Dynamic Transitions in Climate Dynamics

Tian Ma and Shouhong Wang
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I. Motivations and Objectives

The thorough understanding of climate low frequency variability (LFV) is a challenging problem with important practical implications for geophysical efforts to quantify predictability, analyze error growth in dynamical models, and develop efficient forecast methods.

We focus in this talk two sources of such LFV:

- thermohaline circulation (THC) (Stommel 61; Rooth 82; Welander 86; Salmon 86; Colin de Verdi‘ere 88; Cessi and Young 92; Quon and Ghil 92; Thual and McWilliams 92; Dijkstra and Molemaker 97; Dijkstra and Neelin 99; Djikstra 00; .....)
- El Nino Southern Oscillation (ENSO)

The main technical approach is the dynamic transition theory of dissipative dynamical systems, that we developed to identify the transition states and to classify them both dynamically and physically:

- With this theory, all transitions of dissipative systems are classified into three categories: 1) continuous (Type-I), 2) catastrophic (Type-II), and 3) random (Type-III).

- **Key philosophy of the theory** is to search for the complete set of transition states, represented by a local attractor.

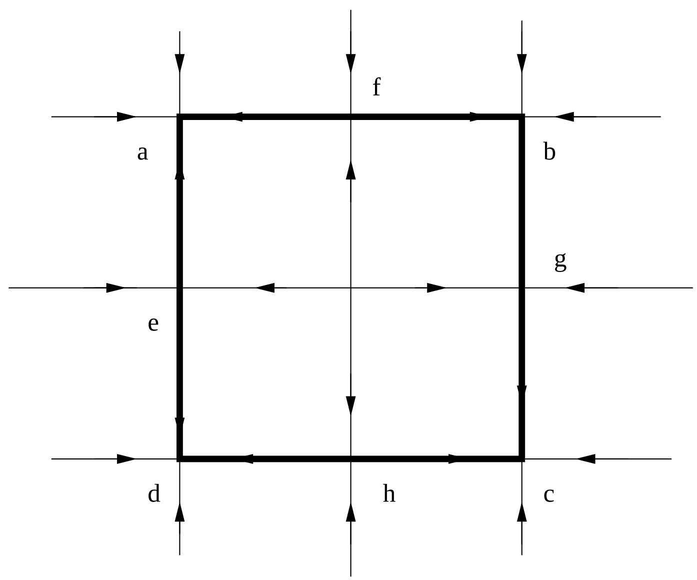
- With this philosophy, the theory is developed under close links to the physics. In return the theory is applied to the physical problems, leading to numerous physical predictions:

  T. Ma & S. Wang, Phase Transition Dynamics in Nonlinear Sciences, Springer-Verlag, to appear, 2012, 679pp.

Examples:

