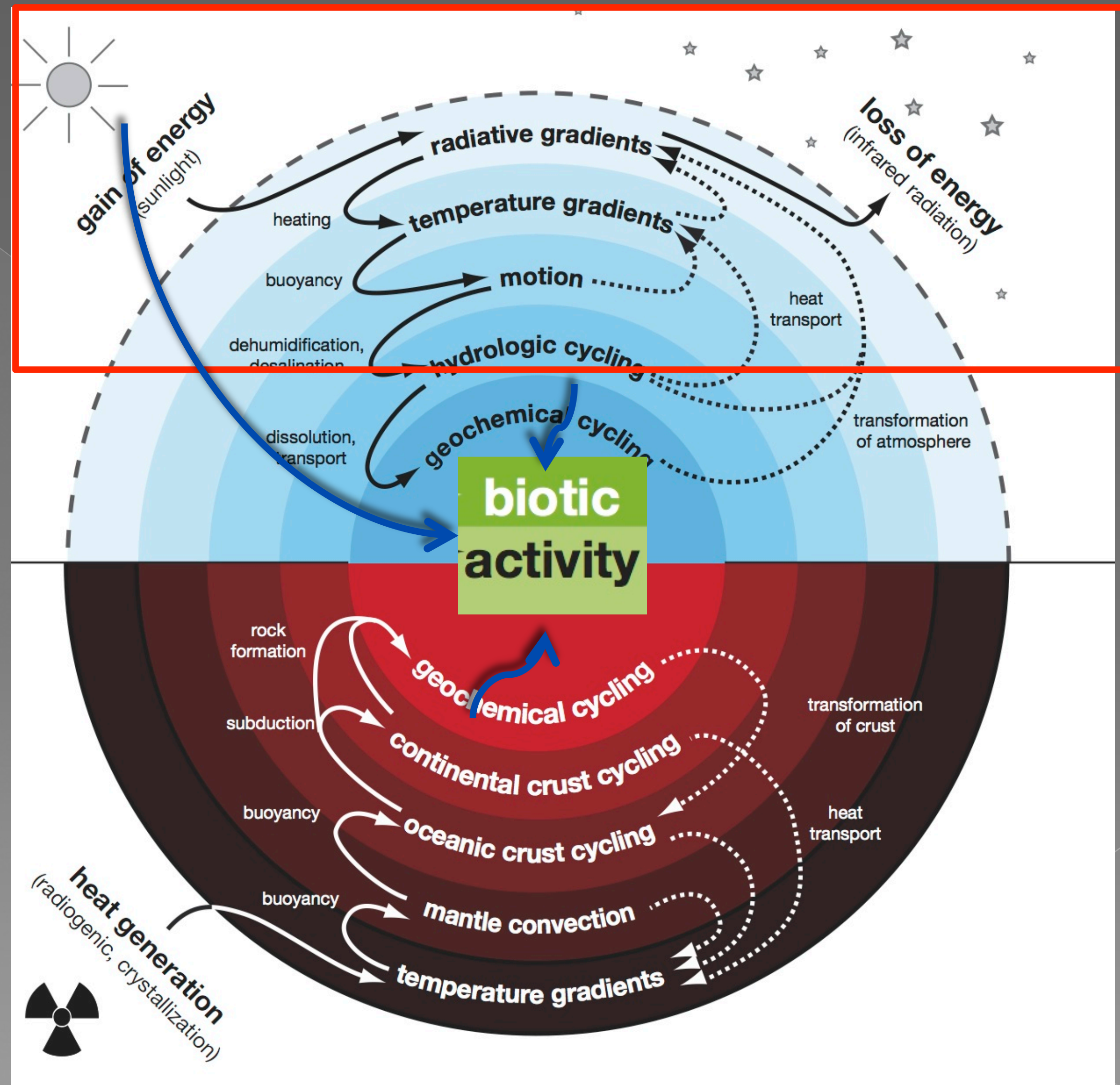
A complex system: our Earth

Nonlinearity

Irreversibility

Disequilibrium

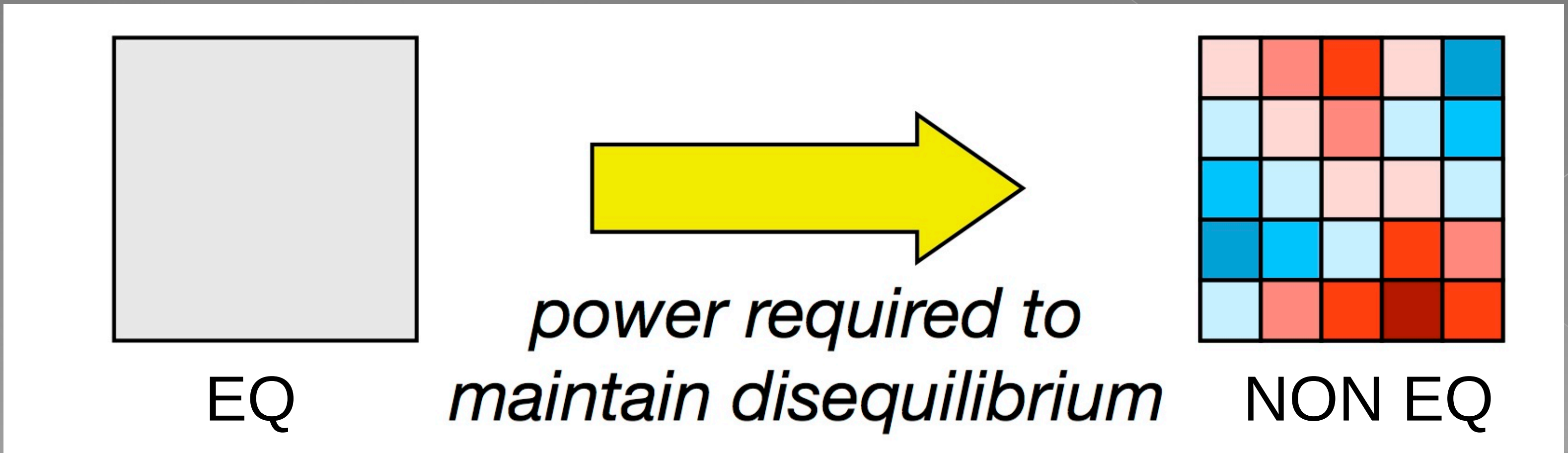
Multiscale



(Kleidon, 2011)

Looking for the big picture

- Global structural properties (Saltzman 2002).
- Deterministic & stochastic dynamical systems
  - Example: stability of the thermohaline circulation
  - Stochastic forcing: ad hoc “closure theory” for noise
- Stat Mech & Thermodynamic perspective
  - Planets are non-equilibrium thermodynamical systems
  - Thermodynamics: large scale properties of climate system; definition of robust metrics for GCMs, data
  - Ergodic theory and much more
  - Stat Mech for Climate response to perturbations



Response theory

- The response theory formalizes a Gedankenexperiment: a system, a measuring device, a clock, and a set of turnable knobs.
- Changes of the statistical properties of a system in terms of the unperturbed system
- Divergence in the response is informative of “tipping points”, where phase transitions occur
- This seems a suitable environment for developing a climate change theory
  - “Blind” use of several CM experiments
  - We struggle with concepts and computations of climate sensitivity and climate response
- We can use the theory for deriving parametrizations!

Perturbations to NESS Systems

Axiom A systems

- Axiom A dynamical systems are very special
  - Include Anosov flows (hyperbolic, struct. stable, dense)
  - Non-wandering set is hyperbolic
  - Periodic points are dense
  - SRB invariant measure: time averages converge in a Lebesgue sense to the ensemble averages for measurable observables

- When we perform numerical simulations, we more or less implicitly set ourselves in these hypotheses
  - Not generic systems, but, following the chaotic hypothesis by Gallavotti and Cohen (1995, 1996), systems with many d.o.f. can be treated as if they were Axiom A systems when macroscopic averages are considered.
  - Extension of ergodic hypothesis
  - These are, in some sense, good physical models!!!

