Climate Change: Multidecadal +, National Taiwan University (Taida) Taipei, 18 Sept. 2012

# Mathematical Theory of Climate Sensitivity

### **Michael Ghil**

Ecole Normale Supérieure, Paris, and University of California, Los Angeles Joint work with M.D. Chekroun, D. Kondrashov, J. C. McWilliams and J. D. Neelin (UCLA) + A. Bracco (Georgia Tech), E. Simonnet (INLN, Nice), S. Wang (Indiana U.) and I. Zaliapin (U. Nevada, Reno)





Please visit these sites for more info. http://www.atmos.ucla.edu/tcd/ http://www.environnement.ens.fr/

# Motivation

- The *climate system* is highly *nonlinear and* quite *complex*.
- The system's *major components* the atmosphere, oceans, ice sheets *evolve* on many time and space scales.
- Its predictive understanding has to rely on the system's physical, chemical and biological modeling, but also on the thorough mathematical analysis of the models thus obtained: the forest vs. the trees.
- The *hierarchical modeling* approach allows one to give proper weight to the understanding provided by the models vs. their realism: back-and-forth between *"toy"* (conceptual) and *detailed* ("realistic") *models*, and between *models* and *data*.
- How do we disentangle *natural variability* from *the anthropogenic forcing*: *can we & should we, or not?*

### **Climate and Its Sensitivity**

Let's say CO<sub>2</sub> doubles: How will "climate" change?

- Climate is in stable equilibrium (fixed point); if so, mean temperature will just shift gradually to its new equilibrium value.
- 2. Climate is purely periodic; if so, mean temperature will (maybe) shift gradually to its new equilibrium value.
  But how will the period, amplitude and phase of the limit cycle change?
- 3. And how about some "real stuff" now: chaotic + random?

Ghil (in *Encycl. Global Environmental Change*, 2002)



### A little more on natural variability, I

nature 28 March 1991

### letters to nature

Nature 350, 324 - 327 (1991); doi:10.1038/350324a0

### Interdecadal oscillations and the warming trend in global temperature time series

M. Ghil & R. Vautard

THE ability to distinguish a warming trend from natural variability is critical for an understanding of the climatic response to increasing

greenhouse-gas concentrations. Here we use singular spectrum analysis<sup>1</sup> to analyse the time series of global surface air tem-peratures for the past 135 years<sup>2</sup>, allowing a secular warming trend and a small number of oscillatory modes to be separated from the noise. The trend is flat until 1910, with an increase of 0.4 °C since then. The oscillations exhibit interdecadal periods of 21 and 16 years, and interannual periods of 6 and 5 years. The interannual oscillations are probably related to global aspects of the El Niño-Southern Oscillation (ENSO) phenomenon<sup>3</sup>. The interdecadal oscillations could be associated with changes in the extratropical ocean circulation<sup>4</sup>. The oscillatory components have combined (peak-to-peak) amplitudes of 0.2 °C, and therefore limit our ability to predict whether the inferred secular warming trend of 0.005 °Cyr<sup>-1</sup> will continue. This could postpone incontrovertible detection of the greenhouse warming signal for one or two decades.

### A little more on natural variability, II

There are several natural modes of variability, internal to the climate system: El Niño/Southern Oscillation (*ENSO*), North Atlantic Oscillation (*NAO*), Pacific Decadal Oscillation (*PDO*), etc.

It is the chaotic interaction of these modes that is forced by us, not some dumb equilibrium.

Plaut, Ghil & Vautard (Science, 1995)



### Outline

- The IPCC process: results and uncertainties
- Natural climate variability as a source of uncertainties
  - sensitivity to initial data  $\rightarrow$  error growth
  - sensitivity to model formulation → see below!
- Uncertainties and how to fix them
  - structural stability and other kinds of robustness
  - non-autonomous and random dynamical systems (NDDS & RDS)
- Two illustrative examples
  - the Lorenz convection model
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## **GO2 IN THE ATMOSPHERE**



### **Temperatures and GHGs**

# Greenhouse gases (GHGs) go up, temperatures go up:

It's gotta do with us, at least a bit, doesn't it?