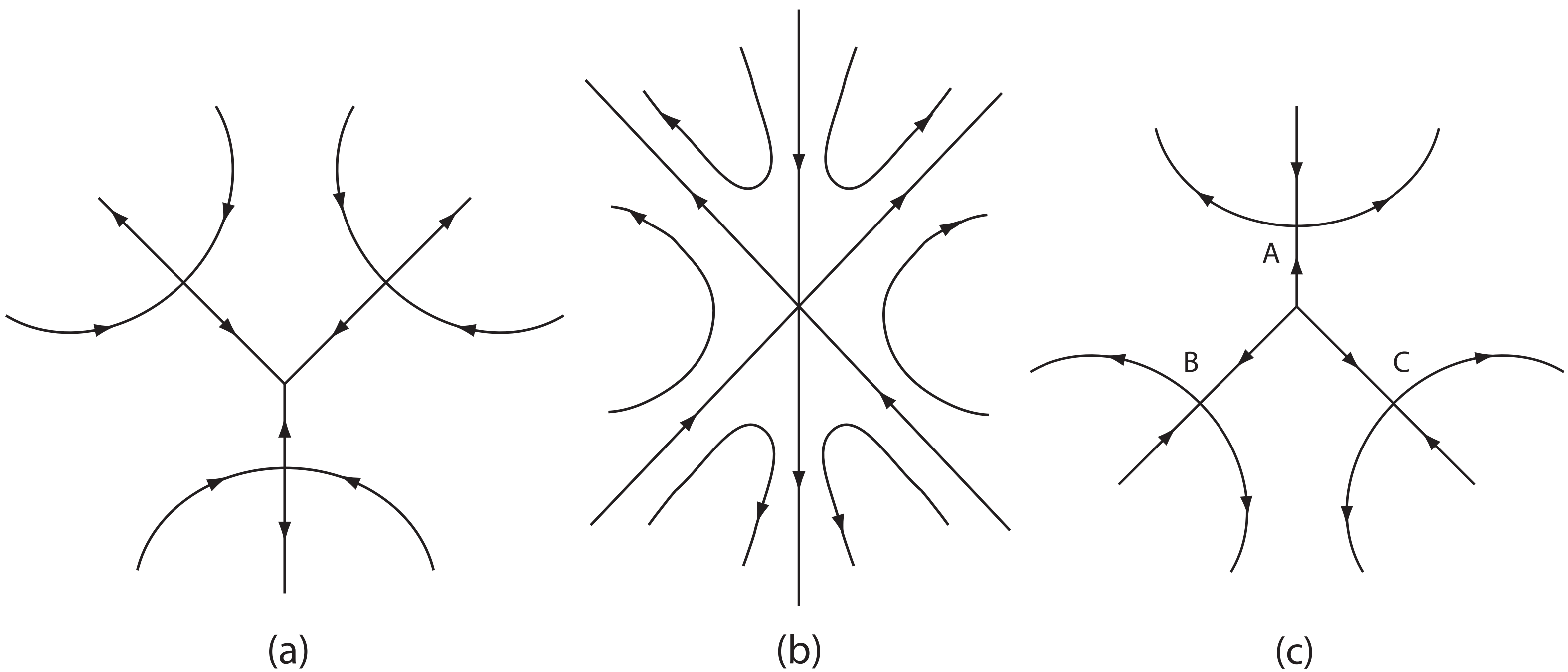
$$\frac{dx_1}{dt} = \lambda x_1 - x_1^3 + o(|x|^3),$$
$$\frac{dx_2}{dt} = \lambda x_2 - x_2^3 + o(|x|^3)$$

The system undergoes a dynamic transition at $\lambda = 0$:

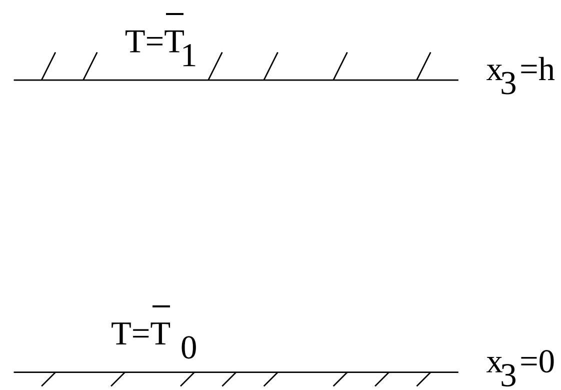


(1)

$$\dot{x}_1 = \lambda x_1 + x_1^2 + x_1 x_2 - 10x_1^3,$$
$$\dot{x}_2 = \lambda x_2 - 2x_1 x_2 - x_2^2 - 10x_2^3.$$



Consider the Bénard convection on a nondim. domain $\Omega = D \times (0,1)$ with a set of physically sound BCs. Let Rayleigh number be $\boxed{R = g\alpha(\bar{T}_0 - \bar{T}_1)h^3/(\kappa\nu)}$.



Thm (Ma & W., 04): As R crosses the first critical Rayleigh number R_c, the system always undergoes a continuous transition to an attractor $\Sigma_R \simeq S^{m-1}$, where m is the multiplicity of R_c.

In addition, Σ_R attracts every bounded set of $H \setminus \Gamma$, where Γ is the stable manifold of the basic solution.