SRB measure

- The invariant measure of the unperturbed system is not absolutely continuous w.r.t. Lebesgue
  - it is so only along the unstable (and neutral) manifold
  - it is singular in the stable directions (effect of the contraction!)
  - Locally, “Cantor set times a smooth manifold”.
  - Kolmogorov measure…
- For deterministic, dissipative chaotic etc. systems FDT does not work
  - It is not possible to write the response as a correlation integral, there is an additional term
  - The system, by definition, never explores the stable directions, whereas a perturbations has components also outside the unstable manifold
- But…

Applicability of FDT

- For deterministic, dissipative chaotic etc. systems FDT does not work
  - It is not possible to write the response as a correlation integral, there is an additional term
  - The system, by definition, never explores the stable directions, whereas a perturbations has components also outside the unstable manifold
- Recent studies (Branstator et al.) suggest that, nonetheless, information can be retrieved
- What is the time needed to build up a statistics such that the FDT gives useful results?
  - Probably, numerical noise also helps
  - The choice of the observable is surely also crucial

Ruelle ('98) Response Theory

- If the Axiom A flow is perturbed as: $\dot{x} = F(x) + e(t)X(x)$
- We can express the expectation value of an observable Φ as: $\langle\Phi\rangle(t) = \langle\Phi\rangle_0 + \sum_{n=1}^{\infty}\langle\Phi\rangle^{(n)}(t)$
- where the n^{th} order perturbation can be expressed as:

$$\langle\Phi\rangle^{(n)}(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} \mathrm{d}\sigma_1\mathrm{d}\sigma_2\dots\mathrm{d}\sigma_n G^{(n)}(\sigma_1,\dots,\sigma_n)e(t-\sigma_1)e(t-\sigma_2)\dots e(t-\sigma_n).$$

This is a perturbative theory…

- with a causal Green function:

$$G^{(n)}(\sigma_1,\ldots,\sigma_n) = \int \rho_{SRB}(\mathrm{d}x) \quad \Theta(\sigma_1)\Theta(\sigma_2-\sigma_1)\ldots\Theta(\sigma_n-\sigma_{n-1}) \times$$
$$\times \Lambda\Pi(\sigma_n-\sigma_{n-1})\ldots\Lambda\Pi(\sigma_2-\sigma_1)\Lambda\Pi(\sigma_1)\Phi(x)$$

  - Expectation value of an operator evaluated over the invariant measure $\rho_{SRB}(\mathrm{d}x)$ of the unperturbed flow!

- where: $\Lambda(\bullet) = X(x)\nabla(\bullet)$ and $\Pi(\tau)A(x) = A(x(\tau))$

  Projection on the perturbation flow

  Unperturbed evolution operator

- Kubo theory (Equil.) is a special case… L. 2008

Linear case

- Perturbation to Φ: $\langle \Phi \rangle^{(1)}(t) = \int d\sigma G_{\Phi}^{(1)}(\sigma) e(t-\sigma)$
- Linear Green: $G_{\Phi}^{(1)}(t) = \int \rho_0(dx)\Theta(t)\Lambda\Pi(t)\Phi$

- Short-term for *G* $G_{\Phi}^{(1)}(t) \approx \alpha\Theta(t)t^{\beta} + o(t^{\beta})$
- Asymptotics for χ $\chi_{\Phi}^{(1)}(\omega) \approx \alpha i^{\beta+1}\beta!\omega^{-\beta-1} + o(\omega^{-\beta-1})$

- If β is even, Im $\{\chi\}$ dominates
- If β is odd, Re $\{\chi\}$ dominates

- Short term behaviour of G → asymptotic behaviour of χ!

Kramers-Kronig relations

- The in-phase and out-of-phase responses are connected by Kramers-Kronig relations:
  - Measurements of the real (imaginary) part of the susceptibility → K-K → best estimate of the imaginary (real)
- Every causal linear model obeys these constraints
- K-K exist also for nonlinear susceptibilities

$$\Re\left\{\chi^{(1)}(\omega)\right\} = \frac{2}{\pi}\wp\int_0^\infty \frac{\omega'\Im\left\{\chi^{(1)}(\omega')\right\}}{\omega'^2-\omega^2}\mathrm{d}\omega$$

$$\Im\left\{\chi^{(1)}(\omega)\right\} = -\frac{2\omega}{\pi}\wp\int_0^\infty \frac{\Re\left\{\chi^{(1)}(\omega')\right\}}{\omega'^2-\omega^2}\mathrm{d}\omega'$$

with $\chi^{(1)}(\omega) = [\chi^{(1)}(-\omega)]^*$

Kramers, 1926; Kronig, 1927

What if we add some noise?

Noise in Numerical Modelling

- Deterministic numerical models are supplemented with additional stochastic forcings.
- Overall practical goals:
  - an approximate but convincing representation of the spatial and temporal scales which cannot be resolved;
  - faster exploration of the attractor of the system, due to the additional “mixing”;
  - Especially desirable when computational limitations
- Fundamental reasons:
  - A good (“physical”) invariant measure of a dynamical system is robust with respect to the introduction of noise
    - exclusion of pathological solutions;
  - Limit of zero noise → statistics of the deterministic system?
  - Noise makes the invariant measure smooth
- A very active, interdisciplinary research sector

Stochastic forcing

- $e(t) = \varepsilon\eta(t) = \varepsilon dW(t)/dt$ where $W(t)$ is a Wiener process
- Therefore, $\langle\eta(t)\rangle = 0$ and $\langle\eta(t)\eta(t')\rangle = \delta(t-t')$
- We obtain:

$$\delta_{\varepsilon,t}\rho(\Phi) = \varepsilon\int d\tau G_\Phi^{(1)}(\tau)\eta(t-\tau) +$$
$$+\varepsilon^2\int d\tau_1 d\tau_2 G_\Phi^{(2)}(\tau_1,\tau_2)\eta(t-\tau_1)\eta(t-\tau_2) + o(\varepsilon^3)$$
$$\ldots.$$
$$\langle\delta_\varepsilon\rho(\Phi)\rangle = \varepsilon^2\int d\tau_1 G_\Phi^{(2)}(\tau_1,\tau_1) + o(\varepsilon^4) =$$
$$= 1/2\,\varepsilon^2\int\rho_0(dx)\int d\tau_1\Theta(\tau_1)X_i\partial_i X_j\partial_j\Phi(f^{\tau_1}x) + o(\varepsilon^4)$$

- The linear correction vanishes; only even orders of perturbations give a contribution
- No time-dependence

Some observations

- The correction to the expectation value of any observable ~ variance of the noise
  - Stochastic system → deterministic system
  - Convergence of the statistical properties is fast
- We have an explicit formula!

Correlations...

- Ensemble average over the realisations of the stochastic processes of the expectation value of the time correlation of the response of the system:

$$\left\langle \int d\sigma \delta_{\varepsilon,\sigma} \rho(\Phi) \delta_{\varepsilon,\sigma-t} \rho(\Phi) \right\rangle = \varepsilon^2 \int d\sigma d\tau_1 d\tau_2 G_\Phi^{(1)}(\sigma-\tau_1) G_\Phi^{(1)}(\sigma-t-\tau_2) \langle \eta(\tau_1) \eta(\tau_2) \rangle + o(\varepsilon^4)$$
$$= \varepsilon^2 \int d\sigma G_\Phi^{(1)}(\sigma) G_\Phi^{(1)}(\sigma-t) + o(\varepsilon^4)$$

- Leading order is proportional to ε^2
- It is the convolution product of the linear Green function!