Wikicommons, from Hansen *et al.* (*PNAS*, 2006); see also http://data.giss.nasa.gov/ gistemp/graphs/

# Unfortunately, things aren't all that easy!

### What to do?

Try to achieve better interpretation of, and agreement between, models ...

**Ghil, M**., 2002: Natural climate variability, in *Encyclopedia of Global Environmental Change*, T. Munn (Ed.), Vol. 1, Wiley Natural variability introduces additional complexity into the anthropogenic climate change problem

The most common interpretation of observations and GCM simulations of climate change is still in terms of a scalar, linear Ordinary Differential Equation (ODE)



Linear response to CO<sub>2</sub> vs. observed change in T

Hence, we need to consider instead a system of nonlinear Partial Differential Equations (PDEs), with parameters and multiplicative, as well as additive forcing (deterministic + stochastic)

 $\frac{dX}{dt} = N(X, t, \mu, \beta)$ 

# Global warming and its socio-economic impacts

**Temperatures rise:** 

- What about impacts?
- How to adapt?

The answer, my friend, is blowing in the wind, *i.e.*, it depends on the accuracy and reliability of the forecast ...



Source : IPCC (2007), AR4, WGI, SPM

Figure SPM.5. Solid lines are multi-model global averages of surface warming (relative to 1990–1999) for the scenarios A2, A1B and B1, shown as continuations of the 20th century simulations. Shading denotes the ±1 standard deviation range of individual model annual averages. The orange line is for the experiment where concentrations were held constant at year 2000 values. The grey bars at right indicate the best estimate (solid line within each bar) and the likely range assessed for the six SRES marker scenarios. The assessment of the best estimate and likely ranges in the grey bars includes the AOGCMs in the left part of the figure, as well as results from a hierarchy of independent models and observational constraints. (Figures 10.4 and 10.29)

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Courtesy Tim Palmer, 2009

### **Exponential divergence vs. "coarse graining"**

The classical view of dynamical systems theory is: positive Lyapunov exponent → trajectories diverge exponentially

But the presence of multiple regimes implies a much more structured behavior in phase space

Still, the probability distribution function (pdf), when calculated forward in time, is pretty smeared out

L. A. Smith (Encycl. Atmos. Sci., 2003)



### So what's it gonna be like, by 2100?

Table SPM.2. Recent trends, assessment of human influence on the trend and projections for extreme weather events for which there is an observed late-20th century trend. {Tables 3.7, 3.8, 9.4; Sections 3.8, 5.5, 9.7, 11.2–11.9}

Phenomenon <sup>a</sup> and direction of trend	Likelihood that trend occurred in late 20th century (typically post 1960)	Likelihood of a human contribution to observed trend <sup>b</sup>	Likelihood of future trends based on projections for 21st century using SRES scenarios
Warmer and fewer cold days and nights over most land areas	Very likely°	Likely <sup>d</sup>	Virtually certain <sup>d</sup>
Warmer and more frequent hot days and nights over most land areas	Very likely*	Likely (nights) <sup>a</sup>	Virtually certain <sup>d</sup>
Warm spells/heat waves. Frequency increases over most land areas	Likely	More likely than not <sup>r</sup>	Very likely
Heavy precipitation events. Frequency (or proportion of total rainfall from heavy falls) increases over most areas	Likely	More likely than not	Very likely
Area affected by droughts increases	Likely in many regions since 1970s	More likely than not	Likely
Intense tropical cyclone activity increases	Likely in some regions since 1970	More likely than not	Likely
Increased incidence of extreme high sea level (excludes tsunamis)9	Likely	More likely than not <sup>th</sup>	Likely <sup>i</sup>

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# How important are different sources of uncertainty?

Varies, but typically no single source dominates.



## Can we, nonlinear dynamicists, help?

The uncertainties might be *intrinsic*, rather than mere "tuning problems"

If so, maybe *stochastic structural stability* could help!

Might fit in nicely with recent taste for "stochastic parameterizations"



The DDS dream of structural stability (from Abraham & Marsden, 1978)

# **Non-autonomous Dynamical Systems**

### A linear, dissipative, forced example: forward vs. pullback attraction

Consider the scalar, linear ordinary differential equation (ODE)

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 $\dot{x} = -\alpha x + \sigma t \,, \ \alpha > 0 \,, \ \sigma > 0 \,.$ 

The autonomous part of this ODE,  $\dot{x} = -\alpha x$ , is dissipative and all solutions  $x(t; x_0) = x(t; x(0) = x_0)$  converge to 0 as  $t \to +\infty$ .