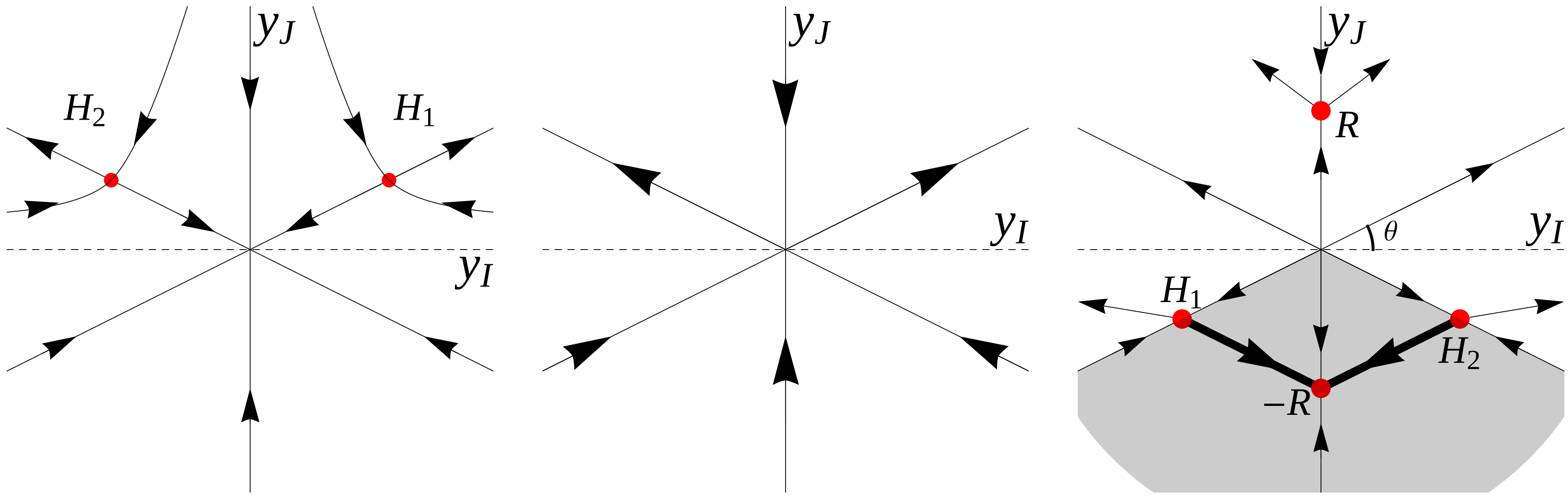
Thm (Ma & W., 07): Consider the 3D Bénard convection in $\Omega = (0, L_1) \times (0, L_2) \times (0, 1)$ with free top-bottom and periodic horizontal BC's, and with

$$\frac{k_1^2}{L_1^2} + \frac{k_2^2}{L_2^2} = \frac{1}{8} \qquad \text{for some } k_1, k_2 \in \mathbb{Z}.$$

Then

$$\Sigma_R = \begin{cases} S^5 & \text{if} \quad L_2 = \sqrt{k^2 - 1} L_1, \quad k = 2, 3, \cdots, \\ S^3 & \text{otherwise.} \end{cases}$$

(joint with H. Dijkstra, T. Sengul, 2011): For the surface tension driven convection, the system can undergo a random transition as the Marangoni number crosses the threshold:



(a) $\lambda < \lambda_c$ (b) $\lambda = \lambda_c$ (c) $\lambda > \lambda_c$

(2)
$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = \mathsf{Pr}\,(-\nabla p + \Delta\mathbf{u})$$
$$\frac{\partial \theta}{\partial t} + (\mathbf{u}\cdot\nabla)\,\theta = w + \Delta\theta$$
$$\nabla\cdot\mathbf{u} = 0$$

BC: Free-Slip on the lateral of $\Omega = (0, L_1)\times(0, L_2)\times(0,1)$ and

$u = v = w = \theta = 0$ at $z = 0$

$\dfrac{\partial (u, v)}{\partial z} + \lambda\nabla_H\theta = w = \dfrac{\partial\theta}{\partial z} + \mathsf{Bi}\theta = 0$ at $z = 1$

$\mathsf{Bi} \geq 0$ the Biot number

$\lambda = \dfrac{\xi_0\gamma_T(\theta_0 - \theta_1)d^2}{\rho_0\nu\kappa} > 0$ the Marangoni number