So what?

- Computing the Fourier Transform we obtain:

$$\left\langle \left| \delta_{\varepsilon,\omega} \rho(\Phi) \right|^2 \right\rangle \approx \varepsilon^2 \left| \chi_{\Phi}^{(1)}(\omega) \right|^2$$

- Interestingly, we end up with the linear susceptibility...
- Let's rewrite he equation:

$$\left\langle P_{\varepsilon,\omega}(A) \right\rangle - P_{\omega}(A) \approx \left\langle \left| \delta_{\varepsilon,\omega} \rho(A) \right|^2 \right\rangle \approx \varepsilon^2 \left| \chi_1(\omega) \right|^2$$

- So: difference between the power spectrum of the signal in the cases with and without noise → linear susceptibility
  - Stoch forcing enhances the Power Spectrum

With some complex analysis

- We know that $\chi_{\Phi}^{(1)}(\omega)$ is analytic in the upper complex plane
- So is $\log\left[\chi_{\Phi}^{(1)}(\omega)\right] = \log\left[\left|\chi_{\Phi}^{(1)}(\omega)\right|\right] + i\arg\left[\chi_{\Phi}^{(1)}(\omega)\right]$
  - Apart from complex zeros...
- The real ($\log\left|\chi_{\Phi}^{(1)}(\omega)\right|$) and imag ($\arg\left[\chi_{\Phi}^{(1)}(\omega)\right]$) obey KK relations
  - From the observation of the power spectra we obtain the real part
  - With KK analysis we obtain the imaginary part
- We can reconstruct the linear susceptibility!
- And from it, the Green function

Extensions: Δ (PS)>0 (noise)

- If $dx_i/dt = F_i(x) \to dx_i/dt = F_i(x) + \sum_{j=1}^{p} \varepsilon_j X_i^j(x)\eta_j(t)$ $\quad \langle \eta_i(t)\eta_j(t')\rangle = C_{ij}\delta(t-t')$

$$\langle P_{\{\varepsilon\},\omega}(A)\rangle - P_\omega(A) \approx \left\langle \left|\delta_{\{\varepsilon\},\omega}\rho(A)\right|^2 \right\rangle \approx \sum_{l=1}^{p} \varepsilon_k^2 \left|\chi_1^k(\omega)\right|^2 + \sum_{l>m=1}^{p} \varepsilon_l \varepsilon_m C_{l,m} \left[\chi_1^l(\omega)\left(\chi_1^m(\omega)\right)^* + \left(\chi_1^l(\omega)\right)^* \chi_1^m(\omega)\right]$$

- If $dx_i/dt = F_i(x) \to dx_i/dt = F_i(x) + X_i(x)\eta(t)$ $\quad \langle \eta(\tau_1)\eta(\tau_2)\rangle = D(\tau_2 - \tau_1)$

$$\langle P_{\varepsilon,\omega}(A)\rangle - P_\omega(A) \approx \left\langle \left|\delta_{\varepsilon,\omega}\rho(A)\right|^2 \right\rangle \approx \left|\chi_1(\omega)\right|^2 D(\omega)$$

- If $dx_i/dt = F_i(x) \to dx_i/dt = F_i(x) + X_i(x,t)$ $\to$ Schauder Dec.

$$\langle P_{\varepsilon,\omega}(A)\rangle - P_\omega(A) \approx \left\langle \left|\delta_{\{\varepsilon\},\omega}\rho(A)\right|^2 \right\rangle \approx \sum_{l=1} \left|\chi_l^1(\omega)\right|^2 D_{l,l}(\omega) + \sum_{l>m=1} D_{l,m}(\omega)\chi_1^l(\omega)\left[\chi_1^m(\omega)\right]^* + D_{m,l}(\omega)\left[\chi_1^l(\omega)\right]^* \chi_1^m(\omega)$$

Lorenz 96 model

- Excellent toy model of the atmosphere
  - Advection
  - Dissipation
  - Forcing
- Test Bed for Data assimilation schemes
- Becoming popular in the community of statistical physicists
  - Scaling properties of Lyapunov & Bred vectors
- Evolution Equations

$$\dot{x}_i = x_{i-1}\left(x_{i+1} - x_{i-2}\right) - x_i + F \qquad i = 1,\ldots,N \quad x_i = x_{i+N}$$

- Spatially extended, 2 Parameters: N & F

Some properties

- Let $$E = \sum_{j=1}^{N} \frac{x_j^2}{2} \qquad M = \sum_{j=1}^{N} x_j$$
- and $$e = \frac{E}{N} \qquad m = \frac{M}{N}$$
- Stationary State: $$2\langle e \rangle_0 = F \langle m \rangle_0$$
- Closure: $$\langle m \rangle_0 \approx \lambda F^{\gamma} \quad F \geq 5;\ \lambda \approx 1.15;\ \gamma \approx 0.35$$
  - System is extended, in chaotic regime the properties are intensive
- We perform simulations with specific F=8 and N=40, but results are “universal”

Global Perturbation

- $$F \to F + \varepsilon e(t)$$

- Observable: $e = E/N$

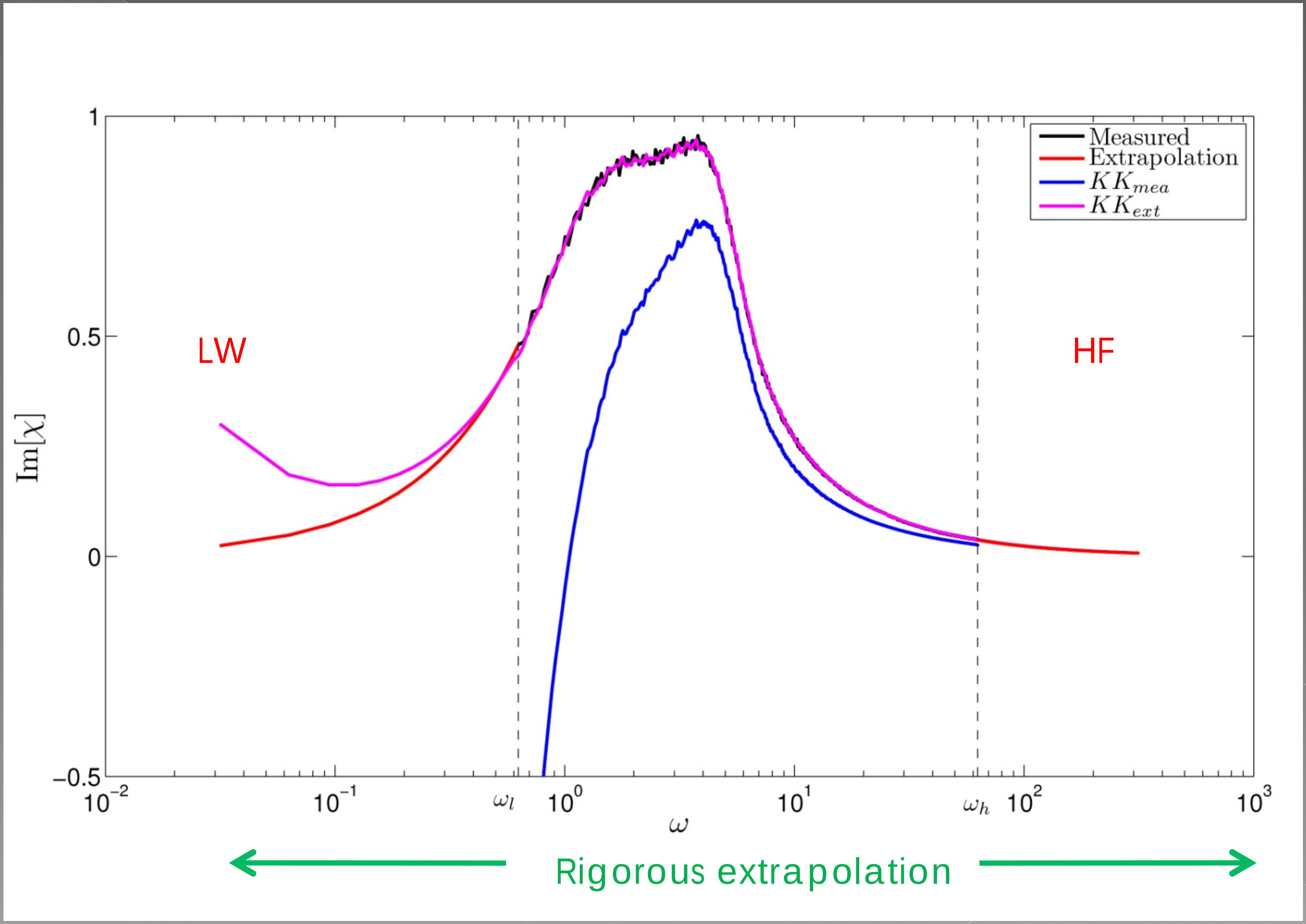
$$G_e^{(1)}(t) \approx \Theta(t)\langle m\rangle_0 + \Theta(t)t\left(F - 2\langle m\rangle_0\right) + \ldots$$