What about the non-autonomous, forced ODE? As the energy being put into the system by the forcing is dissipated, we expect things to change in time. In fact, if we "pull back" far enough, replace  $x(t, x_0)$  by  $x(s, t; x_0) = x(s, t; x(s) = x_0)$ ,





 $x(s,t;x_0)$ , with  $x_0$  varying

 $a(t) = \frac{\sigma}{\alpha}(t - t)$ 

#### Remarks

We've just shown that:

$$|x(t,s;x_0)-a(t)| \underset{s
ightarrow -\infty}{\longrightarrow} 0$$
 ; for every  $t$  fixed,

and for all initial data  $x_0$ , with  $a(t) = \frac{\sigma}{\alpha}(t - 1/\alpha)$ .

 We've just encountered the concept of pullback attraction; here {a(t)} is the pullback attractor of the system (1).

#### What does it mean physically?

The pullback attractor provides a way to assess an asymptotic regime at time t — the time at which we observe the system — for a system starting to evolve from the remote past s, s << t.

- This asymptotic regime evolves with time: it is a dynamical object.
- Dissipation now leads to a dynamic object rather than to a static one, like the strange attractor of an autonomous system.

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#### Random Dynamical Systems (RDS), I - RDS theory

This theory is the counterpart for randomly forced dynamical systems (RDS) of the *geometric theory* of ordinary differential equations (ODEs). It allows one to treat stochastic differential equations (SDEs) — and more general systems driven by noise — as flows in (phase space)×(probability space). SDE~ODE, RDS~DDS, L. Arnold (1998)~V.I. Arnol'd (1983).

#### Setting:

- (i) A phase space *X*. **Example**:  $\mathbb{R}^n$ .
- (ii) A probability space (Ω, F, P). Example: The Wiener space Ω = C<sub>0</sub>(ℝ; ℝ<sup>n</sup>) with Wiener measure P.
- (iii) A model of the noise  $\theta(t) : \Omega \to \Omega$  that preserves the measure  $\mathbb{P}$ , i.e.  $\theta(t)\mathbb{P} = \mathbb{P}; \theta$  is called the driving system. **Example:**  $W(t, \theta(s)\omega) = W(t + s, \omega) - W(s, \omega);$ it starts the noise at *s* instead of t = 0.
- (iv) A mapping  $\varphi : \mathbb{R} \times \Omega \times X \to X$  with the cocycle property. **Example**: The solution operator of an SDE.

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#### RDS, III- Random attractors (RAs)

Pullback attraction to  $A(\omega)$ 

A random attractor  $A(\omega)$  is both *invariant* and "pullback" *attracting*:

- (a) Invariant:  $\varphi(t,\omega)\mathcal{A}(\omega) = \mathcal{A}(\theta(t)\omega)$ .
- (b) Attracting:  $\forall B \subset X$ ,  $\lim_{t\to\infty} \operatorname{dist}(\varphi(t, \theta(-t)\omega)B, \mathcal{A}(\omega)) = 0$  a.s.

 $B(\theta(-\tau_1)\omega)$  $\{\omega\}_X X$  $\{\theta(t)\omega\} X$  $B(\theta(-\tau, )\omega)$  $\varphi(t,\omega)A(\omega) = A(\theta(t)\omega)$  $A(\omega)$ 0  $\theta(t)\omega$ Ω  $\theta(-\tau_i)\omega$  $\theta(-\tau_{2})\omega$ 

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#### Random attractor of the stochastic Lorenz system

#### Snapshot of the random attractor (RA)



- A snapshot of the RA, A(ω), computed at a fixed time t and for the same realization ω; it is made up of points transported by the stochastic flow, from the remote past t T, T >> 1.
- We use multiplicative noise in the deterministic Lorenz model, with the classical parameter values b = 8/3,  $\sigma = 10$ , and r = 28.
- Even computed pathwise, this object supports meaningful statistics.