$\xi = \xi_0(1 - \gamma_T\theta)$ surface tension on the top

II. Dynamic Transitions for the Boussinesq Model

(3)
$$
\begin{aligned}
&\frac{\partial u}{\partial t} = \mathrm{Pr}\ (\Delta u - \nabla p) + \mathrm{Pr}\ \left[RT - \mathsf{sign}(S_0 - S_1)\tilde{R}S\right]\vec{k} - (u\cdot\nabla)u\\
&\frac{\partial T}{\partial t} = \Delta T + u_3 - (u\cdot\nabla)T,\\
&\frac{\partial S}{\partial t} = \mathrm{Le}\ \Delta S + \mathsf{sign}(S_0 - S_1)u_3 - (u\cdot\nabla)S\\
&\mathsf{div}\ u = 0,
\end{aligned}
$$

with Free Slip Boundary Conditions in $\Omega = (0, L_1) \times (0, L_2) \times (0, 1)$

$R = \frac{\alpha_T g(T_0 - T_1)h^3}{\kappa_T \nu}$ the thermal Rayleigh number,

$\tilde{R} = \frac{\alpha_S g(S_0 - S_1)h^3}{\kappa_T \nu}$ the saline Rayleigh number,

$\mathrm{Pr} = \frac{\nu}{\kappa_T}$, $\quad$ Le $= \frac{\kappa_S}{\kappa_T}$ the Prandtl and the Lewis numbers.

Main Results (Ma-Wang, 07, 08, 09): Let

$$(4) \qquad K = \text{sign}(1 - \text{Le}\ ) \left[\frac{\text{Le}^2}{1 - \text{Le}} \left(1 + \frac{1}{\text{Pr}} \right) \sigma_c - \tilde{R} \right],$$

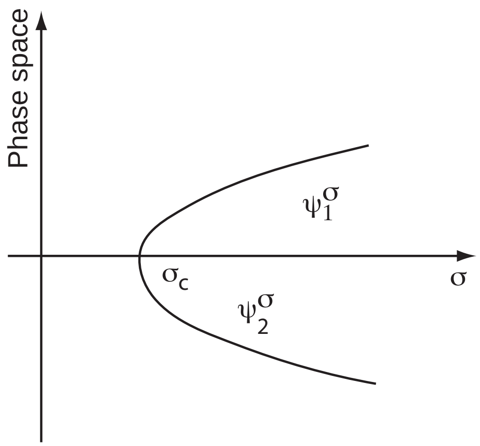
$$(5) \qquad \sigma_c = \min_{\substack{(j,k) \in \mathbb{Z}^2, j,k \geq 0, \\ j^2 + k^2 \neq 0, l \geq 1}} \frac{\pi^4 (j^2 L_1^{-2} + k^2 L_2^{-2} + 1)^3}{j^2 L_1^{-2} + k^2 L_2^{-2}} = \frac{\pi^4 (j_1^2 L_1^{-2} + k_1^2 L_2^{-2} + 1)^3}{j_1^2 L_1^{-2} + k_1^2 L_2^{-2}}.$$

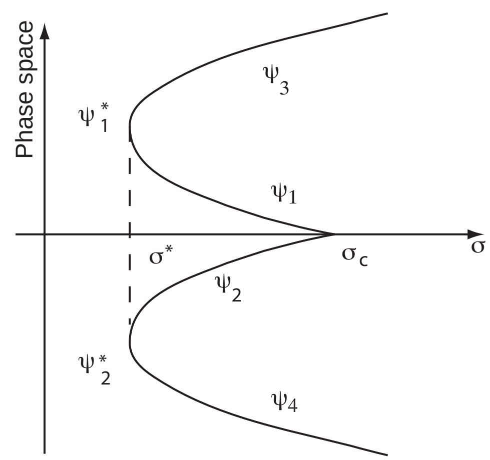
for some integer pair (j_1, k_1) such that $j_1 \geq 0$, $k_1 \geq 0$, $j_1^2 + k_1^2 \neq 0$.

Double diffusive flow analysis is also joint with Chun-Hsiung Hsia.