$$\chi_e^{(1)}(\omega) \approx i\frac{\langle m\rangle_0}{\omega} - \frac{\left(F - 2\langle m\rangle_0\right)}{\omega^2} + \ldots$$

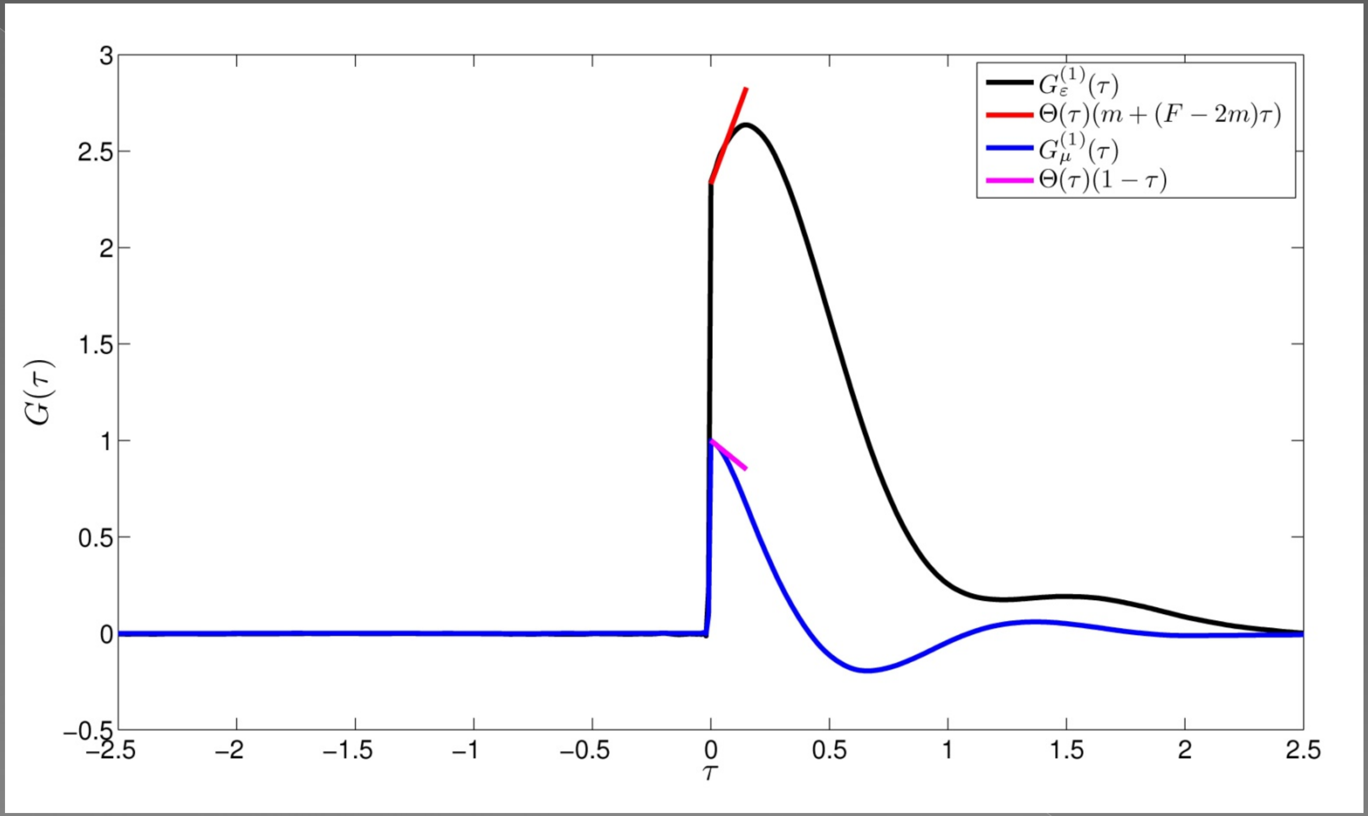
- We can compute the leading order for both the real and imaginary part

L & Sarno, 2011

Imag part of the susceptibility



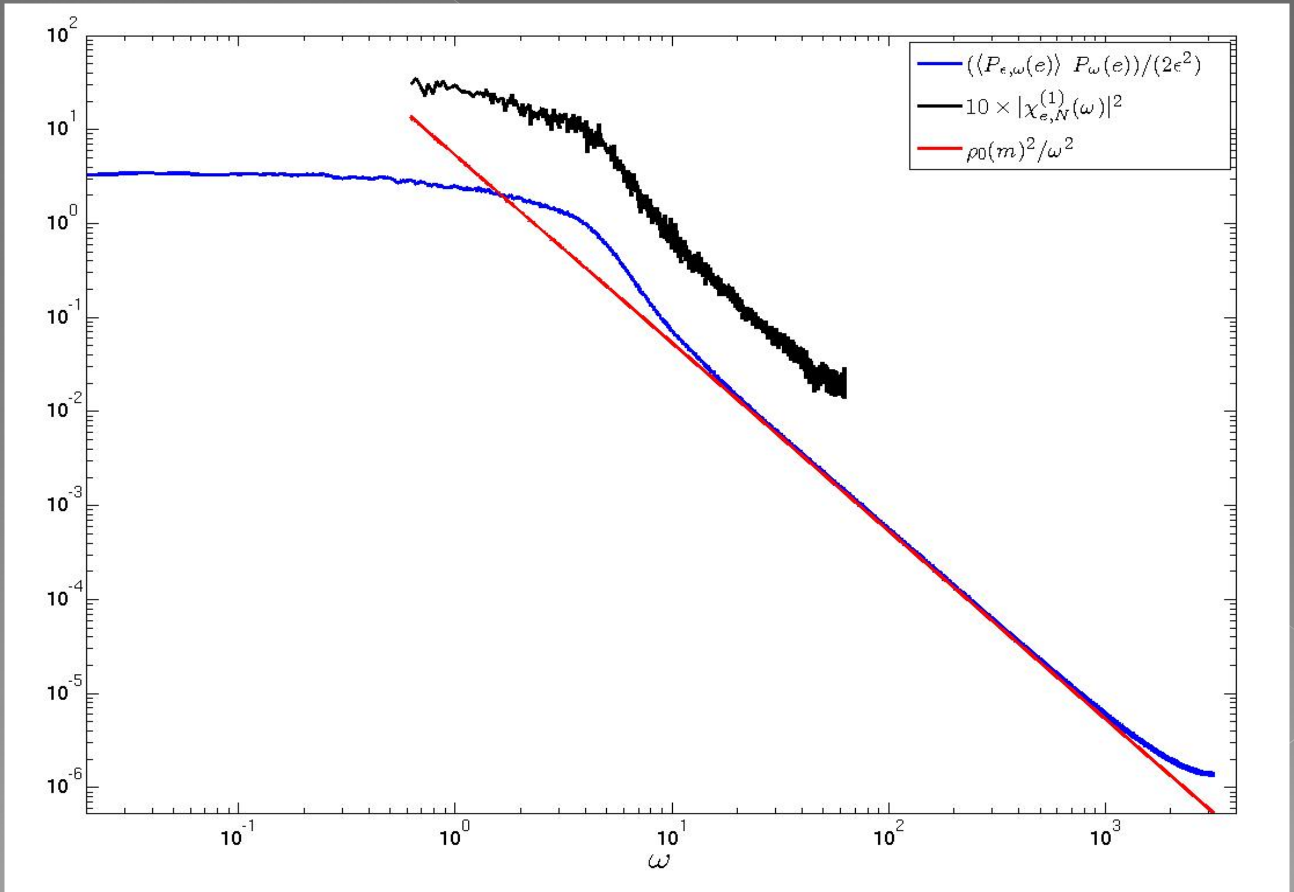
Green Function!