#### Sample measures supported by the R.A.



- We compute the probability measure on the R.A. at some fixed time *t*, and for a fixed realization ω. We show a "projection", ∫ μ<sub>ω</sub>(x, y, z)dy, with multiplicative noise: dx<sub>i</sub>=Lorenz(x<sub>1</sub>, x<sub>2</sub>, x<sub>3</sub>)dt + α x<sub>i</sub>dW<sub>t</sub>; i ∈ {1, 2, 3}.
- 10 million of initial points have been used for this picture!

#### Sample measure supported by the R.A.



• Still 1 Billion I.D., and  $\alpha = 0.5$ . Another one?

Michael Ghil Climate Change and Climate Sensitivity

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#### Sample measures evolve with time.

 Recall that these sample measures are the frozen statistics at a time *t* for a realization ω.

• How do these frozen statistics evolve with time?

#### • Action!



A day in the life of the Lorenz (1963) model's random attractor, or LORA for short; see SI in Chekroun, Simonnet & Ghil (2011, *Physica D*)

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Ghil (in *Encycl. Global Environmental Change*, 2002)



### **Classical Strange Attractor**

Physically closed system, modeled mathematically as autonomous system: neither deterministic (anthropogenic) nor random (natural) forcing.

The attractor is strange, but still fixed in time ~ "irrational" number.

Climate sensitivity ~ change in the average value (first moment) of the coordinates (*x*, *y*, *z*) as a parameter  $\lambda$  changes.



### **Random Attractor**

Physically open system, modeled mathematically as non-autonomous system: allows for deterministic (anthropogenic) as well as random (natural) forcing.

The attractor is "pullback" and evolves in time ~ "imaginary" or "complex" number.

Climate sensitivity ~ change in the statistical properties (first and higher-order moments) of the attractor as one or more parameters ( $\lambda$ ,  $\mu$ , ...) change.

Ghil (*Encyclopedia of Atmospheric Sciences*, 2<sup>nd</sup> ed., 2012)



# Parameter dependence –

It can be smooth or it can be rough: Niño-3 SSTs from intermediate coupled model for ENSO (Jin, Neelin & Ghil, 1994, 1996)

Skewness & kurtosis of the SSTs: time series of 4000 years,



### $\Delta \delta = 3 \cdot 10^{-4}$





δ

### M. Chekroun & D. Kondrashov (work in progress)

δ

## Parameter dependence – II

When it is smooth, one can optimize a GCM's single-parameter dependence



ICTP AGCM (Neelin, Bracco, Luo, McWilliams & Meyerson, PNAS, 2010)



Multi-objective algorithms avoid arbitrary weighting of criteria in a unique cost function:



Optimization algorithms that are O(N) and  $O(N^2)$ , rather than  $O(S^N)$ , where N is the number of parameters and S is the sampling density. ICTP AGCM (Neelin, Bracco, Luo, McWilliams & Meyerson, *PNAS*, 2010)

### Sample measures for an NDDE model of ENSO

### The Galanti-Tziperman (GT) model (JAS, 1999)

$$\frac{dT}{dt} = -\epsilon_T T(t) - M_0(T(t) - T_{sub}(h(t))),$$

$$\begin{split} h(t) &= M_1 e^{-\epsilon_m (\tau_1 + \tau_2)} h(t - \tau_1 - \tau_2) & \text{models for EN} \\ &- M_2 \tau_1 e^{-\epsilon_m (\frac{\tau_1}{2} + \tau_2)} \mu(t - \tau_2 - \frac{\tau_1}{2}) T(t - \tau_2 - \frac{\tau_1}{2}) \\ &+ M_3 \tau_2 e^{-\epsilon_m \frac{\tau_2}{2}} \mu(t - \frac{\tau_2}{2}) T(t - \frac{\tau_2}{2}). \end{split}$$

Seasonal forcing given by  $\mu(t) = 1 + \epsilon \cos(\omega t + \phi).$ The pullback attractor and its invariant measures were computed.

Figure shows the changes in the mean,  $2^{nd} \& 4^{th}$  moment of h(t), along with the Wasserstein distance  $d_W$ , for changes of 0–5% in the delay parameter  $\tau_{\kappa,0}$ .

Note intervals of both smooth & rough dependence!