Case $K > 0$: The first dynamic transition of the system occurs as the R-Rayleigh number $\sigma = R - \mathrm{Le}^{-1}\tilde{R}$ crosses the critical number σ_c, leading to **multiple equilibria**:

- If $b_1 = \sigma_c - \frac{1-\mathrm{Le}^2}{\mathrm{Le}^3}\tilde{R} > 0$, the transition is of continuous type as shown.
- If $b_1 < 0$, then the transition is of jump type, leading to the existence of metastable stables, saddle-node bifurcations and the hysteresis associated with it.





Case $K < 0$: the first transition of the system occurs as the C-Rayleigh number

$$\eta = R - \frac{\Pr + \text{Le}}{\Pr + 1}\tilde{R}$$

crosses its first critical value

$$\eta_c = \frac{(\Pr + \text{Le})(1 + \text{Le})}{\Pr}\sigma_c,$$

leading to spatiotemporal oscillations (periodic solutions).

The transition can be either continuous or jump, dictated by the sign of a nondimensional parameter b_2.

III. Scaling Law

For the classical Bénard convection with free-slip BC, the critical temperature gradient is given by

(6) $\triangle T_c \simeq \frac{27\kappa_T \nu \pi^4}{4g\alpha_T h^3} \to \begin{cases} \text{large} & \text{if } h \to 0, \\ \text{small} & \text{if } h \to \infty. \end{cases}$ critical horizontal scale $= \sqrt{2}h$.

The resolution of this discrepancy is carried out by adding to the momentum eqs turbulent friction terms as $F = (\sigma_0 u_1, \sigma_0 u_2, \sigma_1 u_3)$.

Based on our analysis to ensure the independence of $\triangle T_c$ on the vertical scale h, we propose the following scaling law:

$\sigma_0 = C_0 h^2, \qquad \sigma_1 = C_1 h^2$ with C_0 and C_1 independent of h

$\delta_0 = C_0 h^4/\nu, \qquad \delta_1 = C_1 h^4/\nu$ (nondim form with vert length scaled to 1)

IV. Dynamic Transitions for An Idealized THC Model

$h = 4 \times 10^3\text{m}$, $L_c = 10^4\text{m}$, $\alpha_T = 2.1 \times 10^{-4}\ K^{-1}$

$\text{Pr} = 8$, $\text{Le} = 10^{-2}$, $\alpha_S \cong 0.92 \times 10^{-3}(psu)^{-1}$

$\nu = 1.1 \times 10^{-6}\ m^2 s^{-1}$, $\kappa_T = 1.4 \times 10^{-7} m^2 s^{-1}$

$$R = \frac{g\alpha_T(T_0 - T_1)}{\kappa_T \nu} h^3 = 0.86 \times 10^{21}(T_0 - T_1)[^\circ C^{-1}],$$

$$\tilde{R} = \frac{g\alpha_S(S_0 - S_1)}{\kappa_T \nu} h^3 = 3.75 \times 10^{21}(S_0 - S_1)(psu)^{-1}.$$

To fit the length scale of the THC, we need to consider the Boussinesq Equation with added friction term in its nondim form with vertical length scaled to 1:

MODEL: BE $+ (\delta_0 u_1, \delta_0 u_2, \delta_1 u_3)$ with $\delta_0 = 1.17 \times 10^8$, $\delta_1 = 2.33 \times 10^{23}$

Results

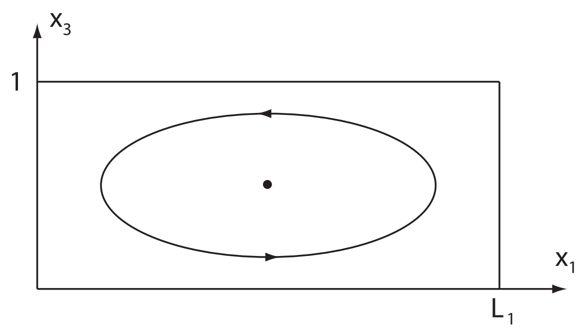
1. Deduced Critical Parameters:

$\alpha_c^2 = \pi^2 \left[\frac{\delta_0}{\delta_1}\right]^{1/2} = 2.24 \times 10^{-7}$	critical wave number
$L_c = \frac{\pi}{\alpha_c} = \left[\frac{\delta_1}{\delta_0}\right]^{1/4} = 0.67 \times 10^4$	critical horizontal length scale
$\sigma_c = (\pi^2 + \alpha_c^2)\delta_1 + \frac{\pi^4 \delta_0}{\alpha_c^2} = 2.33 \times 10^{24}$	critical R-Rayleigh number
$\eta_c = (1 + \mathrm{Le}\ )\sigma_c$	critical C-Rayleigh number
$\sigma = R - \mathrm{Le}^{-1}\tilde{R}, \qquad \eta = R - \tilde{R}$	R- and C-Rayleigh numbers

2. Case $\tilde{R} < 2.35 \times 10^{20}$: The system undergoes a dynamic transition at σ_c to a local attractor consisting of multiple equilibria and their unstable manifolds:

- If $\tilde{R} < 2.33\times 10^{18}$, the transition is continuous. In fact, the problem bifurcates to two stable steady state solutions ψ_1^σ and ψ_2^σ for $\sigma > \sigma_c$ with basin of attractions of U_1 and U_2.

  In addition, the initial value $\tilde{\psi} \in U_i$, then there is a time t_0 such that as $t > t_0$, the flow structure of the solution $\psi(t, \psi_0)$ is topologically equivalent to:



$$v^{\pm} = \left(\pm C\beta^{1/2}(\sigma) L_1 \sin\frac{\pi x_1}{L_1}\cos\pi x_3, 0, \mp C\beta^{1/2}(\sigma)\cos\frac{\pi x_1}{L_1}\sin\pi x_3\right)$$

$$T^{\pm} = T_0 + (T_1 - T_0)x_3 \mp \frac{C\beta^{1/2}(\sigma)}{\alpha_c^2 + \pi^2}\cos\frac{\pi x_1}{L_1}\sin\pi x_3,$$

$$S^{\pm} = S_0 + (S_1 - S_0)x_3 \mp \frac{\mathsf{sign}(S_0 - S_1)C\beta^{1/2}}{\mathrm{Le}\ (\alpha_c^2 + \pi^2)}\cos\frac{\pi x_1}{L_1}\sin\pi x_3,$$

$$\beta(\sigma) = k(\sigma - \sigma_c) + o(|\sigma - \sigma_c|)$$

$$v_{\mathsf{max}} = CL_1\kappa h^{-1}\beta^{1/2} = 0.64 \times 10^{-7}\beta^{1/2}(\sigma)\mathsf{m/s}, \quad \frac{v_3}{v_1} = 1.56 \times 10^{-4}$$

- If $2.33 \times 10^{18} < \tilde{R} < 2.35 \times 10^{20}$, the transition is jump, and there are two saddle-node bifurcations from (ψ_1^*, σ^*) and (ψ_2^*, σ^*) with $\sigma^* < \sigma_c$.

3. Case $\tilde{R} > 2.35 \times 10^{20}$: The system undergoes a dynamic transition at η_c to an attractor consisting of spatiotemporal oscillations (Hopf bifurcation).

- For $2.35 \times 10^{20} < \tilde{R} < 2.35 \times 10^{24}$, the transition is continuous, leading to a stable periodic solution. The period is about 1.1×10^{6} s. NOT realistic.

- For $\tilde{R} > 2/35 \times 10^{24}$, the transition is jump.

Conclusions

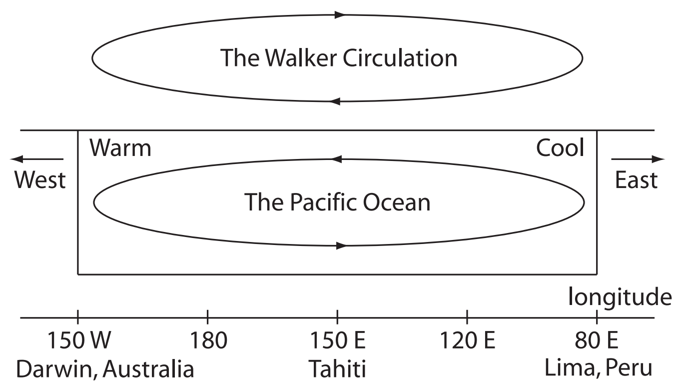
- A nondimensional parameter K is introduced to distinguish the multiple steady state and oscillatory spatiotemporal patterns, which play an important role in understanding the mechanism of THC in different oceanic basins.