- Inverse FT of the susceptibility
- Response to any forcing with the same spatial pattern but with general time pattern

Using stochastic forcing…

- Squared modulus of $\chi_e^{(1)}(\omega)$
- Blue: Using stoch pert; Black: deter forcing
- … And many many many less integrations



A Climate Change experiment

- Observable: globally averaged TS
- Forcing: increase of CO_2 concentration
- Linear response: $\langle T_S \rangle^{(1)}(t) = \int d\sigma G_{T_S}^{(1)}(\sigma) e(t-\sigma)$
- Let's perform an ensemble of experiments
  - Concentration is increased at t=0 and brought back to initial value at t= τ $e(t) = \varepsilon[\Theta(t) - \Theta(t-\tau)]$
- Fantastic, we estimate $\langle T_S \rangle^{(1)}(\omega)$
- …and we obtain: $\chi^{(1)}(\omega) = \dfrac{\langle T_S \rangle^{(1)}(\omega)}{\varepsilon(\sin(\omega\tau) + i(1-\cos(\omega\tau)))}$
- … Now we can predict future T_S

What is a Parametrization?

Consider a two-level system

$$\begin{cases} \dot{X} = F_X(X) + \Psi_X(X, Y) \\ \dot{Y} = F_Y(Y) + \Psi_Y(X, Y) \end{cases}$$

- unperturbed = uncoupled
- perturbation = coupling Ψ
- modelling = a perturbation of X that mimics this response

- Surrogating the coupling: Fast→Slow Variables
- Optimising Computer Resources
- Underlining Mechanisms of Interaction

How to Construct a Parametrization?

- We try to match the evolution of the single trajectory of the X variables
  - Mori-Zwanzig Projector Operator technique: needs to be made explicit
  - Accurate “Forecast”

- We try to match the statistical properties of a general observable A=A(X)
  - Ruelle Response theory
  - Accurate “Climate”

- Match between these two approaches?

That's the result!

- This system has the same expectation values as the original system (up to 2nd order)
- We have explicit expression for the three terms a (deterministic), b (stochastic), c (memory)- 2nd order expansion

$$\frac{dX(t)}{dt} = F_X(X(t)) + \underbrace{M_X(X(t))}_{a} + \underbrace{\sigma(t)}_{b} + \underbrace{\int_0^\infty d\tau\, h(\tau, X(t-\tau))}_{c}$$