Neutral delay-differential equation (NDDE), derived from Cane-Zebiak and Jin-Neelin models for ENSO: *T* is East-basin SST and *h* is thermocline depth.


# Pullback attractor and invariant measure of the GT model

The time-dependent pullback attractor of the GT model supports an invariant measure  $\nu = \nu(t)$ , whose density is plotted in 3-D perspective.

The plot is in delay coordinates h(t+1) vs. h(t) and the density is highly concentrated along 1-D filaments and, furthermore, exhibits sharp, near–0-D peaks on these filaments.

The Wasserstein distance *d*<sub>W</sub> between one such configuration, <sup>0.4</sup> at given parameter values, and <sup>0.2</sup> another one, at a different set of values, is proportional to the work <sup>0</sup> <sup>30</sup> needed to move the total probability mass from one configuration to the other.

Climate sensitivity  $\gamma\,$  can be defined as  $\gamma=\partial d_{\rm W}/\partial \tau$ 



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## **Concluding remarks, I – RDS and RAs**

## **Summary**

- A change of paradigm from closed, autonomous systems to open, non-autonomous ones.
- Random attractors are (i) spectacular, (ii) useful, and (iii) just starting to be explored for climate applications.

## Work in progress

- Study the effect of specific stochastic parametrizations on model robustness.
- Applications to intermediate models and GCMs.
- Implications for climate sensitivity.
- Implications for predictability?

## Yet another (grand?) unification

Lorenz (JAS, 1963)

Climate is deterministic and autonomous, but highly nonlinear.

Trajectories diverge exponentially,

forward asymptotic PDF is multimodal.

Hasselmann (*Tellus*, 1976)Climate is stochastic and noise-driven, but quite linear.Trajectories decay back to the mean,

forward asymptotic PDF is unimodal.

### **Grand unification (?)**

Climate is deterministic + stochastic, as well as highly nonlinear. Internal variability and forcing interact strongly, change and sensitivity refer to both mean and higher moments.



## Concluding remarks, II – Climate change & climate sensitivity

### What do we know?

- It's getting warmer.
- We do contribute to it.
- So we should act as best we know and can!

### What do we know less well?

- By how much?
  - Is it getting warmer ...
  - Do we contribute to it ...
- How does the climate system (atmosphere, ocean, ice, etc.) really work?
- How does natural variability interact with anthropogenic forcing?

## What to do?

- Better understand the system and its forcings.
- Explore the models', and the system's, robustness and sensitivity

- stochastic structural and statistical stability!

– linear response = response function + susceptibility function!!

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## **Some general references**

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## **Reserve slides**

## Atmospheric CO<sub>2</sub> at Mauna Loa Observatory



**GHGs rise!** 

It's gotta do with us, at least a bit, ain't it? But just how much?

*IPCC* (2007)







AR4 adjustment of 20<sup>th</sup> century simulation

www.lseca





Grantham Research Institute on Climate Change and the Environment

L.A. ("Lenny") Smith (2009) personal communication

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SDE~ODE, RDS~DDS, L. Arnold (1998)~V.I. Arnol'd (1983).

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- (iv) A mapping  $\varphi : \mathbb{R} \times \Omega \times X \to X$  with the cocycle property. **Example**: The solution operator of an SDE.

### **RDS, II - A Geometric View of SDEs**



- $\varphi$  is a random dynamical system (RDS)
- $\Theta(t)(x,\omega) = (\theta(t)\omega, \varphi(t,\omega)x)$  is a flow on the bundle

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### Non-autonomous Dynamical Systems - I

### A linear example as a paradigm

Let us first start with a very difficult problem:

Study the "dynamics" of 
$$\dot{x} = -\alpha x + \sigma t$$
,  $\alpha, \sigma > 0$ . (1)

#### First remarks:

- The system  $\dot{x} = -\alpha x$ , i.e. the autonomous part of (1), is dissipative. All the solutions of  $\dot{x} = -\alpha x$  converge to 0 as  $t \to +\infty$ .
- Is it the case for (1)? Certainly not! The autonomous part is forced; we even introduce an infinite energy over an infinite time interval: ∫<sub>0</sub><sup>+∞</sup> t dt = +∞! Forward attraction seems to be ill adapted to time-dependent forcing.