- For both the multiple equilibria and periodic solutions transitions, both Type-I (continuous) and Type-II (jump) transitions can occur, depending respectively on the signs of two computable nondimensional parameters b_1 and b_2.

- A convection scale law is introduced, providing a method to introduce proper friction terms in the model in order to derive the correct circulation length scale.

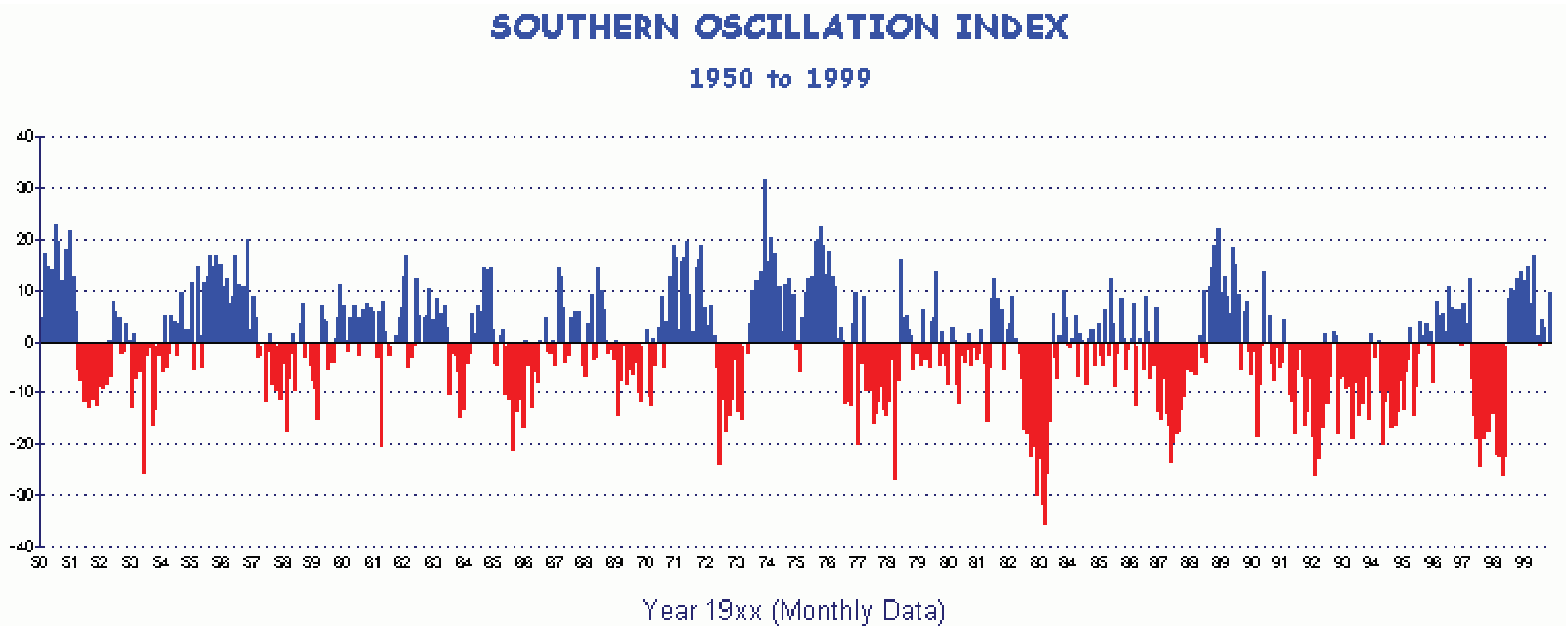
- The analysis of the idealized model with the proper friction terms shows that the THC appears to be associated with the continuous transitions to stable multiple equilibria.

V. New Metastable State Oscillation Theory of ENSO



- The behavior of the **Walker cell** is a key factor giving rise to the ENSO.
- When the convective activity weakens or reverses, an **El Niño** phase takes place, causing the ocean surface to be warmer than average, reducing or terminating the upwelling of cold water.
- A particularly strong Walker circulation causes a **La Niña** event, resulting in cooler sea-surface temperature (SST) due to stronger upwelling.