Deterministic ✓

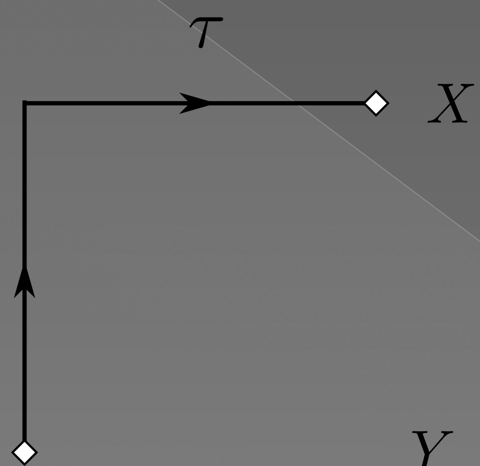
Stochastic ✓

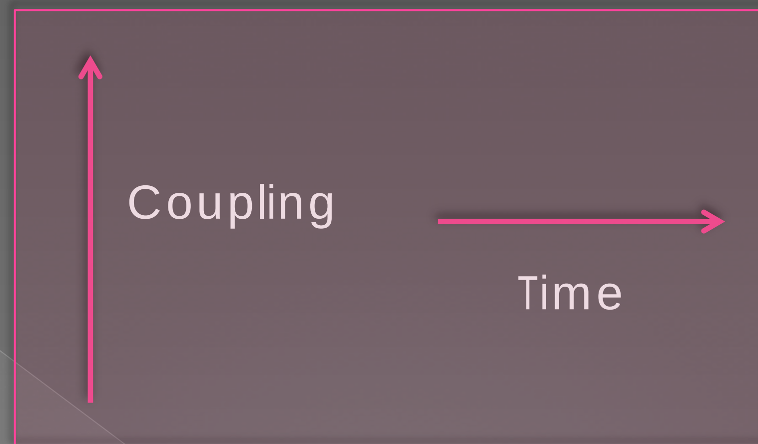
Memory ✖

TODAY

Diagrams: 1st and 2nd order

- 1





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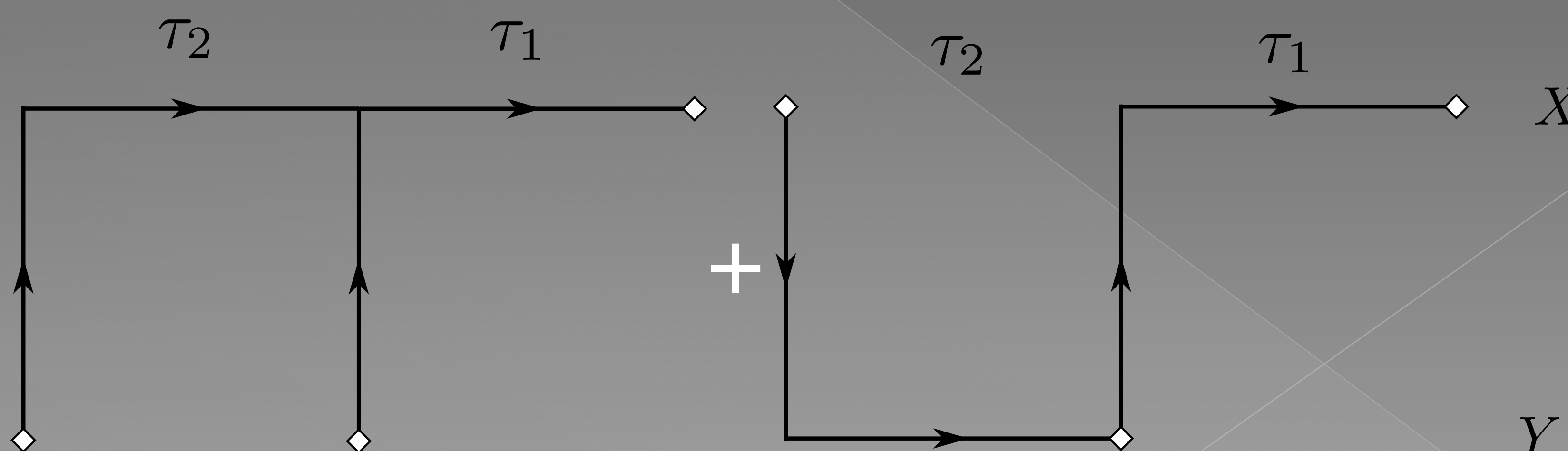
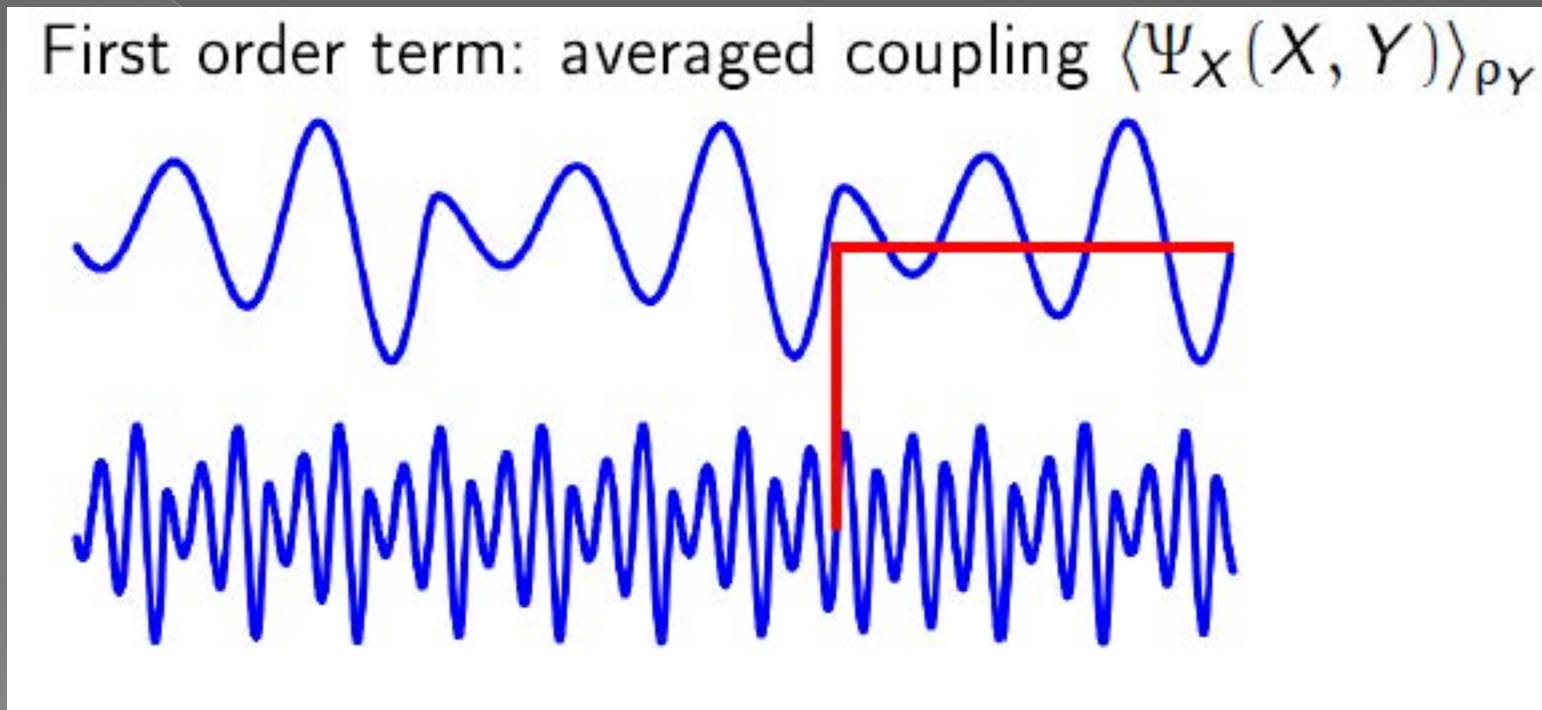