#### Goal:

Find a concept of attraction that is:

- (i) compatible with the forward concept, when there is no forcing; and
- (ii) provides a way to assess the effect of dissipation in some sense.

For that let's do some computations...

### Non-autonomous Dynamical Systems - I

#### A linear example as a paradigm

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### A French garden near the castle of La Roche-Guyon



Michael Ghil, Mickaël D. Chekroun, Eric Simonnet, Ilya Zaliapin

### Devil's quarry for a coupling parameter $\varepsilon = 0.15$ : a web of resonances



Michael Ghil, Mickaël D. Chekroun, Eric Simonnet, Ilya Zaliapin

### Effect of the noise on Devil's quarry



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#### Chekroun, Simonnet and Ghil, 2011

Timmerman & Jin (*Geophys. Res. Lett.*, 2002) have derived the following low-order, tropical-atmosphere–ocean model. The model has three variables: thermocline depth anomaly h, and

SSTs  $T_1$  and  $T_2$  in the western and eastern basin.

$$\begin{aligned} \dot{T}_1 &= -\alpha(T_1 - T_r) - \frac{2\varepsilon u}{L}(T_2 - T_1), \\ \dot{T}_2 &= -\alpha(T_2 - T_r) - \frac{w}{H_m}(T_2 - T_{sub}), \\ \dot{h} &= r(-h - bL\tau/2). \end{aligned}$$

The related diagnostic equations are:

$$T_{sub} = T_r - \frac{T_r - T_{r0}}{2} [1 - \tanh(H + h_2 - z_0)/h^*]$$
  
$$\tau = \frac{a}{\beta} (T_1 - T_2) [\xi_t - 1].$$

- $\tau$ : the wind stress anomalies,  $w = -\beta \tau / H_m$ : the equatorial upwelling.
- $u = \beta L \tau / 2$ : the zonal advection,  $T_{sub}$ : the subsurface temperature.

Wind stress bursts are modeled as white noise  $\xi_t$  of variance  $\sigma$ , and  $\varepsilon$  measures the strength of the zonal advection.

### Random attractors: the frozen statistics

### Random Shil'nikov horseshoes



### $\sigma = 0.005$



- Horseshoes can be noise-excited, left: a weakly-perturbed limit cycle, right: the same with larger noise.
- Golden: most frequently-visited areas; white 'plus' sign: most visited.

### An episode in the random's attractor life



Michael Ghil Climate Change and Climate Sensitivity

## **Devil's Bleachers in a 1-D ENSO Model**

## Ratio of ENSO frequency to annual cycle



F.-F. Jin, J.D. Neelin & M. Ghil, *Physica D*, **98**, 442-465, 1996

### Sample measure supported by the R.A.



- 1 Billion I.D., and a different color palette!
- Intensity is  $\alpha = 0.2$ .
- Do you want different noise intensities?

### Sample measure supported by the R.A.

### Another proj. of the sample measure, "friendlier"



 The next slides are similar, with different noise level α and more I.D....

Michael Ghil Toward a Mathematical Theory of Climate Sensitivity

### Sample measure supported by the R.A.



- Here  $\alpha = 0.4$ . The sample measure is approximated for another realization  $\omega$  of the noise, starting from 8 billion I.D.
- Now more serious stuff is coming...

#### Sample measures evolve with time.

 Recall that these sample measures are the frozen statistics at a time *t* for a realization ω.

• How do these frozen statistics evolve with time?

#### • Action!

Michael Ghil Toward a Mathematical Theory of Climate Sensitivity

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### Property of $\mu_{\omega}$ for chaotic stochastic systems-I

### The Sinai-Ruelle-Bowen (SRB) property

- RDS theory offers a rigorous way to define random versions of stable and unstable manifolds, via the Lyapunov spectrum, the Oseledec multiplicative theorem, and a random version of the Hartman-Grobman theorem.
- When the sample measures  $\mu_{\omega}$  of an RDS have absolutely continuous conditional measures on the random unstable manifolds, then  $\mu_{\omega}$  is called a *random SRB measure*.
- If the sample measure of an RDS φ is SRB, then its a "physical" measure in the sense that:

$$\lim_{s \to -\infty} \frac{1}{t-s} \int_{s}^{t} G \circ \varphi(s, \theta_{-s}\omega) x \, \mathrm{d}s = \int_{\mathcal{A}(\theta_{t}\omega)} G(x) \mu_{\theta_{t}\omega}(\mathrm{d}x), \quad (3)$$

for almost every  $x \in X$  (in the Lebesgue sense), and for every continuous *observable*  $G : X \to \mathbb{R}$ .