- SOI gives a simple measure of the strength and phase of the Southern Oscillation: In the **El Niño** phase, the SOI is **negative or zero**, when in **La Niña** phase, the SOI is **strongly positive**, and in the **normal state** the SOI is **small and positive**.

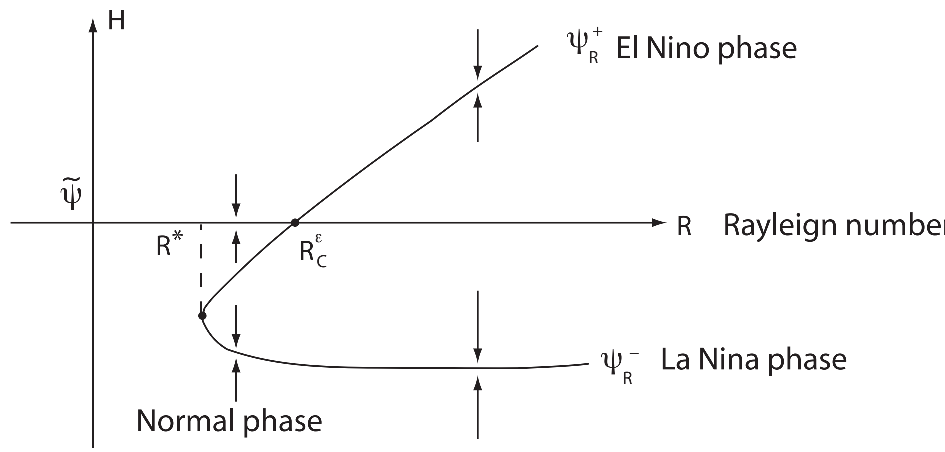


- There have been extensive studies in recent years, following the pioneering work of (Dijkstra 00, Ghil 00, Jin 96, Jin-Neelin-Ghil 96, Sardeshmukh-Compo-Penland 00, Zebiak-Cane 87, ...).

- An interesting current debate is whether ENSO is best modeled as a stochastic or chaotic system - linear and noise-forced, or nonlinear oscillatory and unstable system (G. Philander and A. Fedorov 03)?

- A careful fundamental level examination of the problem is crucial.

Consider the basic atmospheric model over the tropics:



Mechanism of ENSO: ENSO is a self-organizing and self-excitation system, with two highly coupled oscillatory processes:

- the oscillation between the two metastable warm (El Nino phase) and cold events (La Nina phase), and
- the spatiotemporal oscillation of the sea surface temperature (SST) field.

The interplay between these two processes

- gives rises the climate variability associated with ENSO,
- leads to both the random and deterministic features of ENSO, and
- defines a natural feedback mechanism, driving the sporadic oscillation of ENSO.

The randomness is closely related to the uncertainty/fluctuations of the initial data between the narrow basins of attractions of the corresponding metastable events.

The deterministic feature is represented by a deterministic coupled atmospheric and oceanic model predicting the basins of attraction and the sea-surface temperature (SST).

VI. Remarks

The theory has been applied to a wide range of problems in nonlinear sciences, leading to a number of **physical predictions**:

- **Equilibrium phase transitions**: Gas-liquid transition (the nature and theory of the critical point), ferromagnetism (asymmetry principle of fluctuations), binary systems (existence of 2nd-order transitions), superconductivity (characterization of 1st and 2nd order transitions), and superfluidity (prediction of a new superfluid phase)

- **Classical Fluid Dynamics:** Bénard convection (richness of the transients), Taylor problem, and Taylor-Couette-Poiseuille flows (mechanism of the formation of the Taylor vortices)

- **Geophysical Fluid Dynamics and Climate Dynamics:** double-diffusive flows, thermohaline circulation (scaling law and criteria for flow regimes), ENSO (a new metastable states oscillation theory of ENSO), ...

- **Biology and chemistry:** Chemotaxis, Belosov-Zhabotinsky chemical reaction

- **Pattern formation:** formation and mechanism of different patterns in Marengoni flow (with H. Dijkstra and T. Sengul), Magnetohydrodynamic convection (with T. Sengul), diblock copolymer melts (Cahn-Hilliard model with long range repulsive interactions) (with H. Liu, T. Sengul, Pingwen Zhang).