Mean Field



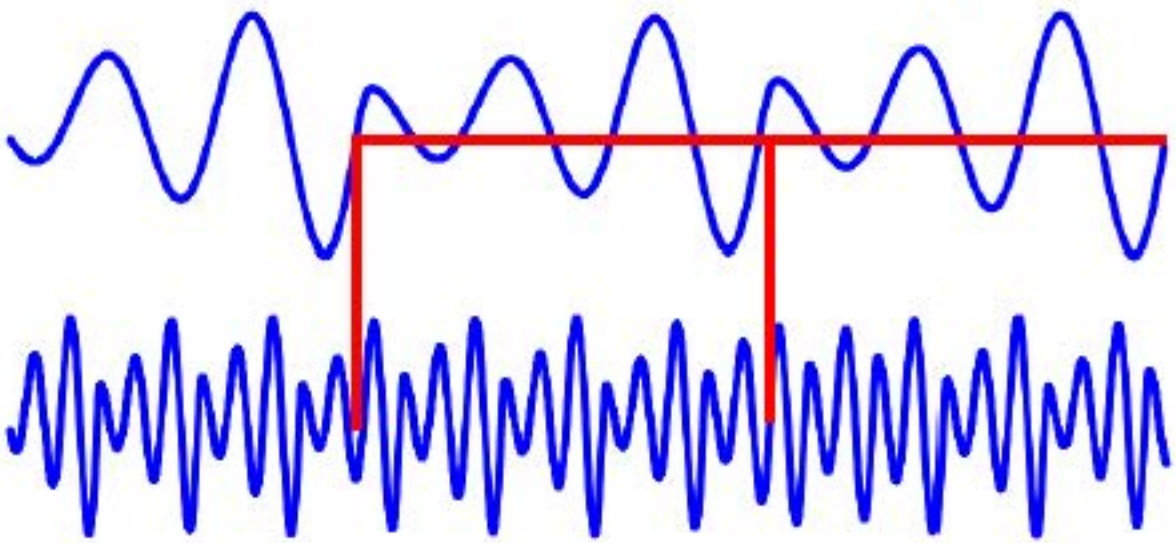
$$M_X(X(t))$$

- Deterministic Parametrization
  - This is the “average” coupling

Fluctuations

Second order term (1/2): fluctuations around the mean

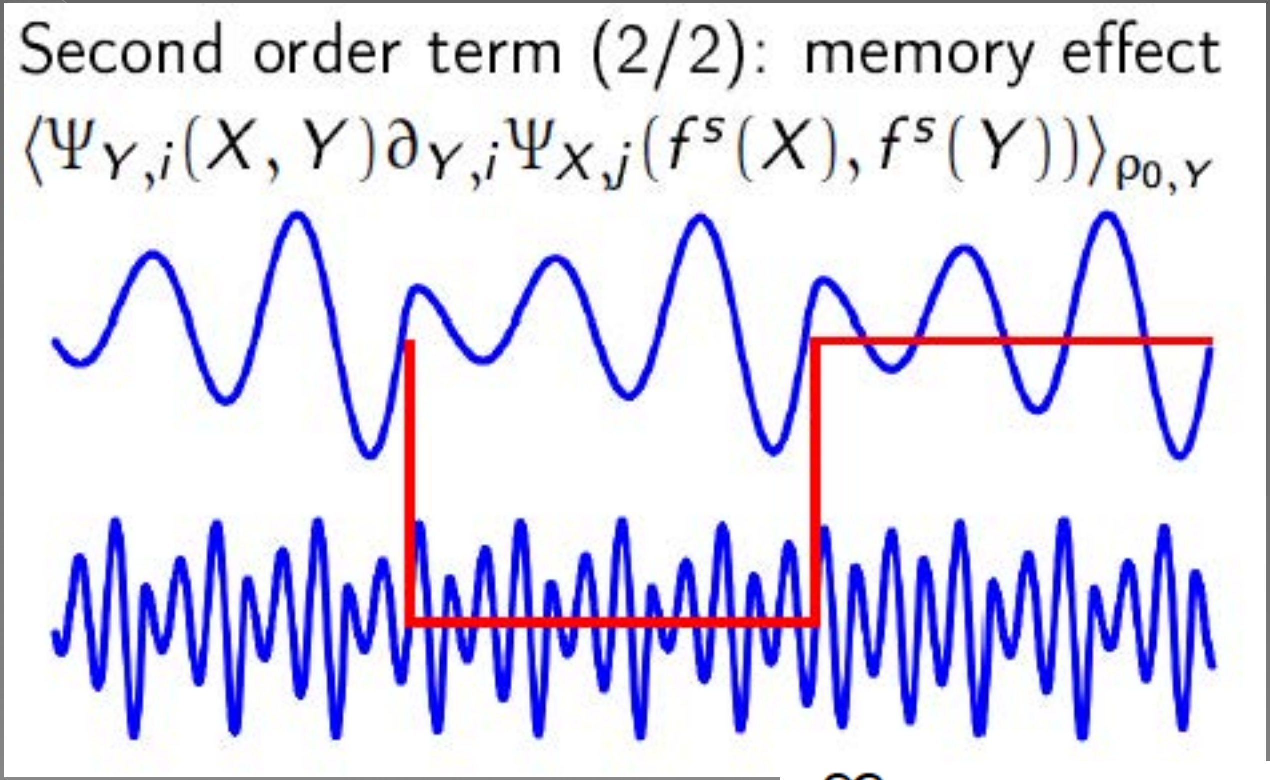
$$\langle \delta\Psi_X(X,Y)\,\delta\Psi_X(f^\tau(X), f^\tau(Y)) \rangle_{\rho_Y}$$



- Stochastic Parametrization
  - Expression for correlation properties

$$\sigma(t)$$

Memory



- New term, small for vast scale separation
  - This is required to match local vs global

$$\int_0^\infty d\tau\, h(\tau, X(t-\tau))$$

Two words on Mori-Zwanzig

- Answers the following question
  - what is the effective X dynamics for an ensemble of initial conditions Y(0), when ρ_Y is known?
  - We split the evolution operator using a projection operator P on the relevant variables
- Effective dynamics has a deterministic correction to the autonomous equation, a term giving a stochastic forcings (due to uncertainty in the the initial conditions Y(0)), a term describing a memory effect.
- One can perform an approximate calculation, expanding around the uncoupled solution…

Mori-Zwanzig: a simple example (Gottwald, 2010)

- Just a 2X2 system:

$$\dot{x} = L_{11}x + L_{12}y$$
$$\dot{y} = L_{21}x + L_{22}y.$$

- *x* is "relevant"
- We solve with respect to y:

$$y(t) = \mathrm{e}^{L_{22}t}y(0) + \int_0^t \mathrm{e}^{L_{22}(t-s)}L_{21}x(s)\,\mathrm{d}s,$$

- We plug the result into *x*:

$$\dot{x} = L_{11}x + L_{12}\int_0^t \mathrm{e}^{L_{22}(t-s)}L_{21}x(s)\,\mathrm{d}s + L_{12}\mathrm{e}^{L_{22}t}y(0)$$

- Markov Memory Noise

Same result!

- Optimal forecast in a probabilistic sense
- 2nd order expansion
- <u>Same as obtained with Ruelle</u>
  - Parametrizations are “well defined” for CM & NWP

$$\frac{dX(t)}{dt} = F_X\big(X(t)\big) + \underbrace{M_X\big(X(t)\big)}_{a} + \underbrace{\sigma(t)}_{b} + \underbrace{\int_0^\infty d\tau\, h\big(\tau, X(t-\tau)\big)}_{c}$$

- Memory required to match local vs global

Conclusions

- We have used Ruelle response theory to study the impact of deterministic and stochastic forcings to non-equilibrium statistical mechanical systems
- Frequency-dependent response obeys strong constraints
  - We can reconstruct the Green function!
- Δ expectation value of observable ≈ variance of the noise
  - SRB measure is robust with respect to noise
- Δ power spectral density ≈ to the squared modulus of the linear susceptibility
  - More general case: Δ power spectral density >0
  - The method is VERY parsimonious
- What is a parametrization? I hope I gave a useful answer
  - We have ground for developing new and robust schemes
- Application to more interesting models
- OPENING A POST-DOC POSITION SOON

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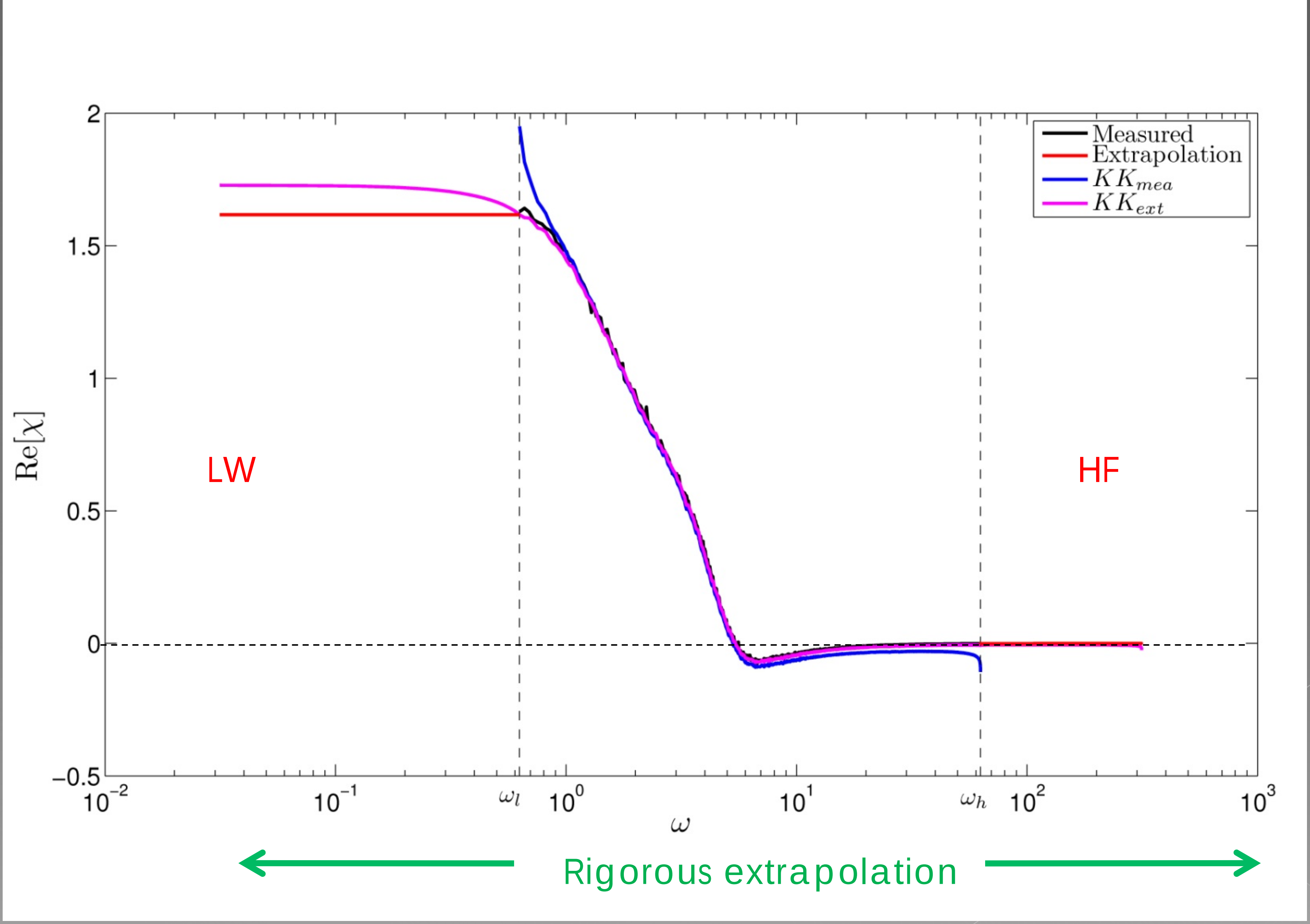
Real part of the susceptibility