The measure μ<sub>ω</sub> is also the image of the Lebesgue measure under the stochastic flow φ: for each region of A(ω), it gives the probability to end up on that region, when starting from a volume.

### Property of $\mu_{\omega}$ for chaotic stochastic systems-II

### A remarkable theorem of Ledrappier and Young (1988)

Ledrappier and Young have proved that, that if the stationary solution, ρ, of the Fokker-Planck equation associated to an SDE presenting a Lyapunov exponent > 0, has a density w.r.t. the Lebesgue measure, then:

#### $\mu_{\omega}$ is a random SRB measure.

- This theorem applies to a large class of dissipative stochastic systems, namely the hypoelliptic ones that exhibit a Lyapunov exponent > 0: they all support a random SRB measure.
- Furthermore, we have the important relation:

$$\mathbb{E}(\mu_{\bullet}) = \rho, \tag{4}$$

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where  $\rho$  is the stationary solution of the Fokker-Planck equation, when the latter is unique.

### The Ruelle response formula

 Physically, the challenge is to find the trade-off between the physics present in the model and the stochastic parameterizations of the missing physics.

From a mathematical point of view, climate sensitivity can be related to sensitivity of SRB measures.

- The thermodynamic formalism à la Ruelle, in the RDS context, helps to understand the response of systems out-of-equilibrium, to changes in the parameterizations (Gundlach, Kifer, Liu).

$$\delta_t \mu(\mathbf{G}) = \int_{-\infty}^t d\tau \int \mu(d\mathbf{x}) X_{\tau}(\mathbf{x}) \cdot \nabla_{\mathbf{x}}(\mathbf{G} \circ \varphi_{t-\tau}(\mathbf{x})),$$

where  $\varphi_t$  is the flow of the unperturbed system  $\dot{x} = f(x)$ .

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#### The susceptibility function

• In the case  $X_t(x) = \phi(t)X(x)$ , the Ruelle response formula can be written:

$$\delta_t \mu(\mathbf{G}) = \int dt' \kappa(t-t') \phi(t'),$$

where  $\kappa$  is called the response function. The Fourier transform  $\hat{\kappa}$  of the response function is called the susceptibility function.

- In this case δ<sub>t</sub>μ(G)(ξ) = κ̂(ξ)φ̂(ξ) and since the r.h.s. is a product, there are no frequencies in the linear response that are not present in the signal.
- In general, the situation can be more complicated and the theory gives the following criterion of high sensitivity:

 $\mathfrak{C}$ : Poles of the susceptibility function  $\hat{\kappa}(\xi)$  in the upper-half plane  $\Rightarrow$  High sensitivity of the system's response function  $\kappa(t)$ .

 RDS theory offers a path for extending this criterion when random perturbations are considered.

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## climatic uncertainties & moral dilemmas



Thought leaders Rice, top left, spoke of multilateralism, while Bono, left, demanded more action on poverty. Presidents Karzai and Musharraf, right, both face troubles at home

## Feed the world today or...

# • ... keep today's climate for tomorrow?



Davos, Feb. 2008, photos by *TIME Magazine*, 11 Feb. '08; see also Hillerbrand & Ghil, *Physica D*, 2008, **237**, 2132–2138, doi:10.1016/j.physd.2008.02.015.

## The Biofuel Myth

Fine illustration of the moral dilemmas (\*).
Conclusion: "festina lentae," as the Romans (\*\*) used to say..

(\*\*) ~ Han dynasty

(\*) Hillerbrand & Ghil, *Physica D*, 2008 doi:10.1016/j.physd.2008.02.015, available on line.



## Climate Change 1816-2008



## M. Gillot, 2008, Le Monde

## T. Géricault, 1819, Le Louvre