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- Recent studies (Branstator et al.) suggest that, nonetheless, information can be retrieved
- What is the time needed to build up a statistics such that the FDT gives useful results?
  - Probably, numerical noise also helps
  - The choice of the observable is surely also crucial

Some observations

- The correction to the expectation value of any observable ~ variance of the noise
  - Stochastic system → deterministic system
  - Convergence of the statistical properties is fast
- We have an explicit formula!
- If the unperturbed system has an acim:

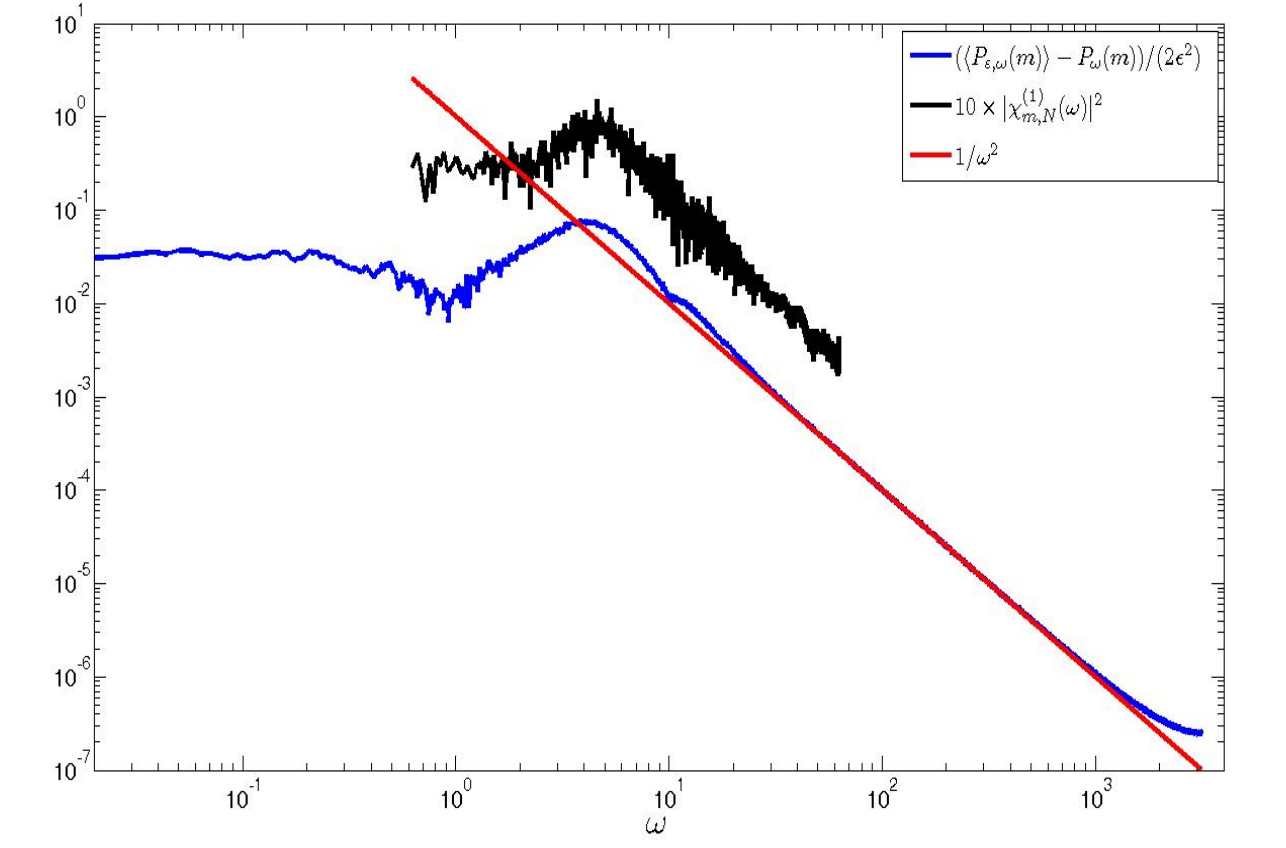
$$\langle \delta_\varepsilon \rho(\Phi) \rangle \approx -\varepsilon^2 \int d\tau \Theta(\tau_1) \int \rho_0(x) dx (\partial_i X_i) X_j \partial_j \Phi(f^{\tau_1} x)$$

- We have a correlation integral, like in a FDT:

$$\langle \delta_\varepsilon \rho(\Phi) \rangle \approx -\varepsilon^2 \int d\tau \int \rho_0(dx) B(x) C(f^{\tau_1} x)$$

Another observable

- Squared modulus of $\chi_m^{(1)}(\omega)$
- Blue: Using stoch pert; Black: deter forcing
- ... And many many many less integrations



Three major experimental challenges in analysing the CS

- Synchronic coherence of data
  - Data feature hugely varying degree of precision
- Diachronic coherence of data
  - Technology and prescriptions for data collection have changed with time
- Space-time coverage
  - Data density (Antarctica vs Germany)
  - We have "direct" data only since Galileo time
  - Before, we have to rely on indirect (proxy) data
- Merging of Data and Models

A rigorous view on Climate Change

- The analysis of how systems respond to external perturbations to their steady state constitutes one of the crucial subjects in physics and mathematics
- Sometimes, we use “blindly” several CM experiments in order to understand the response
- The natural variability blurs the signal, in order to be rigorous we should repeat many times the same experiment (and with various CMs)
- In climate science, we struggle considerably with concepts and computations of climate sensitivity and climate response
- We would like to be able to define more rigorously Climate Change!

Linear case

- The input $I(t)=e(t)$ and the output $O(t)=\langle \Phi^{(1)}(t) \rangle$ are connected by the following linear relationship involving $a(t)=G^{(1)}(t)$:

$$O(t) = \int_{-\infty}^{\infty} a(t-t')I(t')dt'$$

- By applying Fourier Transform to both members we obtain:

$$O(\omega) = a(\omega)I(\omega)$$

- Is there a connection between the properties of $a(t)=G^{(1)}(t)$ and those of $a(\omega)= \chi^{(1)}(\omega)$?

Background

- In quasi-equilibrium statistical mechanics, the Kubo theory ('50s) allows for an accurate treatment of perturbations to the canonical equilibrium state
  - In the linear case, the FDT bridges the properties of the forced and free fluctuations of the system
- When considering general dynamical systems (e.g. forced and dissipative), the situation is more complicated (no FDT, in general)
- Recent advances (Ruelle, mostly): for a class of dynamical systems it is possible to define a perturbative theory of the response to small perturbations
  - We follow this direction…
- We apply the theory also for stochastic forcings